Research Article

Multiple Solutions for a Nonlinear Fractional Boundary Value Problem via Critical Point Theory

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This paper is concerned with the existence of multiple solutions for the following nonlinear fractional boundary value problem: $D_{T-}^{\alpha}(a(x)D_{0+}^{\alpha}u(x)) = f(x, u(x))$, $x \in [0, T]$, $u(0) = u(T) = 0$, where $\alpha \in (1/2, 1]$, $a(x) \in L^{\infty}[0, T]$ with $a_0 = \text{ess inf}_{x \in [0, T]} a(x) > 0$, D_{T-}^{α} and D_{0+}^{α} stand for the left and right Riemann-Liouville fractional derivatives of order α , respectively, and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. The existence of infinitely many nontrivial high or small energy solutions is obtained by using variant fountain theorems.

1. Introduction

Consider the following nonlinear fractional boundary value problem (BVP for short):

$$\begin{aligned} D_{T-}^{\alpha}(a(x)D_{0+}^{\alpha}u(x)) &= f(x, u(x)), \quad x \in [0, T], \\ u(0) &= u(T) = 0, \end{aligned} \quad (1)$$

where $\alpha \in (1/2, 1]$, $a(x) \in L^{\infty}[0, T]$ with $a_0 = \text{ess inf}_{x \in [0, T]} a(x) > 0$, D_{T-}^{α} and D_{0+}^{α} stand for the left and right Riemann-Liouville fractional derivatives of order α , respectively, and $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Fractional calculus has numerous applications in various fields, including signal processing and control, fractal theory, neural network model, mechanics and engineering, and chemical physics. For details, see [1–5] and the references therein. Fractional differential operators have attracted many researchers' attention because of their applications in modeling physical phenomena exhibiting anomalous diffusion. Recently, the existence of solutions for nonlinear fractional differential equations was obtained by using nonlinear analysis, including some fixed theorems, coincidence degree theory, the monotone iterative method, and the method of upper-lower solutions. It is worth mentioning that the methods above cannot be used to deal with BVP (1) because it is very difficult to find the equivalent integral equation.

On the other hand, variational methods have proved to be a very effective approach in dealing with the existence and multiple solutions for fractional boundary value problem including both left and right fractional derivatives; see [6–16].

For example, in [7], by using mountain pass theorem, Jiao and Zhou studied the existence of at least one nontrivial solution for the following fractional boundary value problem:

$$\begin{aligned} D_{T-}^{\alpha}(D_{0+}^{\alpha}u(t)) &= \nabla F(t, u(t)), \quad t \in [0, T], \\ u(0) &= u(T) = 0, \end{aligned} \quad (2)$$

where $\alpha \in (0, 1]$, $F : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a suitable given function, and $\nabla F(t, x)$ is the gradient of F with respect to x .

In [8], by means of variational methods and critical point theory due to Bonanno and Marano, the existence of three distinct weak solutions is obtained for the following fractional boundary value problem:

$$\begin{aligned} D_{T-}^{\alpha}(a(t)D_{0+}^{\alpha}u(t)) &= \lambda F_u(t, u(t), v(t)), \quad t \in [0, T], \\ D_{T-}^{\beta}(b(t)D_{0+}^{\beta}u(t)) &= \lambda F_v(t, u(t), v(t)), \quad t \in [0, T], \\ u(0) &= u(T) = 0, \\ v(0) &= v(T) = 0, \end{aligned} \quad (3)$$

where λ is a positive real parameter, $\alpha, \beta \in (0, 1]$, $a, b \in L^\infty[0, T]$ with $a_0 = \text{ess inf}_{x \in [0, T]} a(x) > 0$ and $b_0 = \text{ess inf}_{x \in [0, T]} b(x) > 0$, $F : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function such that $F(\cdot, x, y)$ is continuous in $[0, T]$, and $F(t, \cdot, \cdot)$ is a C^1 function.

In [9], by using fountain theorem and dual fountain theorem, the existence of infinitely many solutions is established for the following fractional boundary value problem:

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} D_{0+}^{-\beta} (u'(t)) + \frac{1}{2} D_{T-}^{-\beta} (u'(t)) \right) + \nabla F(t, u(t)) \\ = 0, \quad t \in [0, T], \\ u(0) = u(T) = 0, \end{aligned} \quad (4)$$

where $D_{0+}^{-\beta}$ and $D_{T-}^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta \leq 1$, respectively.

To the best of our knowledge, there are fewer results on multiple solutions for system (1). The purpose of this paper is to fill this gap. Inspired by the above references, we apply variant fountain theorems to study the existence of infinitely many small or high energy solutions for BVP (1). It should be noted that the (P.S.) condition is not needed, which is an important condition assumed in [7, 9]. As is well known, the so-called Ambrosetti-Rabinowitz condition

$$0 < \theta F(x, u) \leq u f(x, u) \quad (5)$$

for some $\theta > 2$ is often assumed to deduce the boundedness of the (P.S.) sequences. By taking advantage of the variant fountain theorems, the Ambrosetti-Rabinowitz condition is not needed.

The rest of this paper is organized as follows. Section 2 contains preliminary results. In Section 3, we apply variant fountain theorems to prove the existence of infinitely many small or high energy solutions for BVP (1). In Section 4, two examples are presented to illustrate the main results.

2. Preliminaries

To obtain multiple solutions for BVP (1), it is necessary to construct a suitable function space and introduce some definitions, preliminary lemmas.

