Research Article

On Similarity and Reducing Subspaces of the n -Shift plus Certain Weighted Volterra Operator

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Let $g(z)$ be an n -degree polynomial ($n \geq 2$). Inspired by Sarason's result, we introduce the operator T_1 defined by the multiplication operator M_g plus the weighted Volterra operator V_g on the Bergman space. We show that the operator T_1 is similar to M_g on some Hilbert space $S_g^2(\mathbb{D})$. Then for $g(z) = z^n$, by using matrix manipulations, the reducing subspaces of the corresponding operator T_2 on the Bergman space are characterized.

1. Introduction

The invariant subspace and reducing subspace problems are interesting and important themes in operator theory. The conjecture is that every bounded linear operator T on a separable Hilbert space H has a nontrivial closed invariant subspace. A closed linear nontrivial subspace X of H is called an invariant subspace for T if X is different from $\{0\}$ and H such that $TX \subset X$. If X and $X^\perp$ are both invariant subspaces for T , then X is said to be a reducing subspace for T . The invariant subspace and reducing subspace problems on the Hardy space and the Bergman space have been studied extensively in the literature. We mention here the papers [1–13] and the books [14–17] which include a lot of the information on the corresponding operator theory.

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, and let $L_a^2(\mathbb{D})$ denote the Bergman space of analytic functions which belongs to $L^2(\mathbb{D})$, where $L^2(\mathbb{D})$ is the space of square integrable functions on $\mathbb{D}$. It is well known that $L_a^2(\mathbb{D})$ is a Hilbert space. If $f \in L_a^2(\mathbb{D})$, then

$$\|f\|_2^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z), \quad (1)$$

where dA is the normalized area measure on $\mathbb{D}$, and

$$dA(z) = \frac{1}{\pi} dx dy = \frac{r}{\pi} dr d\theta. \quad (2)$$

For $f, g \in L_a^2(\mathbb{D})$, let

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} a_k z^k, \\ g(z) &= \sum_{k=0}^{\infty} b_k z^k, \end{aligned} \quad (3)$$

and then the inner product of f and g is defined by

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} dA(z) = \sum_{k=0}^{\infty} \frac{a_k \bar{b}_k}{k+1}. \quad (4)$$

In this inner product, $L_a^2(\mathbb{D})$ has an orthonormal basis $\{e_k\}_{k=0}^{\infty}$, where

$$e_k(z) = \sqrt{k+1} z^k, \quad (k = 0, 1, \dots). \quad (5)$$

Let $H^\infty(\mathbb{D})$ denote the algebra of bounded analytic functions on $\mathbb{D}$. For $\varphi \in H^\infty(\mathbb{D})$, M_φ is an analytic multiplication operator on the Bergman space defined by

$$M_\varphi f = \varphi f, \quad \text{for } f \in L_a^2(\mathbb{D}). \quad (6)$$

M_φ is a bounded linear operator on $L_a^2(\mathbb{D})$ with

$$\|M_\varphi\| = \|\varphi\|_\infty = \sup \{|\varphi(z)| : z \in \mathbb{D}\}. \quad (7)$$

Over the years it has been shown that many familiar classes of operators do have invariant subspaces. The lattice of shift operator acting on the Hardy space is completely described by Beurling's Theorem [4]. Sarason (see [11]) characterized all closed invariant subspaces of the Volterra operator

$$(Vf)(x) = \int_0^x f(y) dy, \quad (8)$$

for $f \in L^2(0, 1)$, $0 < y \leq x < 1$.

In [1], Aleman characterized boundedness and compactness of the integral operator

$$T_g f(z) = \int_0^z g'(w) f(w) dw \quad (9)$$

between Hardy space H^p and H^q for $p, q > 0$. In [2], using Beurling's Theorem, Aleman and Korenblum studied the complex Volterra operator in the Hardy space $H^2(\mathbb{D})$ defined by

$$(Vf)(z) = \int_0^z f(w) dw. \quad (10)$$

Then they characterized the lattice of closed invariant subspaces of V . Sarason (see [11]) studied the lattice of closed invariant subspaces of $M_x + V$ acting on $L^2(0, 1)$. Montes-Rodriguez et al. (see [9]) and Cowen et al. (see [5]) used the idea of Sarason to study the invariant subspaces of certain classes of composition operators on the Hardy space. Following Sarason's work, Čučković and Paudyal (see [6]) characterized the lattice of closed invariant subspaces of the shift plus complex Volterra operator on the Hardy space. In their paper, the operator T is defined by

$$(Tf)(z) = zf(z) + \int_0^z f(w) dw, \quad (11)$$

for $f \in H^2(\mathbb{D})$, $z \in \mathbb{D}$.

