Research Article

Iterative Schemes for Nonconvex Quasi-Variational Problems with $\mathcal{V}$-Prox-Regular Data in Banach Spaces

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In this paper, we propose an extension of quasi-equilibrium problems from the convex case to the nonconvex case and from Hilbert spaces to Banach spaces. The proposed problem is called quasi-variational problem. We study the convergence of some algorithms to solutions of the proposed nonconvex problems in Banach spaces.

1. Introduction

Let $X$ be a Banach space and let $X^*$ be the dual space of $X$. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing of $X^*$ and $X$. Let $C : X \rightrightarrows X$ be a set-valued mapping with nonempty closed values and let $F : X \times X \to \mathbb{R}$ be a bifunction satisfying $F(x, x) = 0$ for all $x \in \text{Fix}(C) = \{x \in X : x \in C(x)\}$. We associate with a closed convex valued set-valued mapping $C$ and a convex bifunction $F$ the following well known quasi-equilibrium problem:

\[
\text{Find } x \in C(x) \text{, such that } F(x, x) \geq 0, \quad (\text{QEP}[C, F])
\]

\[
\forall x \in C(x).
\]

In this paper we propose the following appropriate extensions of $(\text{QEP}[C, F])$ from the convex case to the nonconvex case in Banach spaces setting. We associate with $C$ and $F$ the following nonconvex quasi-variational problem equilibrium problems:

\[
\text{Find } x \in C(x) \text{, s.t. } \left[ -\partial^\mathcal{V} F(x, \cdot)(x) \right] \cap N^\mathcal{V} (C(x); x) \neq \emptyset, \quad (\text{NQVP}[C, F])
\]

where $\partial^\mathcal{V}$ and $N^\mathcal{V}$ are the usual proximal subdifferential and proximal normal cone in Hilbert spaces. This problem has been introduced and studied in Bounkhel et al. [2]. Since then it has been studied and extended in various ways in Hilbert spaces by the authors in [3] and in Noor [4] and many works (see for instances Noor et al. [5, 6]).

1. If $X$ is a Hilbert space, the proposed $(\text{NQVP}[C, F])$ becomes

\[
\text{Find } x \in C(x), \text{ such that } \left[ -\partial^\mathcal{V} F(x, \cdot)(x) \right] \cap N^\mathcal{V} (C(x); x) \neq \emptyset, \quad (1)
\]

where $\partial^\mathcal{V}$ and $N^\mathcal{V}$ are the usual proximal subdifferential and proximal normal cone in Hilbert spaces. This problem has been introduced and studied in Bounkhel et al. [2]. Since then it has been studied and extended in various ways in Hilbert spaces by the authors in [3] and in Noor [4] and many works (see for instances Noor et al. [5, 6]).
(NQVP\([C, F]\)) becomes the following well known convex equilibrium problem:

\[
\begin{align*}
\text{Find } & \bar{x} \in C, \\
\text{such that } & F(\bar{x}, x) \geq 0, \\
\forall x & \in C,
\end{align*}
\]

(2)

which has been studied in various works (see for instance Moudafi [7], M. A. Noor and K. I. Noor [5], and the references therein).

(3) If \(F(x, y) = \langle T(x), y - x \rangle\), with \(T : X \to X^{*}\), is a nonlinear operator then (NQVP\([C, F]\)) reduces to

\[
\begin{align*}
\text{Find } & \bar{x} \in C(\bar{x}), \\
\text{s.t. } & -T(\bar{x}) \in N^\pi(C(\bar{x}); \bar{x})
\end{align*}
\]

which will be shown in Section 4 to be equivalent in the uniform V-prox-regular case, for some \(\rho \geq 0\), to the following quasi-variational inequality:

\[
\begin{align*}
\text{Find } & \bar{x} \in C(\bar{x}), \\
\text{s.t. } & \langle T(\bar{x}), x - \bar{x} \rangle + \rho V(J(\bar{x}), x) \geq 0, \\
\forall x & \in C(\bar{x}).
\end{align*}
\]

(4)

This inequality is new in Banach spaces. However, it has been studied, in Hilbert spaces, in Bounkhet al. [2], when \(C\) is a uniformly \(V\)-prox-regular set (see also Bounkhet and Al-Sinan [8] and Noor et al. [5, 6]).

When \(\rho = 0\) and \(C(\bar{x}) \equiv C\) the last inequality becomes

\[
\begin{align*}
\text{Find } & \bar{x} \in C, \\
\text{such that } & \langle T(\bar{x}), x - \bar{x} \rangle \geq 0, \\
\forall x & \in C,
\end{align*}
\]

(5)

which is known as the classical variational inequality introduced and studied in Stampacchia [9].

Our main objective of the present paper is to prove the convergence of some algorithms to solutions of the proposed nonconvex quasi-variational problem (NQVP\([C, F]\)).