Definition 1 (see [17]). Let f be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional derivatives of order α for function f , denoted by $D_{a+}^\alpha f(x)$ and $D_{b-}^\alpha f(x)$, are defined by

$$\begin{aligned} D_{a+}^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-s)^{n-\alpha-1} f(s) ds, \\ D_{b-}^\alpha f(x) &= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (s-x)^{n-\alpha-1} f(s) ds, \end{aligned} \quad (6)$$

where $x \in [a, b]$, $n-1 \leq \alpha < n$, and $n \in \mathbb{N}$.

Lemma 2 (see [18]). If $f(a) = f(b) = 0$, $f' \in L^\infty([a, b], \mathbb{R})$, $g \in L^1([a, b], \mathbb{R})$, or $g(a) = g(b) = 0$, $g' \in L^\infty([a, b], \mathbb{R})$, $f \in L^1([a, b], \mathbb{R})$, then

$$\int_a^b [D_{a+}^\alpha f(x)] g(x) dx = \int_a^b [D_{b-}^\alpha g(x)] f(x) dx. \quad (7)$$

Denote

$$\begin{aligned} \|u\|_{L^p} &= \left(\int_0^T |u(x)|^p dx \right)^{1/p}, \\ \|u\|_\infty &= \max_{x \in [0, T]} |u(x)|. \end{aligned} \quad (8)$$

Definition 3 (Let $0 < \alpha \leq 1$). The fractional derivative space E_0^α is defined by the closure of $C_0^\infty([0, T], \mathbb{R})$ with respect to the norm

$$\|u\|_\alpha = \left(\int_0^T |u(x)|^2 dx + \int_0^T a(x) |D_{0+}^\alpha u(x)|^2 dx \right)^{1/2}, \quad (9)$$

$\forall u \in E_0^\alpha$.

Remark 4. (i) The fractional derivative space E_0^α is the space of functions $u \in L^2([0, T], \mathbb{R})$ having an α -order Riemann-Liouville fractional derivative $D_{0+}^\alpha u \in L^2([0, T], \mathbb{R})$, $a(x) \in L^\infty[0, T]$ with $a_0 = \text{ess inf}_{x \in [0, T]} a(x) > 0$, and $u(0) = u(T) = 0$. (ii) E_0^α is a reflexive and separable Banach space.

Lemma 5 (see [18], let $1/2 < \alpha \leq 1$). For any $u \in E_0^\alpha$, one has

$$\begin{aligned} \|u\|_{L^2} &\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|D_{0+}^\alpha u\|_{L^2}, \\ \|u\|_\infty &\leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha) \sqrt{2\alpha-1}} \|D_{0+}^\alpha u\|_{L^2}. \end{aligned} \quad (10)$$

Remark 6.

(i) By (10), we can get

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1) \sqrt{a_0}} \left(\int_0^T a(x) |D_{0+}^\alpha u(x)|^2 dx \right)^{1/2}, \quad (11)$$

$\|u\|_\infty$

$$\leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha) \sqrt{a_0} (2\alpha-1)} \left(\int_0^T a(x) |D_{0+}^\alpha u(x)|^2 dx \right)^{1/2}. \quad (12)$$

(ii) According to (11) and (12), for $p \geq 1$, we can obtain

$$\begin{aligned} \|u\|_{L^p} &\leq \frac{T^{\alpha-1/2+1/p}}{\Gamma(\alpha) \sqrt{a_0} (2\alpha-1)} \left(\int_0^T a(x) |D_{0+}^\alpha u(x)|^2 dx \right)^{1/2}, \end{aligned} \quad (13)$$

(iii) From (11), it is easy to see that the norm of E_0^α defined in (9) is equivalent to the norm

$$\|u\|_\alpha = \left(\int_0^T a(x) |D_{0+}^\alpha u(x)|^2 dx \right)^{1/2}, \quad \forall u \in E_0^\alpha. \quad (14)$$

(iv) E_0^α is a separable Hilbert space and the corresponding inner product is

$$(u, v) = \int_0^T a(x) D_{0+}^\alpha u(x) D_{T-}^\alpha v(x) dx, \quad u, v \in E_0^\alpha. \quad (15)$$

Lemma 7 (see [18], let $1/2 < \alpha \leq 1$). *If the sequence u_k converges weakly to u in E_0^α , that is, $u_k \rightharpoonup u$, then $u_k \rightarrow u$ in $C[0, T]$, that is, $\|u - u_k\|_\infty \rightarrow 0$ as $k \rightarrow \infty$.*

In the following, we denote $E = E_0^\alpha$, $\|u\| = \|u\|_\alpha$, $\|u\|_p = \|u\|_{L^p}$ for convenience.

Definition 8. A function $u \in E$ is a weak solution of BVP (1) if

$$\begin{aligned} & \int_0^T a(x) D_{0+}^\alpha u(x) D_{0+}^\alpha v(x) - \int_0^T f(x, u(x)) v(x) dx \\ & = 0, \quad \forall v \in E. \end{aligned} \quad (16)$$

The energy functional $I : E \rightarrow \mathbb{R}$ associated with BVP (1) is defined by

$$\begin{aligned} I(u) &= \frac{1}{2} \int_0^T a(x) |D_{0+}^\alpha u(x)|^2 dx \\ &\quad - \int_0^T F(x, u(x)) dx, \quad \forall u \in E, \end{aligned} \quad (17)$$

where $F(x, u) = \int_0^u f(x, s) ds$. It is easily seen that $I \in C^1(E, \mathbb{R})$.