Ball (see [3]) and Nordgren (see [10]) studied the problem of determining the reducing subspaces for an analytic Toeplitz operator on the Hardy space. In [12], Stessin and Zhu gave a complete description of the weighted unilateral shift operator of finite multiplicity on some Hilbert spaces type I and type II. In [13], Zhu described the properties of the commutant of analytic Toeplitz operators with inner function symbols on the Hardy space and the Bergman space and characterized the reducing subspaces of a class of multiplication operators. In 2011, Douglas and Kim in [7] studied the reducing subspaces for an analytic multiplication operator M_{z^n} on the Bergman space $A_\alpha^2(A_r)$ of the annulus A_r .

Based on the above works, for an n -degree polynomial $g(z)$ ($n \geq 2$), we introduce the operator T_1 defined by the multiplication operator M_g plus the weighted Volterra operator V_g on the Bergman space. We show that the operator T_1 is similar to M_g on some Hilbert space $S_g^2(\mathbb{D})$. Then for $g(z) = z^n$, by using matrix manipulations, the reducing subspaces of corresponding operator T_2 on the Bergman space are characterized.

2. The Similarity of the Operator T_1

For an n -degree polynomial $g(z)$ ($n \geq 2$), the operator T_1 is defined by

$$\begin{aligned} (T_1 f)(z) &= M_{g(z)} f(z) + (V_g f)(z) \\ &= M_{g(z)} f(z) + \int_0^z g'(w) f(w) dw, \end{aligned} \quad (12)$$

for $f \in L_a^2(\mathbb{D})$.

To prove our result, we introduce the space $S_g^2(\mathbb{D})$ defined by

$$S_g^2(\mathbb{D}) = \left\{ h \in H(\mathbb{D}) : h(0) = 0, \frac{Dh}{g'} \in L_a^2(\mathbb{D}) \right\}, \quad (13)$$

where $H(\mathbb{D})$ is the space of holomorphic functions on the unit disk, and $D = d/dz$ is the differentiation operator. From the definition of $S_g^2(\mathbb{D})$, for $h \in S_g^2(\mathbb{D})$, we have

$$\begin{aligned} \int_{\mathbb{D}} |(Dh)(z)|^2 dA(z) &= \int_{\mathbb{D}} \left| g'(z) \frac{Dh(z)}{g'(z)} \right|^2 dA(z) \\ &\leq \|g'(z)\|_\infty^2 \left\| \frac{Dh(z)}{g'(z)} \right\|_2^2. \end{aligned} \quad (14)$$

So $Dh(z) \in L_a^2(\mathbb{D})$. It can be shown that, for any holomorphic function h with $h(0) = 0$ and $Dh(z) \in L_a^2(\mathbb{D})$, then $\|h\|_2^2 \leq \|Dh\|_2^2$. So the norm of $S_g^2(\mathbb{D})$ is defined by

$$\|h\|_{S_g^2(\mathbb{D})}^2 = \|h\|_2^2 + \|Dh\|_2^2. \quad (15)$$

Corresponding inner product is given by

$$\langle h_1, h_2 \rangle_{S_g^2(\mathbb{D})} = \langle h_1, h_2 \rangle + \langle Dh_1, Dh_2 \rangle. \quad (16)$$

In the following, we suppose that $g'(z)$ has no zero points on $\overline{\mathbb{D}}$. This condition guarantees that $S_g^2(\mathbb{D})$ is closed under the given norm.

Theorem 1. *Let V_g be the weighted Volterra operator on $L_a^2(\mathbb{D})$. Then the following statements hold:*

- (i) *Range of V_g is $S_g^2(\mathbb{D})$.*
- (ii) *V_g is a bounded isomorphism from $L_a^2(\mathbb{D})$ to $S_g^2(\mathbb{D})$, and its inverse is $(1/g'(z))D$.*
- (iii) *The operator T_1 acting on $L_a^2(\mathbb{D})$ is similar under V_g to the multiplication operator M_g on $S_g^2(\mathbb{D})$.*

Proof. (i) Let h be in the range of V_g , and then there exists $f \in L_a^2(\mathbb{D})$ such that

$$h(z) = (V_g f)(z) = \int_0^z g'(w) f(w) dw. \quad (17)$$

So $Dh(z) = g'(z)f(z)$, $h(0) = 0$, and $Dh(z)/g'(z) \in L_a^2(\mathbb{D})$. Therefore, $h(z) \in S_g^2(\mathbb{D})$. Conversely, suppose that $h \in S_g^2(\mathbb{D})$; then

$$\begin{aligned} V_g \left(\frac{Dh}{g'(z)} \right) (z) &= \int_0^z g'(w) \frac{Dh}{g'(w)} (w) dw \\ &= h(z) - h(0) = h(z). \end{aligned} \quad (18)$$

Hence h belongs to the range of V_g .