2. Preliminaries

In order to prepare the framework of our study we need to state some concepts and results. First we recall (see for instance [1, 10]) the definition of \(p\)-uniformly convex and \(q\)-uniformly smooth Banach spaces. The space \(X\) is said to be \(p\)-uniformly convex (resp. \(q\)-uniformly smooth) if there is a constant \(c > 0\) such that

\[
\delta_X(e) \geq ce^p \text{ (resp. } \rho_X(t) \leq ct^q),
\]

(6)

where \(\delta_X\) and \(\rho_X\) are defined, respectively, by

\[
\delta_X(e) = \inf \left\{ 1 - \frac{\|x + \frac{y}{2}\|}{\|y\|} : \|x\| = \|y\| = 1, \|x - y\| = e \right\}, 0 \leq e \leq 2,
\]

(7)

\[
\rho_X(t) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : \|x\| = 1, \|y\|\right\},
\]

(8)

\[
t = t > 0.
\]

Notice that the constants \(p\) and \(q\) in the previous definition always satisfy \(p \geq 2\) and \(q \in (1, 2]\). Also we need to recall from [1] the concept of \(V\)-proximal subdifferential \(\partial^v f(x)\) (called in [1] generalised proximal subdifferential). An element \(x^* \in X^*\) belongs to \(\partial^v f(x)\) provided that there exists \(\sigma > 0\) so that

\[
\langle x^*, x' - x \rangle \leq f(x') - f(x) + \sigma V(J(x), x'),
\]

(9)

for \(x'\) very close to \(x\), where \(J : X \to X^*\) is the normalised duality mapping and \(V : X^* \times X \to \mathbb{R}\) is a functional defined by

\[
V(x^*, x) = \|x^*\|^2 - 2 \langle x^*, x \rangle + \|x\|^2,
\]

(10)

for any \(x^* \in X^*, x \in X\).

For a closed nonempty set \(S\) in \(X\) and \(\bar{x} \in S\), the authors in [1] defined the concept of \(V\)-proximal normal cone \(N^\pi(S; \bar{x})\) (called in [1] generalised proximal normal cone) by \(N^\pi(S; \bar{x}) = \partial^v \psi_S(\bar{x})\), where \(\psi_S\) denotes the indicator function associated with \(S\), that is, \(\psi_S(x) = 0\) if \(x \in S\) and \(\psi_S(x) = +\infty\) if \(x \notin S\). We recall, respectively, the concepts of limiting Fréchet subdifferential \(\partial^{LF}\) and limiting \(V\)-proximal subdifferential \(\partial^{L^p}\) (see [11]):

\[
\partial^{L^p} f(x) = \limsup_{x' \to x} \partial^p f(x')
\]

(11)

\[
= \left\{ w - \lim_n x^*_n : x^*_n \in \partial^p f(x_n) \text{ with } x_n \to f x \right\},
\]

(12)

where \(x_n \to f x\) means \(x_n \to x\) with \(f(x_n) \to f(x)\) and

\[
\partial^v f(x) = \left\{ x^* \in X^* : \forall e > 0, \exists \delta > 0 : \langle x^*, x' - x \rangle \leq f(x') - f(x) + e \|x' - x\|, \forall x' \in x + \delta B \right\}.
\]

The limiting Fréchet normal cone is defined similarly, that is,

\[
\partial^{LF} N(S; x) = \limsup_{x' \to x} \partial^F N(S; x')
\]

(13)

\[
= \left\{ w - \lim_n x^*_n : x^*_n \in N^F(S; x_n) \text{ with } x_n \to S x \right\},
\]

(14)
where \(x_n \to^* x\) denotes \(x_n \to x\) with \(x_n \in S\) and \(N^\|f\|(S;x)\) is the Fréchet normal cone which is defined by 
\[
N^\|f\|(S;\overline{x}) = \partial^\|f\|\psi^S_*(\overline{x}).
\]

These all nonconvex objects coincide with their analogues defined in Convex Analysis whenever the data are convex as the following proposition shows (see [11]).

**Proposition 1.** Let \(X\) be a reflexive Banach space.

1. Let \(f : X \to R \cup (+\infty)\) be a l.s.c. convex function and \(\overline{x} \in X\) with \(f(\overline{x}) < +\infty\). Then
\[
\partial f(x) = \partial^\text{con} f(x) = \{x^* \in X^* : \langle x^*, x - \overline{x} \rangle \leq f(x) - f(\overline{x}), \forall x \in X\}.
\]

2. Let \(S\) be a closed convex subset of \(X\) and \(\overline{x} \in S\). Then
\[
N^\|f\|(S;\overline{x}) = N^\text{con} (S;\overline{x}) = \{x^* \in X^* : \langle x^*, x - \overline{x} \rangle \leq 0, \forall x \in S\}.
\]

The following result is needed in our study. It has been proved in [11].

**Theorem 2.** Let \(X\) be a \(q\)-uniformly smooth and \(p\)-uniformly convex Banach space. Assume that \(X\) admits an equivalent norm \(|\cdot|\) such that \(|\cdot|^s\) (for some \(s \geq 2\)) is \(C^2\)-differentiable on \(X \setminus \{0\}\) and let \(V\) be the functional associated with that norm \(|\cdot|\).

1. Let \(f : X \to R \cup [0, +\infty]\) be a l.s.c. function at \(\overline{x} \in \text{dom}\ f\). Then
\[
\partial^L f(\overline{x}) = \partial^\text{FL} f(\overline{x}).
\]

2. Let \(S\) be any nonempty set of \(X\). Then
\[
N^\text{FL}(S;\overline{x}) = N^{L^f}(S;\overline{x}).
\]

We notice that the class of spaces satisfying the assumptions of the previous theorem is very large; it contains obviously any Hilbert space and \(L^p\) spaces and Sobolev spaces \(W^{1,p}\) with \(p \geq 2\) (see Theorem 1.1 in Section 5 in [10, 12]) and for more examples and discussions we refer to [10, 12]. We close this section with the following two concepts of uniform \(V\)-prox-regularity for functions and sets (see [13]).

**Definition 3.** Let \(X\) be a reflexive smooth Banach space. For a given \(r \in (0, +\infty)\), a subset \(S\) is \(V\)-uniformly prox-regular with respect to \(r\) provided that for all \(x \in S\) and all nonzero \(x^* \in N^\|r\|(S;x)\) we have
\[
\langle x^*, x' - x \rangle \leq \frac{1}{2r} V(J(x), x'), \quad \forall x' \in S.
\]

We use the convention \(1/r = 0\) for \(r = +\infty\).