For any $v(x) \in E$, one has

$$\begin{aligned} (I'(u), v) &= \int_0^T a(x) D_{0+}^\alpha u(x) D_{0+}^\alpha v(x) \\ &\quad - \int_0^T f(x, u(x)) v(x) dx. \end{aligned} \quad (18)$$

Moreover, the critical points of I correspond to the weak solutions of BVP (1).

To prove our main results, we need the following two variant fountain theorems in [19].

Let X be a Banach space with the norm $\|\cdot\|$ and $X = \overline{\bigoplus_{j \in \mathbb{N}} X_j}$ with $\dim X_j < \infty$ for each $j \in \mathbb{N}$. Set $W_k = \bigoplus_{j=1}^k X_j$, $Z_k = \bigoplus_{j=k}^\infty X_j$, $B_k = \{u \in W_k : \|u\| \leq \rho_k\}$, $S_k = \{u \in Z_k : \|u\| = r_k\}$, where $\rho_k > r_k > 0$.

Consider a family of C^1 functions $I_\lambda : X \rightarrow \mathbb{R}$ defined by

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2]. \quad (19)$$

Lemma 9. *Assume that the functional I_λ defined above satisfies*

- (A₁) I_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1, 2]$, and $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times X$;
- (A₂) $B(u) \geq 0$ for all $u \in X$, $A(u) \rightarrow \infty$ or $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$; or

(A₃) $B(u) \leq 0$ for all $u \in X$, $B(u) \rightarrow -\infty$ as $\|u\| \rightarrow \infty$;

(A₄) There exist $\rho_k > r_k > 0$ such that

$$\begin{aligned} b_k(\lambda) &= \inf_{u \in Z_k, \|u\|=r_k} I_\lambda(u) > a_k(\lambda) = \max_{u \in W_k, \|u\|=\rho_k} I_\lambda(u), \\ &\quad \forall \lambda \in [1, 2]. \end{aligned} \quad (20)$$

Then

$$b_k(\lambda) \leq c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2], \quad (21)$$

where $\Gamma_k = \{\gamma \in C(B_k, X) : \gamma \text{ is odd, } \gamma|_{\partial B_k} = id\}$. Moreover, for almost every $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}$ such that

$$\begin{aligned} & \sup_n \|u_n^k(\lambda)\| < \infty, \\ & I'_\lambda(u_n^k(\lambda)) \rightarrow 0, \\ & I_\lambda(u_n^k(\lambda)) \rightarrow c_k(\lambda) \\ & \text{as } n \rightarrow \infty. \end{aligned} \quad (22)$$

Lemma 10. *Assume that the functional I_λ defined above satisfies*

- (B₁) I_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1, 2]$, and $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times X$;
- (B₂) $B(u) \geq 0$ for all $u \in X$, $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace of X ;
- (B₃) There exist $\rho_k > r_k > 0$ such that

$$\begin{aligned} a_k(\lambda) &= \inf_{u \in Z_k, \|u\|=\rho_k} I_\lambda(u) \geq 0, \\ b_k(\lambda) &= \max_{u \in W_k, \|u\|=r_k} I_\lambda(u) < 0, \\ &\quad \forall \lambda \in [1, 2], \end{aligned} \quad (23)$$

$$d_k(\lambda) = \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \rightarrow 0$$

as $k \rightarrow \infty$ uniformly for $\lambda \in [1, 2]$.

Then there exist $\lambda_n \rightarrow 1$, $u_n(\lambda_n) \in W_n$ such that

$$\begin{aligned} & I'_{\lambda_n}|_{W_n}(u(\lambda_n)) = 0, \\ & I_{\lambda_n}(u(\lambda_n)) \rightarrow c_k \\ & \text{as } n \rightarrow \infty, \end{aligned} \quad (24)$$

where $c_k \in [d_k(2), b_k(1)]$. In particular, if $\{u(\lambda_n)\}$ has a convergent subsequence for every k , then I_1 has infinitely many nontrivial critical points $\{u_k\} \in X \setminus \{0\}$ satisfying $I_1(u_k) \rightarrow 0^-$ as $k \rightarrow \infty$.

Since E is a separable Hilbert space in terms of Remark 6. Let $\{e_j\}$ be an orthonormal basis of E and define

$X_j = \text{span}\{e_j\}$, $W_k = \bigoplus_{j=1}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. In order to apply Lemmas 9 and 10 to prove the existence of infinitely many solutions of BVP (1), we define A , B , and I_λ on fractional derivative space E by

$$\begin{aligned} A(u) &= \frac{1}{2} \int_0^T (x) (D_{0^+}^\alpha u(x))^2 dx, \\ B(u) &= \int_0^T F(x, u(x)) dx, \\ I_\lambda(u) &= A(u) - \lambda B(u) \\ &= \frac{1}{2} \|u\|^2 - \lambda \int_0^T F(x, u(x)) dx, \end{aligned} \quad (25)$$

$$\forall u \in E, \lambda \in [1, 2].$$

For convenience, we list the following assumptions.

(H₁) There exist constants $c_1 > 0$, $c_2 > 0$, and $p > 2$ such that

$$|f(x, u)| \leq c_1 |u| + c_2 |u|^{p-1}, \quad \forall x \in [0, T], u \in \mathbb{R}. \quad (26)$$

(H₂) There exist $\tau > 2$ and $M > 0$ such that

$$\begin{aligned} -\tau F(x, u) + uf(x, u) &\geq 0, \\ |u| &\geq M, \text{ uniformly for } x \in [0, T]. \end{aligned} \quad (27)$$

(H₃) $F(x, u) \geq 0$, $\forall (x, u) \in [0, T] \times \mathbb{R}$.