(ii) First we want to show V_g is a bounded operator on $L_a^2(\mathbb{D})$. For $f \in L_a^2(\mathbb{D})$, we have

$$\begin{aligned} \|V_g f\|_{S_g^2(\mathbb{D})}^2 &= \|V_g f\|_2^2 + \|D(V_g f)\|_2^2 \leq 2 \|D(V_g f)\|_2^2 \\ &\leq 2 \|g'(z)\|_\infty^2 \|f\|_2^2. \end{aligned} \quad (19)$$

Clearly V_g is linear. To show V_g is one to one, suppose that $f_1, f_2 \in L_a^2(\mathbb{D})$, satisfying that

$$\int_0^z g'(w) f_1(w) dw = \int_0^z g'(w) f_2(w) dw, \quad (20)$$

$\forall z \in \mathbb{D}$.

Differentiating both sides, we obtain that $f_1(z) = f_2(z)$ and hence V_g is one to one. From the definition of V_g we have that $V_g((1/g'(z))Dh)(z) = h(z)$, for $h \in S_g^2(\mathbb{D})$, and $(1/g'(z))D(V_g f)(z) = f(z)$, for $f \in L_a^2(\mathbb{D})$.

Therefore, V_g is a bounded bijective linear operator from $L_a^2(\mathbb{D})$ onto $S_g^2(\mathbb{D})$, and $V_g^{-1} = (1/g')D$.

(iii) Suppose f belongs to $L_a^2(\mathbb{D})$, and $(V_g f)(z) = h(z)$. Note that

$$\begin{aligned} (T_1 f)(z) &= (M_g f)(z) + (V_g f)(z) \\ &= g(z) \frac{h'(z)}{g'(z)} + h(z). \end{aligned} \quad (21)$$

Now applying V_g on both sides of the above equality, we obtain that

$$\begin{aligned} (V_g T_1 f)(z) &= \int_0^z g'(w) \left[\frac{h'(w)}{g'(w)} g(w) + h(w) \right] dw \\ &= \int_0^z [g(w) h'(w) + g'(w) h(w)] dw \\ &= \int_0^z D(g(w) h(w)) dw = g(z) h(z) \\ &= M_g (V_g f)(z). \end{aligned} \quad (22)$$

So $V_g T_1 = M_g V_g$ and $V_g T_1 V_g^{-1} = M_g$. That is to say, V_g transforms the operator T_1 into the multiplication operator M_g on $S_g^2(\mathbb{D})$. $\square$

3. The Reducing Subspaces of the Operator T_2

In this section, for fixed $n \geq 2$, we consider the case of $g(z) = z^n$. That is, the operator T_2 is defined by

$$\begin{aligned} (T_2 f)(z) &= M_{z^n} f(z) + (V_{z^n} f)(z) \\ &= M_{z^n} f(z) + \int_0^z n w^{n-1} f(w) dw, \end{aligned} \quad (23)$$

for $f \in L_a^2(\mathbb{D})$.

Since the n -shift operator is a contraction operator, and the operator V_{z^n} is a bounded operators, T_2 is a bounded operator on $L_a^2(\mathbb{D})$.

Let $\mathcal{A}'(T)$ denote the commutant of T , that is, $\mathcal{A}'(T) = \{S \in L(H) \mid TS = ST\}$, where $L(H)$ represents the collection of all bounded linear operators on a Hilbert space H . Then we have the following lemma.

Lemma 2. Let $L_a^2(\mathbb{D})$ be the Bergman space. If P is a bounded operator on $L_a^2(\mathbb{D})$, then $P \in \mathcal{A}'(M_z)$ if and only if P admits the following matrix representation:

$$P = \begin{pmatrix} p_{11} & 0 & 0 & \cdots & 0 & \cdots \\ p_{21} & p_{11} & 0 & \cdots & 0 & \cdots \\ p_{31} & \frac{\gamma_1}{\gamma_0} \frac{\gamma_1}{\gamma_2} p_{21} & p_{11} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ p_{k1} & \frac{\gamma_1}{\gamma_0} \frac{\gamma_{k-2}}{\gamma_{k-1}} p_{k-1,1} & \frac{\gamma_2}{\gamma_0} \frac{\gamma_{k-3}}{\gamma_{k-1}} p_{k-2,1} & \cdots & p_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (24)$$

with respect to the orthonormal basis $\{e_k(z) = \gamma_k z^k\}_{k=0}^\infty$ of $L_a^2(\mathbb{D})$, where $p_{jk} \in \mathbb{C}$ ($j, k \geq 1$) and $\gamma_k = \sqrt{k+1}$. Moreover, if P is a projection, then $P \in \mathcal{A}'(M_z)$ if and only if $P = I$ or 0 .