Obviously, this class contains the class of uniformly prox-regular sets ([14, 15]) from Hilbert spaces to Banach spaces since in Hilbert spaces we have \(V(J(x), x') = \|x - x'\|^2\) and the \(V\)-proximal normal cone \(N^\|r\|(S;x)\) coincides with the usual proximal normal cone \(N^\|r\|(S;x)\).

**Definition 4.** Let \(X\) be a reflexive smooth Banach space. Let \(f : X \to R \cup (+\infty)\) be a l.s.c. function and let \(S \subset \text{dom}\ f = \{x \in X : f(x) < +\infty\}\) be a nonempty closed set in \(X\). We recall from [13] that \(f\) is said to be uniformly \(V\)-prox-regular over \(S\) provided that for all \(x \in S\) and all \(x^* \in \partial^\|f\|\psi^S_*(\overline{x})\) we have
\[
\langle x^*, x' - x \rangle \leq f(x') - f(x) + \frac{1}{2r} V(J(x), x'), \quad \forall x' \in S.
\]

The following example is quoted from [13]. For its proof we refer the reader to [13].

**Example 5.** (1) Any l.s.c. proper convex function is uniformly \(V\)-prox-regular over any nonempty closed set \(S\) in its domain with \(r = +\infty\).

(2) Both the indicator function \(\psi^S_*(\overline{x})\) and the function \(d_\tau\) of uniformly \(V\)-prox-regular set \(S\) are uniformly \(V\)-prox-regular over \(S\) with respect to the same constant \(r\).

(3) Any lower-\(C^2\) function \(f\) over convex strongly compact \(K\) in \(X\) is uniformly \(V\)-prox-regular over \(K\) with some \(r \in (0, +\infty)\) (see [13] for the definition of lower-\(C^2\) functions).

The following two lemmas are needed in our proofs in Section 4. The proof of the first one is proved in [1]. The second one is proved in [16].

**Lemma 6.** Let \(X\) be a \(p\)-uniformly convex and \(q\)-uniformly smooth Banach space and \(S\) be a bounded set. Then for some \(\eta, \kappa > 0\) we have
\[
\eta^{-1} \|x - y\|^p \leq V(J(x), y) \leq \kappa^{-1} \|x - y\|^q, \quad \forall x, y \in S.
\]

**Lemma 7.** If \(X\) is a uniformly convex Banach space, then the inequality
\[
V(J(x), y) \geq 8C^2\delta_x \left( \frac{\|x - y\|}{4C} \right)^r
\]
holds for all \(x\) and \(y\) in \(X\), where \(C = \sqrt{/(\|x\|^2 + \|y\|^2)}\).

### 3. Main Results

First we show that in the convex case \((\text{NQVP}[C, F])\) coincides with the quasi-equilibrium problem \((\text{QEP}[C, F])\).

**Proposition 8.** Let \(X\) be a reflexive Banach space. Assume that \(C\) is a closed convex set-valued mapping and \(F\) is a convex bifunction satisfying \(F(x, x) = 0\) for any \(x \in \text{Fix}(C)\). Then we have \((\text{NQVP}[C, F]) \Leftrightarrow (\text{QEP}[C, F]) \).
Proof.

$\Rightarrow$. Let $\overline{x}$ be a solution of (NQVP$[C, F]$); that is, there exists $y^* \in \partial F(\overline{x}, \cdot)(\overline{x})$ such that $-y^* \in N^\mathcal{F}(C(\overline{x}), \overline{x})$. Since $C(\overline{x})$ is a closed convex set, the $V$-proximal normal cone $N^V(\overline{x})$ coincides with the convex normal cone $N^\mathcal{F}(C(\overline{x}), \overline{x})$ (by Proposition 1) and so

$$
\langle y^*; x - \overline{x} \rangle \geq 0, \quad \forall x \in C(\overline{x}).
$$

(21)

On the other hand, the convexity of the bifunction $F$ and Proposition 1 yield

$$
\langle y^*; x - \overline{x} \rangle \leq F(\overline{x}, x) - F(\overline{x}, \overline{x}), \quad \forall x \in X.
$$

(22)

Since $\overline{x} \in C(\overline{x})$ we have $F(\overline{x}, \overline{x}) = 0$ (by assumption) and hence the previous two inequalities ensure

$$
F(\overline{x}, x) \geq 0, \quad \forall x \in C(\overline{x});
$$

(23)

that is, $\overline{x}$ is a solution of (QEP$[C, F]$).

$\Leftarrow$. Let $\overline{x}$ be a solution of (NQEPC$[C, F]$), that is, $F(\overline{x}, x) \geq 0, \forall x \in C(\overline{x})$. Since $C(\overline{x})$ is a closed convex set and $F(\overline{x}, \cdot)$ is a convex function, the function $x \mapsto h(x) = F(\overline{x}, x) + \psi_{C(\overline{x})}(x)$ admits at $\overline{x}$ a global minimum on $X$. It follows that

$$
0 \in \partial^\mathcal{F} h(\overline{x}) = \partial^\mathcal{F} F(\overline{x}, \cdot)(\overline{x}) + \partial^\mathcal{F} \psi_{C(\overline{x})}(\overline{x})
$$

(24)

which is equivalent to $[-\partial^\mathcal{F} F(\overline{x}, \cdot)(\overline{x})] \cap N^\mathcal{F}(C(\overline{x}), \overline{x}) \neq \emptyset$ and hence the proof is complete since $\partial^\mathcal{F} F(\overline{x}, \cdot)(\overline{x}) = \partial^\mathcal{F} F(\overline{x}, \cdot)(\overline{x})$ and $N^\mathcal{F}(C(\overline{x}), \overline{x}) = N^\mathcal{F}(C(\overline{x}), \overline{x})$. $\square$

In the next proposition we establish an inequality characterisation of the proposed nonconvex quasi-variational problem (NQVP$[C, F]$) whenever the data $C$ and $F$ are uniformly $V$-prox-regular.