(H₄) $F(x, -u) = F(x, u)$, $\forall (x, u) \in [0, T] \times \mathbb{R}$.

(H₅) There exist $\eta \in (1, 2)$ and $b(x) \in L^{2/(2-\eta)}[0, T]$ with $b(x) \geq 0$ such that

$$|f(x, u)| \leq b(x) |u|^\eta, \quad \forall (x, u) \in [0, T] \times \mathbb{R}. \quad (28)$$

(H₆) There exist $r \in (1, \eta)$ and $d > 0$ such that

$$\lim_{|u| \rightarrow \infty} \frac{F(x, u)}{|u|^r} > d, \quad \text{uniformly for } x \in [0, T]. \quad (29)$$

(V) $a(x) \in L^\infty[0, T]$, and $\text{ess inf}_{x \in [0, T]} a(x) = a_0 > 0$.

(V') $a(x) \in L^\infty[0, T]$, and $\text{ess inf}_{x \in [0, T]} a(x) = a_0 \geq (\tau/2 - 1)^{-1} 2h(T^\alpha/\Gamma(\alpha + 1))^2 > 0$, where

$$\begin{aligned} h &= \inf \{l : |-\tau F(x, u) + uf(x, u)| \leq lu^2, x \\ &\in [0, T], |u| \leq M\}. \end{aligned} \quad (30)$$

3. Main Results

Theorem 11. Assume that (H₁)–(H₄), (V') hold. Then BVP (1) possesses infinitely many high energy solutions $u^k \in E \setminus \{0\}$ satisfying

$$\begin{aligned} \frac{1}{2} \int_0^T a(x) |D_{0^+}^\alpha u^k(x)|^2 dx - \int_0^T F(x, u^k(x)) dx \\ \longrightarrow +\infty, \text{ as } k \longrightarrow \infty. \end{aligned} \quad (31)$$

Proof. For any $\varepsilon > 0$, it follows from (H₁) that there exist $c_3 > 0$ and $c_4 > 0$ such that

$$|F(x, u)| \leq c_3 |u|^2 + c_4 |u|^p. \quad (32)$$

Combining (32) and (13), it is easily seen that I_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1, 2]$. By (H₄), $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Thus, condition (A₁) holds. Assumption (H₃) implies that $B(u) \geq 0$. Condition (A₂) holds for the fact that $A(u) = (1/2)\|u\|^2 \rightarrow \infty$ as $n \rightarrow \infty$ and $B(u) \geq 0$.

In what follows, we verify condition (A₄). For this sake, we need to prove that there exist two sequences $\rho_k > r_k > 0$ such that

$$b_k(\lambda) = \inf_{u \in Z_k, \|u\| = r_k} I_\lambda(u) > 0, \quad \forall \lambda \in [1, 2], \quad (33)$$

$$a_k(\lambda) = \max_{u \in W_k, \|u\| = \rho_k} I_\lambda(u) < 0, \quad \forall \lambda \in [1, 2]. \quad (34)$$

First, we prove that (33) is true. $\square$

Let $\alpha_r(k) = \sup_{u \in Z_k, \|u\|=1} \|u\|_r$ with $r \geq 2$. We claim that $\alpha_r(k) \rightarrow 0$ as $k \rightarrow \infty$. Indeed, if not, then there exist ε_0 and $\{u_j\} \subset E$ with $u_j \perp W_{k_j-1}$ such that $\|u_j\| = 1$ and $\|u_j\|_r > \varepsilon_0$, where $k_j \rightarrow \infty$ as $j \rightarrow \infty$. For any $v \in E$, we may choose $w_j \in W_{k_j-1}$ such that $w_j \rightarrow v$ as $j \rightarrow \infty$.

$$\begin{aligned} |(u_j, v)| &= |(u_j, w_j - v)| \leq \|u_j\| \|w_j - v\| \longrightarrow 0, \\ j &\longrightarrow \infty. \end{aligned} \quad (35)$$

Thus, $u_j \rightarrow 0$ in E . By Lemma 7, we have $u_j \rightarrow 0$ in $C[0, T]$. Consequently, $\|u_j\|_r \rightarrow 0$. This contradicts the fact that $\|u_j\|_r > \varepsilon_0$.

For $u \in Z_k$, by (32), we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2} \|u\|^2 - 2c_3 \|u\|_2^2 - 2c_4 \|u\|_p^p \\ &\geq \frac{1}{2} \|u\|^2 - 2c_3 \alpha_2^2(k) \|u\|^2 - 2c_4 \alpha_p^p(k) \|u\|^p. \end{aligned} \quad (36)$$

Choose $r_k = 1/(\alpha_2(k) + \alpha_p(k))$. Then $r_k \rightarrow \infty$ as $k \rightarrow \infty$. Then, for any $u \in Z_k$ with $\|u\| = r_k$, we obtain

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2} \|u\|^2 - 2c_3 \frac{\alpha_2^2(k)}{|\alpha_2(k) + \alpha_p(k)|^2} \\ &\quad - 2c_4 \frac{\alpha_p^p(k)}{|\alpha_2(k) + \alpha_p(k)|^p} \geq \frac{1}{2} r_k^2 - 2c_3 - 2c_4 \\ &> 0. \end{aligned} \quad (37)$$

Therefore,

$$b_k(\lambda) = \inf_{u \in Z_k, \|u\| = r_k} I_\lambda(u) > 0, \quad \forall \lambda \in [1, 2]. \quad (38)$$

Next, we prove that (34) is true.