Proof. Denote the orthonormal basis of $L_a^2(\mathbb{D})$ by $\{e_k(z) = \gamma_k z^k\}_{k=0}^\infty$. Note that

$$M_z e_k = z \gamma_k z^k = \frac{\gamma_k}{\gamma_{k+1}} e_{k+1}. \quad (25)$$

So the operator M_z admits the following matrix representation with respect to the above basis:

$$M_z = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots \\ \gamma_0 & 0 & \cdots & 0 & 0 & \cdots \\ \gamma_1 & \frac{\gamma_1}{\gamma_2} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & \frac{\gamma_{k-1}}{\gamma_k} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \quad (26)$$

Suppose that P has the following matrix representation with respect to the orthonormal basis of $L_a^2(\mathbb{D})$:

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1k} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2k} & \cdots \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{k1} & p_{k2} & p_{k3} & \cdots & p_{kk} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (27)$$

From $M_z P = P M_z$, we have

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots \\ \frac{\gamma_0}{\gamma_1} p_{11} & \frac{\gamma_0}{\gamma_1} p_{12} & \cdots & \frac{\gamma_0}{\gamma_1} p_{1k} & \cdots \\ \frac{\gamma_1}{\gamma_2} p_{21} & \frac{\gamma_1}{\gamma_2} p_{22} & \cdots & \frac{\gamma_1}{\gamma_2} p_{2k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\gamma_{k-2}}{\gamma_{k-1}} p_{k-1,1} & \frac{\gamma_{k-2}}{\gamma_{k-1}} p_{k-1,2} & \cdots & \frac{\gamma_{k-2}}{\gamma_{k-1}} p_{k-1,k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\gamma_0}{\gamma_1} p_{12} & \frac{\gamma_1}{\gamma_2} p_{13} & \cdots & \frac{\gamma_{k-2}}{\gamma_{k-1}} p_{1k} & \cdots \\ \frac{\gamma_0}{\gamma_1} p_{22} & \frac{\gamma_1}{\gamma_2} p_{23} & \cdots & \frac{\gamma_{k-2}}{\gamma_{k-1}} p_{2k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ \frac{\gamma_0}{\gamma_1} p_{k2} & \frac{\gamma_1}{\gamma_2} p_{k3} & \cdots & \frac{\gamma_{k-2}}{\gamma_{k-1}} p_{kk} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (28)$$

Thus

$$\begin{aligned} p_{ij} &= 0, \quad i < j, \\ p_{ii} &= p_{11}, \quad i = 2, 3, \dots, \\ \frac{\gamma_{j+k-1}}{\gamma_{j-1}} p_{j+k,j} &= \frac{\gamma_{j+k-2}}{\gamma_{j-2}} p_{j+k-1,j-1}, \quad (j \geq 2, k \geq 1). \end{aligned} \quad (29)$$

So we obtain

$$P = \begin{pmatrix} p_{11} & 0 & 0 & \cdots & 0 & \cdots \\ p_{21} & p_{11} & 0 & \cdots & 0 & \cdots \\ p_{31} & \frac{\gamma_1}{\gamma_0} \frac{\gamma_1}{\gamma_2} p_{21} & p_{11} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ p_{k1} & \frac{\gamma_1}{\gamma_0} \frac{\gamma_{k-2}}{\gamma_{k-1}} p_{k-1,1} & \frac{\gamma_2}{\gamma_0} \frac{\gamma_{k-3}}{\gamma_{k-1}} p_{k-2,1} & \cdots & p_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (30)$$

Conversely, if P admits the matrix representation (30) with respect to the above basis, simple computation shows that $M_z P = P M_z$. So $P \in \mathcal{A}'(M_z)$. Moreover, if P is a projection, we deduce that $P \in \mathcal{A}'(M_z)$ if and only if P has the following form:

$$P = \begin{pmatrix} p_{11} & & & \\ & p_{11} & & \\ & & \ddots & \\ & & & p_{11} \\ & & & & \ddots \end{pmatrix}, \quad (31)$$

where $p_{11} = 1$ or 0 . $\square$

Remark 3. In fact, since M_z is irreducible on $L_a^2(\mathbb{D})$, any projection in $\mathcal{A}'(M_z)$ is I or 0 .

The following lemma will be used in the proof of main theorem.

Lemma 4. Let $H_j = \overline{\text{span}}\{e_{nk+j} : k \geq 0\}$ ($j = 0, 1, \dots, n-1$). Then

(i) $\{e_{nk+j}\}_{k=0}^\infty$ forms an orthonormal basis of H_j .

(ii) $L_a^2(\mathbb{D}) = H_0 \oplus H_1 \oplus \cdots \oplus H_{n-1}$.

(iii) H_j is a reducing subspace of T_2 .