Proposition 9. Let $X$ be a reflexive Banach space and $\overline{x} \in X$. Assume that $C(\overline{x})$ is uniformly $V$-prox-regular with ratio $r \in (0, \infty]$ and that $F(\overline{x}, \cdot)$ is uniformly $V$-prox-regular over $C(\overline{x})$ with ratio $r' \in (0, \infty]$. Assume also that $F(\overline{x}, \cdot)$ is $\gamma$-Lipschitz around $\overline{x}$ and $F(x, x) = 0$ for any $x \in \text{Fix}(C)$. If $\overline{x}$ is a solution of (NQVP$[C, F]$), then $\overline{x}$ is a solution of the following nonconvex quasi-equilibrium problem. Find $\overline{x} \in C(\overline{x})$ such that

$$
F(\overline{x}, x) + \rho V(J(\overline{x}, x)) \geq 0, \quad \forall x \in C(\overline{x}), \quad \text{(NQEP$[C, F]$)}
$$

for some nonnegative $\rho \geq 0$.

Proof. Assume that $\overline{x}$ is a solution of (NQVP$[C, F]$); that is, $y^* \in \partial^\mathcal{F} F(\overline{x}, \cdot)(\overline{x})$ such that $-y^* \in N^\mathcal{F}(C(\overline{x}); \overline{x})$. By uniform $V$-prox-regularity of the set $C(\overline{x})$ we have

$$
\langle -y^*, x - \overline{x} \rangle \leq \frac{1}{2r} V(J(\overline{x}, x)), \quad \forall x \in C(\overline{x}).
$$

(25)

The $\gamma$-Lipschitz continuity of $F(\overline{x}, \cdot)$ ensures that $\|y^*\| \leq \gamma$ and so we obtain

$$
\langle -y^*, x - \overline{x} \rangle \leq \frac{\gamma}{2r} V(J(\overline{x}, x)), \quad \forall x \in C(\overline{x}).
$$

(26)

On the other hand the uniform $V$-prox-regularity of $F(\overline{x}, \cdot)$ over $C(\overline{x})$ with ratio $r' > 0$; we have

$$
\langle y^*, x - \overline{x} \rangle \leq \frac{1}{2r'} V(J(\overline{x}, x)) + F(\overline{x}, x) - F(\overline{x}, \overline{x}), \quad \forall x \in C(\overline{x}).
$$

(27)

Combining this inequality (27) with (26) we obtain

$$
F(\overline{x}, x) - F(\overline{x}, \overline{x}) + \frac{1}{2r'} V(J(\overline{x}, x)) \geq -\frac{\gamma}{2r} V(J(\overline{x}, x)) \quad \forall x \in C(\overline{x}).
$$

(28)

Since $\overline{x} \in C(\overline{x})$ we have $F(\overline{x}, \overline{x}) = 0$ and so (28) becomes

$$
F(\overline{x}, x) + \rho V(J(\overline{x}, x)) \geq 0 \quad \forall x \in C(\overline{x}),
$$

(29)

with $\rho = \gamma/(2r) + 1/(2r') \geq 0$. Thus the proof is complete. $\square$

It is a natural question to ask whether the converse in the previous proposition is true or not. The answer is positive provided that the space $X$ and the data $C$ and $F$ satisfy some additional assumptions as the following proposition shows.

Proposition 10. Let $X$ be a $q$-uniformly smooth and $p$-uniformly convex Banach space. Assume that $X$ admits an equivalent norm $\| \cdot \|$ such that $\| \cdot \|_s$ (for some $s \geq 2$) is $C^2$-differentiable on $X \setminus \{ 0 \}$ and let $V$ be the functional associated with that norm $\| \cdot \|$. Assume that $C(\overline{x})$ is $V$-prox-regularly regular at $\overline{x}$, that is, $N^2(\overline{x}) = N^2(\overline{x}, \overline{x})$ and that $F(\overline{x}, \cdot)$ is $V$-proximal subdifferentially regular at $\overline{x}$, that is, $\partial^2 F(\overline{x}, \cdot)(\overline{x}) = \partial^2 F(\overline{x}, \cdot)(\overline{x})$. Assume that $F(x, x) = 0$ for any $x \in \text{Fix}(C)$. If $\overline{x}$ is a solution of (NQEP$[C, F]$) for some $\rho \geq 0$, then $\overline{x}$ is a solution of (NQVP$[C, F]$).