We first prove that, for any finite dimensional subspace H of E , there exists a constant $\varepsilon_0 > 0$ such that

$$\text{meas} \{x \in [0, T] : |u(x)| \geq \varepsilon_0 \|u\|\} \geq \varepsilon_0, \quad \forall u \in H \setminus \{0\}. \quad (39)$$

If not, there exists a sequence $\{u_n\} \in H \setminus \{0\}$ such that

$$\text{meas} \left\{x \in [0, T] : |u_n(x)| \geq \frac{1}{n} \|u_n\|\right\} < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \quad (40)$$

Set $v_n = u_n / \|u_n\|$. Then $\|v_n\| = 1$, and

$$\text{meas} \left\{x \in [0, T] : |v_n(x)| \geq \frac{1}{n}\right\} < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \quad (41)$$

On account of the boundedness of $\{v_n\}$ and $\dim H < \infty$, passing to a subsequence if necessary, we may assume that $v_n \rightarrow v$ in E for some $v \in H$. According to the equivalence of any norm in finite dimensional space, we get

$$\int_0^T |v_n(x) - v(x)|^2 dx \rightarrow 0, \quad n \rightarrow \infty. \quad (42)$$

Since $v \neq 0$, there exists a constant $\delta_0 > 0$ such that

$$\text{meas} \{x \in [0, T] : |v(x)| > \delta_0\} \geq \delta_0. \quad (43)$$

Set $\Omega_n = \{x \in [0, T] : |v_n(x)| < 1/n\}$, $\Omega_n^c = \{x \in [0, T] : |v_n(x)| \geq 1/n\}$, and $\Omega_0 = \{x \in [0, T] : |v(x)| \geq \delta_0\}$. Then for n large enough, it follows from (41) and (43) that

$$\begin{aligned} \text{meas}(\Omega_n \cap \Omega_0) &\geq \text{meas}(\Omega_0) - \text{meas}(\Omega_n^c) \geq \delta_0 - \frac{1}{n} \\ &\geq \frac{2}{3}\delta_0. \end{aligned} \quad (44)$$

Consequently,

$$\begin{aligned} \int_0^T |v_n(x) - v(x)|^2 dx &\geq \int_{\Omega_n \cap \Omega_0} |v_n(x) - v(x)|^2 dx \\ &\geq \int_{\Omega_n \cap \Omega_0} [|v(x)|^2 - 2|v_n(x)||v(x)|] dx \\ &\geq \delta_0 \left(\delta_0 - \frac{2}{n}\right) \text{meas}(\Omega_n \cap \Omega_0) \geq \frac{2}{9}\delta_0^3 > 0. \end{aligned} \quad (45)$$

This contradicts (42). Hence, (39) holds.

Since W_k is a finite dimensional subspace of E , from (39), for any $k \in \mathbb{N}$, there exists $\varepsilon_k > 0$ such that

$$\text{meas}(N_u) \geq \varepsilon_k, \quad u \in W_k \setminus \{0\}, \quad (46)$$

where $N_u = \{x \in [0, T] : |u(x)| \geq \varepsilon_k \|u\|\}$. In view of (H_2) , for any $k \in \mathbb{N}$, there exists a constant $L_k > 0$ such that

$$F(x, u) \geq \frac{|u|^2}{\varepsilon_k^3}, \quad |u| \geq L_k. \quad (47)$$

By (46), for any $u \in W_k$ with $\|u\| \geq L_k/\varepsilon_k$, we get $|u(x)| \geq L_k$ for all $x \in N_u$. Therefore,

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} \|u\|^2 - \lambda \int_0^T F(x, u(x)) dx \\ &\leq \frac{1}{2} \|u\|^2 - \varepsilon_k^2 \|u\|^2 \text{meas}(N_u) \frac{1}{\varepsilon_k^3} \\ &\leq -\frac{1}{2} \|u\|^2. \end{aligned} \quad (48)$$

If we take $\rho_k > \max\{r_k, L_k/\varepsilon_k\}$, we get

$$a_k(\lambda) = \max_{u \in W_k, \|u\|=\rho_k} I_\lambda(u) < 0, \quad \forall \lambda \in [1, 2]. \quad (49)$$

Until now, all the conditions of Lemma 9 hold. Hence, for $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that

$$\begin{aligned} \sup_n \|u_n^k(\lambda)\| &< \infty, \\ I'_\lambda(u_n^k(\lambda)) &\rightarrow 0, \\ I_\lambda(u_n^k(\lambda)) &\rightarrow c_k(\lambda), \\ n &\rightarrow \infty. \end{aligned} \quad (50)$$

Furthermore,

$$\begin{aligned} c_k(\lambda) \geq b_k(\lambda) &\geq \frac{1}{2} r_k^2 - 2c_3 - 2c_4 := \bar{b}_k \rightarrow \infty, \\ k &\rightarrow \infty. \end{aligned} \quad (51)$$

$$c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_\lambda(\gamma(u)) \leq \max_{u \in B_k} I_1(u) := \bar{c}_k.$$

Thus,

$$\bar{b}_k \leq c_k(\lambda) \leq \bar{c}_k, \quad \lambda \in [1, 2]. \quad (52)$$