Proof. (i) and (ii) are obvious. We only need to show (iii). Note that

$$\begin{aligned} T_2 e_{nk+j}(z) &= M_{z^n} e_{nk+j}(z) + (V_{z^n}) e_{nk+j}(z) \\ &= z^n \gamma_{nk+j} z^{nk+j} + \int_0^z n w^{n-1} \gamma_{nk+j} w^{nk+j} dw \\ &= \left(1 + \frac{n}{n(k+1)+j}\right) \frac{\gamma_{nk+j}}{\gamma_{n(k+1)+j}} e_{n(k+1)+j}. \end{aligned} \quad (32)$$

So we have $T_2 H_j \subset H_j$ and

$$\begin{aligned} T_2 (H_0 \oplus H_1 \oplus \cdots \oplus H_{j-1} \oplus H_{j+1} \oplus \cdots \oplus H_{n-1}) \\ \subset H_0 \oplus H_1 \oplus \cdots \oplus H_{j-1} \oplus H_{j+1} \oplus \cdots \oplus H_{n-1}, \end{aligned} \quad (33)$$

as desired. $\square$

Set $T_{2j} = T_2|_{H_j}$ ($j = 0, 1, \dots, n-1$). Then we have the following theorem.

Theorem 5. If $Q : L_a^2(\mathbb{D}) \rightarrow L_a^2(\mathbb{D})$ is a projection, then

$$Q \begin{pmatrix} T_{20} & & & \\ & T_{21} & & \\ & & \ddots & \\ & & & T_{2,n-1} \end{pmatrix} = \begin{pmatrix} T_{20} & & & \\ & T_{21} & & \\ & & \ddots & \\ & & & T_{2,n-1} \end{pmatrix} Q \quad (34)$$

if and only if

$$Q = \begin{pmatrix} Q_0 & & & \\ & Q_1 & & \\ & & \ddots & \\ & & & Q_{n-1} \end{pmatrix}, \quad (35)$$

where Q_i ($i = 0, 1, \dots, n-1$) is I_{H_i} or 0.

Proof. If $Q \in \mathcal{A}'(\text{diag}(T_{20}, T_{21}, \dots, T_{2,n-1}))$, note that $L_a^2(\mathbb{D}) = H_0 \oplus H_1 \oplus \dots \oplus H_{n-1}$, and then the operator Q can be decomposed in the following form:

$$Q = \begin{pmatrix} Q_{00} & Q_{01} & \cdots & Q_{0,n-1} \\ Q_{10} & Q_{11} & \cdots & Q_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n-1,0} & Q_{n-1,1} & \cdots & Q_{n-1,n-1} \end{pmatrix}, \quad (36)$$

where $Q_{ij} : H_j \rightarrow H_i$ ($i, j = 0, 1, \dots, n-1$). The condition $Q \in \mathcal{A}'(\text{diag}(T_{20}, T_{21}, \dots, T_{2,n-1}))$ yields that

$$\begin{pmatrix} Q_{00}T_{20} & Q_{01}T_{21} & \cdots & Q_{0,n-1}T_{2,n-1} \\ Q_{10}T_{20} & Q_{11}T_{21} & \cdots & Q_{1,n-1}T_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n-1,0}T_{20} & Q_{n-1,1}T_{21} & \cdots & Q_{n-1,n-1}T_{2,n-1} \end{pmatrix} = \begin{pmatrix} T_{20}Q_{00} & T_{20}Q_{01} & \cdots & T_{20}Q_{0,n-1} \\ T_{21}Q_{10} & T_{21}Q_{11} & \cdots & T_{21}Q_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ T_{2,n-1}Q_{n-1,0} & T_{2,n-1}Q_{n-1,1} & \cdots & T_{2,n-1}Q_{n-1,n-1} \end{pmatrix}. \quad (37)$$

Suppose that Q_{ij} has the following matrix representation with respect to the orthonormal basis of H_j :

$$Q_{ij} = \begin{pmatrix} q_{11}^{ij} & q_{12}^{ij} & q_{13}^{ij} & \cdots & q_{1k}^{ij} & \cdots \\ q_{21}^{ij} & q_{22}^{ij} & q_{23}^{ij} & \cdots & q_{2k}^{ij} & \cdots \\ q_{31}^{ij} & q_{32}^{ij} & q_{33}^{ij} & \cdots & q_{3k}^{ij} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{k1}^{ij} & q_{k2}^{ij} & q_{k3}^{ij} & \cdots & q_{kk}^{ij} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}, \quad (38)$$

where $q_{t_1+1,t_2+1}^{ij} = \langle Q_{ij}e_{t_2n+j}, e_{t_1n+i} \rangle$ ($t_1, t_2 = 0, 1, \dots$).

From (32), we know the operator T_{2j} admits the following matrix representation with respect to the basis of H_j :

$$T_{2j} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots \\ \left(1 + \frac{n}{n+j}\right) \frac{\gamma_j}{\gamma_{n+j}} & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \left(1 + \frac{n}{2n+j}\right) \frac{\gamma_{n+j}}{\gamma_{2n+j}} & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & \left(1 + \frac{n}{n(k+1)+j}\right) \frac{\gamma_{nk+j}}{\gamma_{n(k+1)+j}} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \end{pmatrix}. \quad (39)$$

Case 1 ($i = j$). From (37), we get $Q_{jj}T_{2j} = T_{2j}Q_{jj}$ ($j = 0, 1, \dots, n-1$). Applying Lemma 2, we know Q_{jj} has the form

similar to (30). Note that Q is a projection, so $Q_{jj} = Q_{jj}^*$. Hence $Q_{jj} = q_{11}^{jj}I_{H_j}$ ($q_{11}^{jj} \in \mathbb{R}$).