Proof. Let $\overline{x}$ be a solution of (NQEP$[C, F]$) for some $\rho \geq 0$; that is,

$$
F(\overline{x}, x) + \rho V(J(\overline{x}, x)) \geq 0 \quad \forall x \in C(\overline{x}).
$$

(30)

Then $\overline{x}$ is a global minimum of the function $x \mapsto h(x) = F(\overline{x}, x) + \rho V(J(\overline{x}, x)) + \psi_{C(\overline{x})}(x)$ over $X$ and hence

$$
0 \in \partial^2 h(\overline{x}) \subset \partial^2 \psi_{C(\overline{x})} + \partial^2 F(\overline{x}, \cdot)(\overline{x}) = \partial^2 F(\overline{x}, \cdot)(\overline{x}) + \partial^2 \psi_{C(\overline{x})}(\overline{x}) \quad (31)
$$

Note that the function $x \mapsto V(J(\overline{x}, x))$ is differentiable and its gradient is given by $\text{grad}(V(J(\overline{x}, \cdot)))(x) = 2(J(x) - J(\overline{x}))$. Using the fact that the limiting $V$-proximal subdifferential
Fix any $y \in \mathbb{X} + (\delta/2)B$. Clearly $y \in x_n + (\delta/2)B + (\delta/2)B \subset x_n + \delta B$ and hence (34) ensures
\[
\langle x^*, y - \mathbb{x} \rangle \leq \frac{1}{2r} V(J(x_n), y) + f(y) - f(x_n).
\]
Using now the fact that $x_n \to \mathbb{x}$, the continuity of $J$ and $V$, and the weak convergence of $x_n^*$ to $x^*$ to pass to the limit as $n$ goes to $\infty$ and to get
\[
\langle x^*, y - \mathbb{x} \rangle \leq \frac{1}{2r} V(J(\mathbb{x}), y) + f(y) - f(\mathbb{x}),
\]
for any $y \in \mathbb{X} + (\delta/2)B$, this means by definition that $x^* \in \partial^* f(\mathbb{x})$ and the proof is complete.
\[\square\]

**Proposition 12.** Let $X$ be a $q$-uniformly smooth and $p$-uniformly convex Banach space and $\mathbb{X} \subset X$. Assume that $X$ admits an equivalent norm $\| \cdot \|$ such that $\| \cdot \|^{s}$ (for some $s \geq 2$) is $C^2$-differentiable on $X \setminus \{0\}$ and let $V$ be the functional associated with that norm $\| \cdot \|$. Assume that $C(\mathbb{X})$ is uniformly $V$-proximal-regular with ratio $\rho \in (0, \infty)$ and that $F(\mathbb{X}, \cdot)$ is uniformly $V$-proximal-regular over $C(\mathbb{X})$ with ratio $r' \in (0, \infty)$. Assume also that $F(\mathbb{X}, \cdot)$ is $\gamma$-Lipschitz around $\mathbb{X}$ and $F(x, x) = 0$ for any $x \in \text{Fix}(C)$. Then $(\text{NQVP}[C, F])$ is equivalent to $(\text{NQE}[C, F])$ for some $\rho \geq 0$.

## 4. Convergence Analysis

### 4.1. Case I: $C$ is a Constant Set-Valued Mapping

In this case the proposed problem becomes as follows:

Find $\mathbb{x} \in C$ such that $\textit{NQVP}[C, F]$ for some $\rho \geq 0$.

In this subsection we propose the following algorithm.

**Algorithm 13.** Let $\rho \geq 0$ and $\lambda_n > 0$ for all $n \geq 1$.

1. Select $x_0 \in C$;
2. For $n \geq 1$ select $x_{n+1} \in C$ such that
\[
\lambda_n^{-1} \langle J(x_{n+1}) - J(x_n), x - x_{n+1} \rangle \leq F(x_n, x) + \rho V(J(x_n), x), \quad \forall x \in C.
\]

**Theorem 14.** Let $X$ be a $q$-uniformly convex Banach space. Let $C$ be a closed nonempty subset of $X$ and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $F(x, x) = 0$ for any $x \in \text{Fix}(C)$. Let $\{x_n\}$ be a sequence generated by Algorithm 13. Assume that
(1) \(C\) is \(V\)-uniformly prox-regular with some \(r \in (0, \infty)\);
(2) \(C\) is ball compact; that is, \(C \cap \eta B\) is compact for any \(\eta > 0\);
(3) The solution set of \((\text{NQVP}[C, F])\) is nonempty;
(4) \(F\) is \(W\)-strongly monotone over \(C\) for some \(\sigma \geq 0\); that is,
\[
F(x, y) + F(y, x) \leq -\sigma W(x, y), \quad \forall x, y \in C,
\]
where \(W(x, y) = (1/2)[V(J(x), y) + V(J(y), x)]\);
(5) \(F\) is upper semicontinuous with respect to the first variable over \(C\); that is,
\[
\limsup_{x' \to x} F(x', y) \leq F(x, y) \quad \forall x, y \in C;
\]
(6) The bifurcation \(F\) is \(\gamma\)-Lipschitz with respect to the second variable and \(F(x_{n+1}^*)\) is \(V\)-uniformly prox-regular with some \(r \in (0, +\infty)\);
(7) There exists \(\lambda > 0\) such that \(\lambda_n \geq \lambda\) for all \(n\);
(8) The parameters \(\gamma, r', \rho, \sigma\) satisfy the inequalities \(2\rho \leq \gamma/2r + 1/2r' \leq \sigma/3\).

Then, there exists subsequence of \(\{x_n\}\) converges to \(\bar{x} \in C\) which solves \((\text{NVP}[C, F])\).