Choose a sequence $\lambda_m \rightarrow 1$ such that (50) holds. Now we show that $\{u_n^k(\lambda_m)\}_{n=1}^\infty$ possesses a strong convergent subsequence. In fact, the sequence $\{u_n^k(\lambda_m)\}_{n=1}^\infty$ is bounded. Then there exists $u^k(\lambda_m) \in E$ such that $u_n^k(\lambda_m) \rightharpoonup u^k(\lambda_m)$ as $n \rightarrow \infty$. By Lemma 7, $u_n^k(\lambda_m) \rightarrow u^k(\lambda_m)$ in $C[0, T]$ as $n \rightarrow \infty$. From $I'_{\lambda_m}(u_n^k(\lambda_m)) \rightarrow 0$, we know

$$\begin{aligned} (I'_{\lambda_m}(u_n^k(\lambda_m)), u_n^k(\lambda_m) - u^k(\lambda_m)) &\rightarrow 0 \\ \text{as } n &\rightarrow \infty. \end{aligned} \quad (53)$$

Also from $u_n^k(\lambda_m) \rightarrow u^k(\lambda_m)$ in E , we get

$$\begin{aligned} (I'_{\lambda_m}(u^k(\lambda_m)), u_n^k(\lambda_m) - u^k(\lambda_m)) &\rightarrow 0 \\ \text{as } n &\rightarrow \infty. \end{aligned} \quad (54)$$

Moreover, by the continuity of f , it follows that

$$\begin{aligned} \int_0^T [f(x, u_n^k(\lambda_m)) - f(x, u^k(\lambda_m))] \\ \cdot (u_n^k(\lambda_m) - u^k(\lambda_m)) dx &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (55)$$

Hence,

$$\begin{aligned} \|u_n^k(\lambda_m) - u^k(\lambda_m)\|^2 &= (I'_{\lambda_m}(u_n^k(\lambda_m)), u_n^k(\lambda_m) \\ &\quad - u^k(\lambda_m)) - (I'_{\lambda_m}(u^k(\lambda_m)), u_n^k(\lambda_m) - u^k(\lambda_m)) \\ &\quad + \lambda_m \int_0^T [f(x, u_n^k(\lambda_m)) - f(x, u^k(\lambda_m))] \\ &\quad \cdot (u_n^k(\lambda_m) - u^k(\lambda_m)) dx \rightarrow 0. \end{aligned} \quad (56)$$

Thus, we conclude that $u_n^k(\lambda_m) \rightarrow u^k(\lambda_m)$ in E as $n \rightarrow \infty$. By (50) and (52), we can get

$$\begin{aligned} I'_{\lambda_m}(u^k(\lambda_m)) &= 0, \\ I_{\lambda_m}(u^k(\lambda_m)) &\in [\bar{b}_k, \bar{c}_k] \end{aligned} \quad (57)$$

for $k \geq k_1$.

Next, we shall prove that $\{u^k(\lambda_m)\}_{m=1}^\infty$ is bounded. By (H_2) , (V') , (11), and (30), we have

$$\begin{aligned} \tau I_{\lambda_m}(u^k(\lambda_m)) - (I'_{\lambda_m}(u^k(\lambda_m)), u^k(\lambda_m)) &= \left(\frac{\tau}{2} - 1\right) \|u^k(\lambda_m)\|^2 + \lambda_m \int_0^T [f(x, u^k(\lambda_m)) u^k(\lambda_m) \\ &\quad - \tau F(x, u^k(\lambda_m))] dx = \left(\frac{\tau}{2} - 1\right) \|u^k(\lambda_m)\|^2 \\ &\quad + \lambda_m \int_{|u^k(\lambda_m)| \leq M} [f(x, u^k(\lambda_m)) u^k(\lambda_m) \\ &\quad - \tau F(x, u^k(\lambda_m))] dx \\ &\quad + \lambda_m \int_{|u^k(\lambda_m)| > M} [f(x, u^k(\lambda_m)) u^k(\lambda_m) \\ &\quad - \tau F(x, u^k(\lambda_m))] dx \geq \left(\frac{\tau}{2} - 1\right) \|u^k(\lambda_m)\|^2 \\ &\quad - \lambda_m \int_{|u^k(\lambda_m)| \leq M} |f(x, u^k(\lambda_m)) u^k(\lambda_m) \\ &\quad - \tau F(x, u^k(\lambda_m))| dx \geq \left(\frac{\tau}{2} - 1\right) \|u^k(\lambda_m)\|^2 \\ &\quad - 2h \|u^k(\lambda_m)\|_2^2 \geq \left(\frac{\tau}{2} - 1\right) \\ &\quad \cdot \left[1 - 2h \left(\frac{\tau}{2} - 1\right)^{-1} \left(\frac{T^\alpha}{\Gamma(\alpha + 1)}\right)^2 \frac{1}{a_0}\right] \|u^k(\lambda_m)\|^2. \end{aligned} \quad (58)$$

Therefore, $\{u^k(\lambda_m)\}_{m=1}^\infty$ is bounded in E . Similar arguments show that $u^k(\lambda_m) \rightarrow u^k$ in E as $m \rightarrow \infty$ ($k \geq k_1$). Then u^k is a critical point of $I = I_1$ with $I(u^k) \in [\bar{b}_k, \bar{c}_k]$. According to $\bar{b}_k \rightarrow \infty$ as $k \rightarrow \infty$, we know that BVP (1) has infinitely many nontrivial high energy solutions.