Case 2 ($i \neq j$). From (37), we have that $T_{2i}Q_{ij} = Q_{ij}T_{2j}$. This equality is equivalent to

$$\begin{pmatrix} 0 & 0 & \cdot & 0 & \cdot \\ \left(1 + \frac{n}{n+i}\right) \frac{\gamma_i}{\gamma_{n+i}} q_{11}^{ij} & \left(1 + \frac{n}{n+i}\right) \frac{\gamma_i}{\gamma_{n+i}} q_{12}^{ij} & \cdot & \left(1 + \frac{n}{n+i}\right) \frac{\gamma_i}{\gamma_{n+i}} q_{1k}^{ij} & \cdot \\ \left(1 + \frac{n}{2n+i}\right) \frac{\gamma_{n+i}}{\gamma_{2n+i}} q_{21}^{ij} & \left(1 + \frac{n}{2n+i}\right) \frac{\gamma_{n+i}}{\gamma_{2n+i}} q_{22}^{ij} & \cdot & \left(1 + \frac{n}{2n+i}\right) \frac{\gamma_{n+i}}{\gamma_{2n+i}} q_{2k}^{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \left(1 + \frac{n}{(k-1)n+i}\right) \frac{\gamma_{(k-2)n+i}}{\gamma_{(k-1)n+i}} q_{k-1,1}^{ij} & \left(1 + \frac{n}{(k-1)n+i}\right) \frac{\gamma_{(k-2)n+i}}{\gamma_{(k-1)n+i}} q_{k-1,2}^{ij} & \cdot & \left(1 + \frac{n}{(k-1)n+i}\right) \frac{\gamma_{(k-2)n+i}}{\gamma_{(k-1)n+i}} q_{k-1,k}^{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \left(1 + \frac{n}{n+j}\right) \frac{\gamma_j}{\gamma_{n+j}} q_{12}^{ij} & \left(1 + \frac{n}{2n+j}\right) \frac{\gamma_{n+j}}{\gamma_{2n+j}} q_{13}^{ij} & \cdot & \left(1 + \frac{n}{(k-1)n+j}\right) \frac{\gamma_{(k-2)n+j}}{\gamma_{(k-1)n+j}} q_{1k}^{ij} & \cdot \\ \left(1 + \frac{n}{n+j}\right) \frac{\gamma_j}{\gamma_{n+j}} q_{22}^{ij} & \left(1 + \frac{n}{2n+j}\right) \frac{\gamma_{n+j}}{\gamma_{2n+j}} q_{23}^{ij} & \cdot & \left(1 + \frac{n}{(k-1)n+j}\right) \frac{\gamma_{(k-2)n+j}}{\gamma_{(k-1)n+j}} q_{2k}^{ij} & \cdot \\ \left(1 + \frac{n}{n+j}\right) \frac{\gamma_j}{\gamma_{n+j}} q_{32}^{ij} & \left(1 + \frac{n}{2n+j}\right) \frac{\gamma_{n+j}}{\gamma_{2n+j}} q_{33}^{ij} & \cdot & \left(1 + \frac{n}{(k-1)n+j}\right) \frac{\gamma_{(k-2)n+j}}{\gamma_{(k-1)n+j}} q_{3k}^{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \left(1 + \frac{n}{n+j}\right) \frac{\gamma_j}{\gamma_{n+j}} q_{k2}^{ij} & \left(1 + \frac{n}{2n+j}\right) \frac{\gamma_{n+j}}{\gamma_{2n+j}} q_{k3}^{ij} & \cdot & \left(1 + \frac{n}{(k-1)n+j}\right) \frac{\gamma_{(k-2)n+j}}{\gamma_{(k-1)n+j}} q_{kk}^{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \quad (40)$$

So we obtain

$$Q_{ij} = \begin{pmatrix} q_{11}^{ij} & 0 & 0 & 0 & \cdots \\ q_{21}^{ij} & \frac{1+n/(n+i)}{1+n/(n+j)} \frac{\gamma_i}{\gamma_j} \frac{\gamma_{n+j}}{\gamma_{n+i}} q_{11}^{ij} & 0 & 0 & \cdots \\ q_{31}^{ij} & \frac{1+n/(2n+i)}{1+n/(n+j)} \frac{\gamma_{n+j}}{\gamma_j} \frac{\gamma_{n+i}}{\gamma_{2n+i}} q_{21}^{ij} & \frac{1+n/(n+i)}{1+n/(n+j)} \frac{1+n/(2n+i)}{1+n/(2n+j)} \frac{\gamma_i}{\gamma_j} \frac{\gamma_{2n+j}}{\gamma_{2n+i}} q_{11}^{ij} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \quad (41)$$