**Proof.** Let \(\bar{x} \in C\) be a solution of \((\text{NVP}[C, F])\). Then by Proposition 9 we have
\[
F(\bar{x}, x) + \rho_0 V(J(\bar{x}), x) \geq 0, \quad \forall x \in C,
\]
for \(\rho_0 = \gamma/2r + 1/2r'\). By the \(W\)-strong monotonicity of \(F\) over \(C\) we have
\[
F(x, \bar{x}) + F(\bar{x}, x) \leq -\sigma W(x, \bar{x}), \quad \forall x \in C.
\]
By setting \(x = x_n\) in these two inequalities we get
\[
F(x_n, \bar{x}) + F(\bar{x}, x_n) \leq -\sigma W(x_n, \bar{x}),
\]
\[
-F(\bar{x}, x_n) \leq \rho_0 V(J(\bar{x}), x_n).
\]
Combining these two inequalities we obtain
\[
F(x_n, \bar{x}) \leq \rho_0 V(J(\bar{x}), x_n) - \sigma W(x_n, \bar{x}) \leq (2\rho_0 - \sigma) W(x_n, \bar{x}).
\]
Using the 8th assumption in Theorem 14 we have \(2\rho_0 - \sigma \leq -\rho_0\) and hence
\[
F(x_n, \bar{x}) \leq -\rho_0 W(x_n, \bar{x}).
\]
This combined with Algorithm 13 gives
\[
\langle x_{n+1}^* - x_n, \bar{x} - x_n \rangle \leq F(x_n, \bar{x}) + \rho V(J(x_n), \bar{x}) \leq -\rho_0 W(x_n, \bar{x}) + \rho V(J(x_n), \bar{x}) \leq (2\rho - \rho_0) W(x_n, \bar{x}),
\]
with \(x_n^* = \lambda_n^{-1}[J(x_n) - J(x_{n+1})]\). Therefore,
\[
\langle J(x_n) - J(x_{n+1}), \bar{x} - x_n \rangle \leq \lambda_n (2\rho - \rho_0) W(x_n, \bar{x}).
\]
Define now a sequence of nonnegative real numbers \(\phi_n = (1/2)V(J(x_n), \bar{x})\). It is not hard to verify that
\[
2 [\phi_n - \phi_n]\frac{1}{\rho} + V(J(x_n), x_{n+1}) = 2 \langle J(x_n) - J(x_{n+1}), \bar{x} - x_n \rangle.
\]
Indeed,
\[
2 [\phi_n - \phi_n] = V(J(x_n), \bar{x}) - V(J(x_n), x_{n+1}) = \langle J(x_n) - J(x_{n+1}), \bar{x} - x_n \rangle - \langle J(x_{n+1}), x_{n+1} \rangle + \langle J(x_{n+1}), \bar{x} \rangle - \langle J(x_{n+1}), x_{n+1} \rangle.
\]
It follows that
\[
\phi_n - \phi_n \leq \langle J(x_n) - J(x_{n+1}), \bar{x} - x_n \rangle,
\]
which ensures with (46) that
\[
\phi_{n+1} - \phi_n \leq \lambda_n (2\rho - \rho_0) W(x_n, \bar{x}).
\]
Using the assumption \(\rho_0 \geq 2\rho\) in the 8th assumption we obtain
\[
\phi_{n+1} \leq \phi_n.
\]
Therefore, the sequence \(\{\phi_n\}\) is a nonincreasing converging sequence to some limit and so it is bounded by some \(\alpha > 0\). Thus by the properties of the functional \(V\) we obtain
\[
\|x_n\| \leq \|\bar{x}\| + \sqrt{2\alpha};
\]
and so
\[
\|x_n\| \leq \|\bar{x}\| + \sqrt{2\alpha}.
\]
that is, \( \{x_n\} \) is bounded and so by the \( q \)-uniform convexity of \( X^* \) (by Lemma 6) we have for some \( \eta > 0 \) depending on \( \alpha \) and on the space \( X^* \) the inequality
\[
\|J(x_{n+1}) - J(x_n)\|^{q'} \leq \eta V_1(J^*(J(x_{n+1})), J(x_n))
\]
\[
= \eta V(J(x_n), x_{n+1}),
\]
where \( J^* : X^* \to X^{**} = X \) is the normalised duality mapping on \( X^* \) and \( V_n : X^{**} \times X^* \to \mathbb{R} \) is the functional defined by
\[
V_n(x^*; x^*) = \|x^*\|^2 - 2\langle x^*; x^* \rangle + \|x^*\|^2,
\]
\[
\forall x^* \in X^*, x^{**} \in X^{**}.
\]
Using now (46) and (47) and the assumption \( \rho_0 \geq 2\rho \) we obtain
\[
\frac{1}{2} V(J(x_n), x_{n+1}) \leq \phi_n - \phi_{n+1}.
\]
Therefore, it follows from the 7th assumption of Theorem 14 that
\[
\|x^*_{n+1}\|^q = \lambda_n^{q'} \|J(x_{n+1}) - J(x_n)\|^q
\]
\[
\leq \lambda \|J(x_{n+1}) - J(x_n)\|^q
\]
\[
\leq \lambda \|J(x_{n+1}) - J(x_n)\|^q \eta V(J(x_n), x_{n+1})
\]
\[
\leq 2\eta \lambda \|\phi_n - \phi_{n+1}\| \to 0 \quad \text{as } n \to \infty,
\]
which ensures that \( \lim_{n \to \infty} x^*_{n+1} = 0 \). On the other hand, since \( \{x_n\} \) is bounded in \( C \) and \( C \) is ball compact then there exists a subsequence \( \{x_{n_k}\} \) which converges to some limit \( \bar{x} \in C \). By Algorithm 13 this subsequence satisfies
\[
\langle x_{n_{k+1}}, x - x_{n_{k+1}} \rangle \leq F(x_{n_{k+1}}, x) + \rho V(J(x_{n_{k+1}}), x),
\]
\[
\forall k, \forall x \in C.
\]
Thus, by letting \( k \to \infty \) in the inequality (58) and by taking into account the upper semicontinuity of \( F \) and the continuity of \( V \) and \( J \), we obtain
\[
0 \leq F(\bar{x}, x) + \rho V(J(\bar{x}), x), \quad \forall x \in C.
\]
This means that \( \bar{x} \) is a solution of (NEP\([C, F]\)). Finally, using now Proposition 12 we get \( \bar{x} \) is a solution of (NVP\([C, F]\)) and so the proof is complete.