Theorem 12. Assume that (H_3) – (H_6) and (V) hold. Then BVP (1) possesses infinitely many small energy solutions $u_k \in E \setminus \{0\}$ satisfying

$$\begin{aligned} \frac{1}{2} \int_0^T a(x) |D_{0^+}^\alpha u_k(x)|^2 dx - \int_0^T F(x, u_k(x)) dx \\ \rightarrow 0^-, \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (59)$$

Proof. By (H_5) , we can get

$$|F(x, u)| \leq \frac{1}{\eta} b(x) |u|^\eta. \quad (60)$$

Combining (60) and (13), it is easily seen that I_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1, 2]$. It follows from (H_4) that $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Thus, condition (B_1) holds. In the following, we verify condition (B_2) . By (H_3) , $B(u) \geq 0$ is obvious. According to (H_6) , there exists $M > 0$ such that

$$F(x, u) \geq d |u|^r, \quad \forall |u| > M. \quad (61)$$

Suppose H is a finite dimensional subspace of E . As in the proof of Theorem 11, there exists $\varepsilon_0 > 0$ such that (39) holds. Then

$$\text{meas}(D_u) \geq \varepsilon_0, \quad (62)$$

where $D_u = \{x \in [0, T] : |u(x)| \geq \varepsilon_0 \|u\|\}$, $\forall u \in H \setminus \{0\}$. Hence, for any $u \in H$ with $\|u\| \geq M/\varepsilon_0$, from (61), we have

$$\begin{aligned} B(u) &= \int_0^T F(x, u(x)) dx \geq \int_{D_u} d |u(x)|^r dx \\ &\geq d \varepsilon_0^r \|u\|^r \text{meas}(D_u) \geq d \varepsilon_0^{1+r} \|u\|^r. \end{aligned} \quad (63)$$

This means that $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace. Hence, condition (B_2) holds.

Next, we claim condition (B_3) holds. By (60) and Hölder inequality, we get

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2} \|u\|^2 - \frac{1}{\eta} \int_0^T b(x) |u|^\eta dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{1}{\eta} \|b(x)\|_{2/(2-\eta)} \|u\|_2^\eta. \end{aligned} \quad (64)$$

According to the definition of $\alpha_r(k)$ in the proof of Theorem 11, we have $\|u\|_2^\eta \leq \alpha_2^\eta(k) \|u\|^\eta$, $\forall u \in Z_k$. Hence,

$$I_\lambda(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{\eta} \|b(x)\|_{2/(2-\eta)} \alpha_2^\eta(k) \|u\|^\eta. \quad (65)$$

Take $\rho_k = ((4/\eta) \|b(x)\|_{2/(2-\eta)} \alpha_2^\eta(k))^{1/(2-\eta)}$. Then $\rho_k \rightarrow 0^+$ as $k \rightarrow \infty$. Therefore,

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq \frac{1}{4} \rho_k^2 > 0. \quad (66)$$

Furthermore, for $\lambda \in [1, 2]$ and $u \in Z_k$ with $\|u\| \leq \rho_k$, we obtain

$$I_\lambda(u) \geq -\frac{1}{\eta} \|b(x)\|_{2/(2-\eta)} \alpha_2^\eta(k) \rho_k^\eta \longrightarrow 0^+, \quad (67)$$

$$k \longrightarrow \infty.$$

So,

$$d_k(\lambda) = \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (68)$$

By (H_5) and (H_6) , we have

$$F(x, u) \geq d|u|^r - \frac{1}{\eta} b(x)|u|^\eta. \quad (69)$$

If $u \in W_k$, by the equivalence of any norm in finite dimensional space and (69), we get

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} \|u\|^2 - \lambda \int_0^T F(x, u(x)) dx \\ &\leq \frac{1}{2} \|u\|^2 - d \int_0^T \|u\|^r dx + \frac{1}{\eta} \int_0^T b(x) |u|^\eta dx \\ &\leq \frac{1}{2} \|u\|^2 - c_1 \|u\|^r + c_2 \|b(x)\|_{2/(2-\eta)} \|u\|^\eta. \end{aligned} \quad (70)$$

Take $r_k > 0$ small enough and $r_k < \rho_k$. It follows that

$$b_k(\lambda) = \max_{u \in W_k, \|u\| = r_k} I_\lambda(u) < 0. \quad (71)$$

Hence, condition (B_3) holds. $\square$

Therefore, by Lemma 10, for every $k \in \mathbb{N}$, there exist $\lambda_n \rightarrow 1$, $u_n(\lambda_n) \in W_n$ such that

$$\begin{aligned} I'_{\lambda_n}|_{W_n}(u(\lambda_n)) &= 0, \\ I_{\lambda_n}(u(\lambda_n)) &\longrightarrow c_k \in [d_k(2), b_k(1)] \\ &\quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (72)$$

Now we show that $\{u(\lambda_n)\}$ has a strong convergent subsequence for every $k \in \mathbb{N}$. In fact by (60), Hölder inequality, and Lemma 5, we get

$$\begin{aligned} \|u(\lambda_n)\|^2 &= 2I_{\lambda_n}(u(\lambda_n)) + 2\lambda_n \int_0^T F(x, u(\lambda_n)) dx \\ &\leq 2c_k + 2\lambda_n \frac{1}{\eta} \|b(x)\|_{2/(2-\eta)} \|u(\lambda_n)\|^\eta. \end{aligned} \quad (73)$$