On the other hand, from (37) we also have $T_{2j}Q_{ji} = Q_{ji}T_{2i}$. In the same way as for $T_{2i}Q_{ij} = Q_{ij}T_{2j}$, we get

$$Q_{ji} = \begin{pmatrix} q_{11}^{ji} & 0 & 0 & 0 & \cdots \\ q_{21}^{ji} & \frac{1+n/(n+j)}{1+n/(n+i)} \frac{\gamma_j}{\gamma_i} \frac{\gamma_{n+i}}{\gamma_{n+j}} q_{11}^{ji} & 0 & 0 & \cdots \\ q_{31}^{ji} & \frac{1+n/(2n+j)}{1+n/(n+i)} \frac{\gamma_{n+i}}{\gamma_i} \frac{\gamma_{n+j}}{\gamma_{2n+j}} q_{21}^{ji} & \frac{1+n/(n+j)}{1+n/(n+i)} \frac{1+n/(2n+j)}{1+n/(2n+i)} \frac{\gamma_j}{\gamma_i} \frac{\gamma_{2n+i}}{\gamma_{2n+j}} q_{11}^{ji} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \quad (42)$$

Q is a projection that yields $Q_{ij}^* = Q_{ji}$. Thus we have $q_{li}^{ij} = 0$ ($l = 2, 3, \dots$). Solving the system of equations

$$\begin{aligned} \bar{q}_{11}^{ij} &= q_{11}^{ji}, \\ \frac{1+n/(n+i)}{1+n/(n+j)} \frac{\gamma_i}{\gamma_j} \frac{\gamma_{n+j}}{\gamma_{n+i}} \bar{q}_{11}^{ij} &= \frac{1+n/(n+j)}{1+n/(n+i)} \frac{\gamma_j}{\gamma_i} \frac{\gamma_{n+i}}{\gamma_{n+j}} q_{11}^{ji}, \end{aligned} \quad (43)$$

we get $q_{11}^{ij} = 0$. Therefore, $Q_{ij} = 0$. $Q^2 = Q$ implies that

$$Q = \begin{pmatrix} Q_0 & & & \\ & Q_1 & & \\ & & \ddots & \\ & & & Q_{n-1} \end{pmatrix}, \quad (44)$$

where Q_i ($i = 0, 1, \dots, n-1$) is I_{H_i} or 0.

Conversely, if

$$Q = \begin{pmatrix} Q_0 & & & \\ & Q_1 & & \\ & & \ddots & \\ & & & Q_{n-1} \end{pmatrix}, \quad (45)$$

where Q_i ($i = 0, 1, \dots, n-1$) is I_{H_i} or 0, it is obvious that $Q \in \mathcal{A}'(\text{diag}(T_{20}, T_{21}, \dots, T_{2,n-1}))$. $\square$

From [14], we know that determining the reducing subspaces of T_2 is equivalent to finding the projection in the commutant of T_2 . Thus we have the following conclusion.

Theorem 6 (main theorem). *Let $L_a^2(\mathbb{D})$ be the Bergman space. For $n \geq 2$, the operator T_2 has 2^n reducing subspaces with minimal reducing subspaces $H_0, H_1, \dots, H_{n-1}$.*

Proof. Suppose that $QT_2 = T_2Q$ for a projection Q . Note that $T_2|_{L_a^2(\mathbb{D})} = T_{20} \oplus T_{21} \oplus \dots \oplus T_{2,n-1}$. Hence

$$\begin{aligned} Q \begin{pmatrix} T_{20} & & & \\ & T_{21} & & \\ & & \ddots & \\ & & & T_{2,n-1} \end{pmatrix} \\ = \begin{pmatrix} T_{20} & & & \\ & T_{21} & & \\ & & \ddots & \\ & & & T_{2,n-1} \end{pmatrix} Q. \end{aligned} \quad (46)$$

By Theorem 5, the projection operator

$$Q = \begin{pmatrix} Q_0 & & & \\ & Q_1 & & \\ & & \ddots & \\ & & & Q_{n-1} \end{pmatrix}, \quad (47)$$

where Q_i ($i = 0, 1, \dots, n-1$) is I_{H_i} or 0. Applying Lemma 4, we have that the reducing subspaces of T_2 are

$$c_0 H_0 \oplus c_1 H_1 \oplus \dots \oplus c_{n-1} H_{n-1}, \quad (48)$$

where $c_i = 0$ or 1 ($i = 0, 1, \dots, n-1$),

and the minimal reducing subspaces are $H_0, H_1, \dots, H_{n-1}$. $\square$

4. Some Consequences

In this section, we use the characterization of the reducing subspaces of T_2 to obtain a description of reducing subspaces of M_{z^n} on $S_n^2(\mathbb{D})$, which is similar to T_2 .