4.2. Case 2: \( C \) Is a General Set-Valued Mapping. In this general case we propose the following algorithm.

Algorithm 15. Let \( \rho \geq 0 \) and \( \lambda_n > 0 \) for all \( n \geq 1 \);

1. Select \( x_0 \in C(x_0) \);
2. For \( n \geq 1 \) select \( x_{n+1} \in C(x_n) \) such that
\[
\lambda_n J(x_n) - J(x_{n+1}) \]
\[
\leq F(x_n, x) + \rho V(J(x_n), x), \quad \forall x \in \text{Im} C,
\]
where \( M > 0 \) is a given positive number and \( \text{Im} C \) is the image of \( C \), that is, \( \text{Im} C = \{y \in X : \exists x \in X \text{ such that } y \in C(x)\} \).

Obviously Algorithm 15 coincides with Algorithm 13 when \( C \) is a constant set-valued mapping. However the assumptions assumed on \( F \) in the previous subsection are not sufficient to prove the convergence of the sequence \( \{x_n\} \) generated by Algorithm 15 to a solution of (NQVP\([C, F]\)). We need to replace the \( W \)-strong monotonicity by a relaxed \( W \)-strong monotonicity of the bifunction \( F \) over \( \text{Im} C \) and we do not assume the nonemptiness of the solution set of the proposed problem. We will say that \( F \) is relaxed \( W \)-strongly monotone over \( \text{Im} C \) provided that for some \( \sigma \geq 0 \) we have
\[
F(x, y) \leq -\sigma W(x, y), \quad \forall x, y \in \text{Im} C.
\]
By symmetry of \( W \), it is clear that any \( W \)-relaxed strongly monotone bifunction with respect to \( \sigma \geq 0 \) is \( W \)-strongly monotone with respect to \( 2\sigma \). This relaxed assumption on \( F \) has been used in Hilbert spaces in [4] and in Banach spaces in [13]. The following theorem is our main result in this subsection.

Theorem 16. Let \( X \) be a \( q \)-uniformly convex Banach space. Let \( C \) be a closed nonempty subset of \( X \) and let \( F : C \times C \to \mathbb{R} \) be a bifunction satisfying \( F(x, x) = 0 \) for any \( x \in \text{Fix}(C) \). Let \( \{x_n\} \) be a sequence generated by Algorithm 15. Assume that

1. The values of \( C \) are \( V \)-uniformly prox-regular with some ratio \( r \in (0, \infty) \);
2. The image of \( C \) is ball compact in \( X \) and its graph is closed;
3. \( F \) is relaxed \( W \)-strongly monotone over \( \text{Im} C \) with some \( \sigma > 0 \);
4. \( F \) is upper semicontinuous with respect to the first variable over \( \text{Im} C \);
5. \( F(x_{n+1}) \) is \( V \)-uniformly prox-regular over \( \text{Im} C \) with some \( r' \in (0, \infty) \);
6. There exists \( \lambda > 0 \) such that \( \lambda_n \geq \lambda \) for all \( n \);
7. The nonnegative parameter \( \rho \) is taken in the interval \([0, \sigma/2]\).

Then, there exists subsequence of \( \{x_n\} \) converging to a solution of (NQVP\([C, F]\)).

Proof. Let \( \bar{x} \in \text{Im} C \). By the relaxed \( W \)-strong monotonicity of \( F \) over \( \text{Im} C \) we have
\[
F(x_n, \bar{x}) \leq -\sigma W(x_n, \bar{x}), \quad \forall n \geq 1.
\]
By Algorithm 15 we have
\[
\langle x_{n+1} - x_{n+1}, \bar{x} - x_{n+1} \rangle \leq F(x_n, \bar{x}) + \rho V(J(x_n), \bar{x}),
\]
with \( x_{n+1} = \lambda_n^{-1}[J(x_n) - J(x_{n+1})] \). Combining these two inequalities we get
\[
\langle x_{n+1} - x_{n+1}, \bar{x} - x_{n+1} \rangle \leq \rho V(J(x_n), \bar{x}) - \sigma W(x_n, \bar{x})
\]
\[
\leq (2\rho - \sigma) W(x_n, \bar{x}).
\]
Therefore,
\[
\langle f(x_n) - f(x_{n+1}), \bar{x} - x_{n+1} \rangle 
\leq \lambda_n (2\rho - \sigma) W(x_n, \bar{x}).
\] (65)

Define now the same nonnegative real sequence \( \phi_n = (1/2)V(f(x_n), \bar{x}) \) used in the proof of Theorem 14. Then we have
\[
\phi_{n+1} - \phi_n \leq \langle f(x_n) - f(x_{n+1}), \bar{x} - x_{n+1} \rangle,
\] (66)
which ensures with (65) that
\[
\phi_{n+1} - \phi_n \leq \lambda_n (2\rho - \sigma) W(x_{n+1}, \bar{x}).
\] (67)

Using the assumption \( \sigma \geq 2\rho \) yields
\[
\phi_{n+1} \leq \phi_n.
\] (68)

Following the same reasoning in the proof of Theorem 14 and the ball compactness of the image of \( C \), we get a subsequence \( \{x_{n_k}\} \) which converges to some limit \( \bar{x} \) satisfying \( \bar{x} \in C(\bar{x}) \) by closedness of the graph of \( C \). By Algorithm 15 this subsequence satisfies
\[
\langle x_{n_k+1}^*, x - x_{n_k+1} \rangle \leq F(x_{n_k+1}, x) + \rho V(f(x_{n_k+1}), x),
\] (69)
\[\forall k, \forall x \in \text{Im} C.
\]