This means $\{u(\lambda_n)\}$ is bounded in E . Without loss of generality, we may assume $u(\lambda_n) \rightharpoonup u$ in E . Since $\{e_j\}$ is the completely orthonormal basis of E , $W_n = L(e_1, e_2, \dots, e_n)$, $u = \sum_{j=1}^\infty (e_j, u)e_j$. Let $P_n : E \mapsto W_n$ be the orthogonal projection operator. We know that $P_n u = \sum_{j=1}^n (e_j, u)e_j$ and $P_n u \rightarrow u$ in E as $n \rightarrow \infty$. Therefore $u(\lambda_n) - P_n u \rightarrow 0$ in E as $n \rightarrow \infty$. Moreover, it follows from $I'_1(u) \in E^*$ that

$$(I'_1(u), u(\lambda_n) - P_n u) \longrightarrow 0 \quad n \longrightarrow \infty. \quad (74)$$

Also since $I'_1 \in C(E \rightarrow E^*)$ and $P_n u \rightarrow u$ in E , we have

$$(I'_1(P_n u) - I'_1(u), u(\lambda_n) - P_n u) \longrightarrow 0 \quad n \longrightarrow \infty. \quad (75)$$

Therefore, by (74) and (75), we get

$$(I'_1(P_n u), u(\lambda_n) - P_n u) \longrightarrow 0 \quad n \longrightarrow \infty. \quad (76)$$

Note that $P_n u(\lambda_n) = u(\lambda_n)$ and $(I'_{\lambda_n}(P_n u(\lambda_n)), P_n(u(\lambda_n) - u)) \rightarrow 0$ since $u(\lambda_n) \in W_n$ and $I'_{\lambda_n}|_{W_n}(u(\lambda_n)) = 0$. By the continuity of f and Lemma 7, it is easily seen that

$$\begin{aligned} \lambda_n \int_0^T f(x, u(\lambda_n))(u(\lambda_n) - P_n u) dx &\longrightarrow 0, \\ n &\longrightarrow \infty. \end{aligned} \quad (77)$$

$$\begin{aligned} \int_0^T f(x, P_n u)(u(\lambda_n) - P_n u) dx &\longrightarrow 0, \\ n &\longrightarrow \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|u(\lambda_n) - P_n u\|^2 \\ &= (I'_{\lambda_n}(u(\lambda_n)), u(\lambda_n) - P_n u) \\ &\quad - (I'_1(P_n u), u(\lambda_n) - P_n u) \\ &\quad + \lambda_n \int_0^T f(x, u(\lambda_n))(u(\lambda_n) - P_n u) dx \\ &\quad - \int_0^T f(x, P_n u)(u(\lambda_n) - P_n u) dx \longrightarrow 0. \end{aligned} \quad (78)$$

So, $u(\lambda_n) - P_n u \rightarrow 0$ in E as $n \rightarrow \infty$. Thus, $u(\lambda_n) \rightarrow u$ in E as $n \rightarrow \infty$. By Lemma 10, we know that $I = I_1$ has infinitely many nontrivial critical points u_k . Consequently, BVP (1) has infinitely many small energy solutions.

4. Examples

In this section, two examples are given to illustrate our results.

Example 1. Consider the following nonlinear fractional boundary value problem:

$$\begin{aligned} D_1^\alpha \left[\left(\frac{3}{(\Gamma(\alpha+1))^2} + x \right) D_0^\alpha u(x) \right] \\ = 3|u|^2 \ln(|u|+1) + \frac{|u|^3}{|u|+1}, \quad x \in [0, 1], \end{aligned} \quad (79)$$

$$u(0) = u(1) = 0,$$

where $\alpha \in (1/2, 1]$.

First, choose $\tau = 3$ and $M = 1$. A direct calculation shows that

$$\begin{aligned} F(x, u) &= |u|^3 \ln(|u| + 1), \\ -3F(x, u) + uf(x, u) &= \frac{|u|^4}{|u| + 1} \geq 0. \end{aligned} \quad (80)$$

Hence, assumption (H_2) holds.

Next, we verify assumption (V') . From (30) we have $h = 1/2$. By direct calculation, one has $a_0 = 3/(\Gamma(\alpha + 1))^2 \geq (\tau/2 - 1)^{-1} 2h(\tau^\alpha/\Gamma(\alpha + 1))^2 > 0$. Hence, assumption (V') holds.

Finally, it is easy to show that (H_1) , (H_3) , and (H_4) hold. So, all the assumptions of Theorem 11 are satisfied. Consequently, BVP (79) has infinitely many high energy solutions.

Example 2. Consider the following nonlinear fractional boundary value problem:

$$\begin{aligned} D_{1-}^\alpha \left[(x^2 + 1) D_{0+}^\alpha u(x) \right] &= \frac{3}{2} e^x |u|^{1/2}, \quad x \in [0, 1], \\ u(0) &= u(1) = 0, \end{aligned} \quad (81)$$

where $\alpha \in (1/2, 1]$.

First, notice that $\text{ess inf}_{x \in [0, 1]} x^2 + 1 = 1 > 0$, which means that assumption (V) holds.

Moreover,

$$|f(x, u)| = \frac{3}{2} e^x |u|^{1/2} \leq 2e^x |u|^{1/2}. \quad (82)$$

Take $\eta = 3/2$ and $b(x) = 2e^x$. It is easy to show that assumption (H_5) are satisfied.

Choose $r = 5/4$. By a direct calculation, one has $F(x, u) = e^x |u|^{3/2}$ and $\lim_{|u| \rightarrow \infty} (e^x |u|^{3/2} / |u|^{5/4}) \rightarrow \infty$. Hence, assumption (H_6) holds.

Finally, it is easy to show that (H_3) and (H_4) hold. So, all the assumptions of Theorem 12 are satisfied. Consequently, BVP (81) has infinitely many small energy solutions.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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