The space $S_n^2(\mathbb{D})$ is defined by

$$\begin{aligned} S_n^2(\mathbb{D}) &= \{f \in H(\mathbb{D}) : Df \in L_a^2(\mathbb{D}), f^{(l)}(0) = 0, l \\ &= 0, 1, \dots, n-1\}. \end{aligned} \quad (49)$$

If $f(z) \in S_n^2(\mathbb{D})$, it follows that $f(z) = \sum_{k=0}^{\infty} a_k z^{n+k}$, and then $Df(z) = \sum_{k=0}^{\infty} (n+k) a_k z^{n+k-1}$. Since $Df \in L_a^2(\mathbb{D})$, we have $\|Df(z)\|_2^2 = \sum_{k=0}^{\infty} (n+k) |a_k|^2$, and

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^2 dA(z) &= \sum_{k=0}^{\infty} \frac{|a_k|^2}{n+k+1} \leq \sum_{k=0}^{\infty} (n+k) |a_k|^2 \\ &= \|Df(z)\|_2^2. \end{aligned} \quad (50)$$

So $f(z) \in L_a^2(\mathbb{D})$. The norm of $S_n^2(\mathbb{D})$ is defined by

$$\|f\|_{S_n^2(\mathbb{D})}^2 = \|f\|_2^2 + \|Df\|_2^2. \quad (51)$$

Corresponding inner product is given by

$$\langle f, g \rangle_{S_n^2(\mathbb{D})} = \langle f, g \rangle + \langle Df, Dg \rangle. \quad (52)$$

The proof of the following theorem is similar to the proof of Theorem 1, so we omit it.

Theorem 7. *Let V_{z^n} be the weighted Volterra operator on $L_a^2(\mathbb{D})$. Then the following statements hold:*

- (i) Range of V_{z^n} is $S_n^2(\mathbb{D})$.
- (ii) V_{z^n} is a bounded isomorphism from $L_a^2(\mathbb{D})$ to $S_n^2(\mathbb{D})$, and its inverse is $(1/nz^{n-1})D$.
(Remark: Note $(1/nz^{n-1})D$ acting on $S_n^2(\mathbb{D})$, and from the definition of $S_n^2(\mathbb{D})$, we know that $z = 0$ is a removable singular point of $(1/nz^{n-1})Dg(z)$, for $g(z) \in S_n^2(\mathbb{D})$. We can define $(1/nz^{n-1})Dg(z)|_{z=0}$ by $\lim_{z \rightarrow 0} (1/nz^{n-1})Dg(z)$. Then $z = 0$ can be viewed as an analytic point of $(1/nz^{n-1})Dg(z)$).
- (iii) The operator T_2 acting on $L_a^2(\mathbb{D})$ is similar under V_{z^n} to the multiplication operator M_{z^n} on $S_n^2(\mathbb{D})$.

Note that $S_n^2(\mathbb{D}) = \overline{\text{span}}\{z^{k+n} : k \geq 0\}$; from (52), we have

$$\begin{aligned} & \langle z^{k+n}, z^{m+n} \rangle_{S_n^2(\mathbb{D})} \\ &= \langle z^{k+n}, z^{m+n} \rangle + \langle (k+n)z^{k+n-1}, (m+n)z^{m+n-1} \rangle \\ &= \begin{cases} \frac{(k+n)(k+n+1)+1}{k+n+1}, & k=m, \\ 0, & k \neq m. \end{cases} \end{aligned} \quad (53)$$

Thus, the orthonormal basis of $S_n^2(\mathbb{D})$ is given by

$$\left\{ f_{k,n}(z) = \sqrt{\frac{k+n+1}{(k+n)(k+n+1)+1}} z^{k+n} \right\}_{k=0}^{\infty}. \quad (54)$$

In order to characterize the reducing subspaces of M_{z^n} on $S_n^2(\mathbb{D})$, we need the following lemma which is similar to Lemma 4.

Lemma 8. Let $L_j = \overline{\text{span}}\{f_{nk+j,n} : k \geq 0\}$ ($j = 0, 1, \dots, n-1$). Then

- (i) $\{f_{nk+j,n}\}_{k=0}^{\infty}$ forms an orthonormal basis of L_j .
- (ii) $S_n^2(\mathbb{D}) = L_0 \oplus L_1 \oplus \dots \oplus L_{n-1}$.
- (iii) L_j is a reducing subspace of M_{z^n} .

Corollary 9. Let $S_n^2(\mathbb{D})$ be described as the above. For $n \geq 2$, the multiplication operator M_{z^n} has 2^n reducing subspaces with minimal reducing subspaces $L_0, L_1, \dots, L_{n-1}$.

Proof. The proof of Corollary 9 is similar to Theorems 5 and 6. Here we omit it. $\square$

Remark 10. In fact, $S_n^2(\mathbb{D})$ is a Hilbert space of type I which is considered by Stessin and Zhu in [12]. So Lemma 8 and Corollary 9 also follow from Stessin and Zhu's paper [12].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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