Thus, by letting \( k \to \infty \) in the inequality (69) and by taking into account the upper semicontinuity of \( F \) and the continuity of \( V \) and \( f \), we obtain
\[
0 \leq F(\bar{x}, x) + \rho V(f(\bar{x}), x), \quad \forall x \in C(\bar{x}).
\] (70)

This means that \( \bar{x} \) is a solution of \( \text{(NQEP}[C, F]) \) which ensures by Proposition 12 that under the assumptions of our theorem the solution \( \bar{x} \) is also a solution of \( \text{(NQVP}[C, F]) \). Thus completing the proof. \( \square \)

4.3. Case 3: \( F \) Has the Form: \( F(x, y) = \langle T(x), y - x \rangle \). In this subsection we restrict our attention to the following form of the bifunction \( F \):
\[
F(x, y) = \langle T(x), y - x \rangle,
\] (71)
where \( T : X \to X^* \) is a nonlinear operator. In this case \( \partial^* F(x)\cdot(x) = \{ f(T(x)) \} \) and so \( \text{(NQVP}[C, F]) \) becomes:

Find \( \bar{x} \in C(\bar{x}), \) such that \( T(\bar{x}) \in -N^\infty(C(\bar{x}), \bar{x}) \).

We suggest the following algorithm to solve \( \text{(NQVP}[C, T]) \) under some natural and appropriate assumptions on \( C \) and \( T \).

Algorithm 17. Let \( \delta_n \downarrow 0 \) with \( \delta_0 \) be too small.

(i) Select \( x_0 \in C(x_0), \) \( y_0^* = T(x_0) \) and \( \rho > 0; \)
(ii) For \( n \geq 0, \)

(a) Compute \( z_{n+1} := f^*(f(x_n) - \rho y_n^*); \)
(b) Compute \( x_{n+1} := \pi_C(x_n) \) and \( y_{n+1} := T(x_{n+1}), \)

where \( \pi_S \) is the generalised projection defined in terms of the functional \( V \) instead of the norm square (introduced and studied in the convex case in [16] and for the nonconvex case we refer to the recent paper [11]). A point \( \bar{x} \in S \) is called the generalised projection of a given \( x^* \in X^* \) provided that
\[
V(x^*, \bar{x}) = \inf_{s \in S} V(x^*, s).
\] (72)

The following characterisation of the \( V \)-proximal normal cone in terms of the generalised projection is proved in [1].

Proposition 18. For any closed nonempty set \( S \) in a reflexive Banach space \( X \) and for any point \( \bar{x} \in X \) we have
\[
N^\infty(S; \bar{x}) = \{ x^* \in X^*: \exists \lambda > 0 \text{ such that } \bar{x} \in \pi_S(J(\bar{x}) + \lambda x^*) \}.
\] (73)

We need the following lemma:

Lemma 19. Let \( S \) be a closed set in \( X, \bar{x} \in S, \ y^* \in X^* \), and \( r > 0 \). If \( \bar{x} \in \pi_S(J(\bar{x}) - r y^*); \) then \( \bar{x} \in \pi_S(J(\bar{x}) - r y^*) \). Assume that \( \rho \in [0, r] \). Let \( \lambda = \rho/r \in [0, 1] \). We claim that
\[
V(J(\bar{x}) - \rho y^*; \bar{x}) = \inf_{s \in S} V(J(\bar{x}) - \rho y^*; s).
\] (74)

First, observe that for any \( s \in S \) we have
\[
2 \langle J(\bar{x}) - \rho y^* - J\bar{x}; s - \bar{x} \rangle = 2 \langle \lambda (J(\bar{x}) - r y^*) + (1-\lambda) (J(\bar{x}) - J\bar{x}; s - \bar{x}) \rangle
\] (75)
\[
= 2 \lambda \langle (J(\bar{x}) - r y^*) - J(\bar{x}; s - \bar{x}) \rangle.
\]

If \( (J(\bar{x}) - r y^*) - J(\bar{x}; s - \bar{x}) < 0 \), then obviously we have
\[
2 \langle J(\bar{x}) - \rho y^* - J\bar{x}; s - \bar{x} \rangle < 0 \leq V(J(\bar{x}), s).
\] (76)

Otherwise, we have \( (J(\bar{x}) - r y^*) - J(\bar{x}; s - \bar{x}) \geq 0 \). Then since \( 0 \leq \lambda \leq 1 \) we have
\[
2 \lambda \langle (J(\bar{x}) - r y^*) - J(\bar{x}; s - \bar{x}) \rangle \leq 2 \langle (J(\bar{x}) - r y^*) - J(\bar{x}; s - \bar{x}) \rangle
\] (77)
and so we obtain
\[ 2 \langle J(\bar{x}) - \rho y^* - J\bar{x}; s - \bar{x} \rangle \]
\[ \leq 2 \langle (J(\bar{x}) - ry^*); s - \bar{x} \rangle \]
\[ \leq \| J(\bar{x}) - ry^* \|^2 - 2 \langle (J(\bar{x}) - ry^*); \bar{x} \rangle + \| \bar{x} \|^2 \]
\[ + 2 \langle (J(\bar{x}) - ry^*); s \rangle - \| J(\bar{x}) - ry^* \|^2 - \| s \|^2 \]
\[ + \| s \|^2 - 2 \langle J(\bar{x}); s - \bar{x} \rangle - \| \bar{x} \|^2 \]
\[ \leq V(J(\bar{x}) - ry^*; \bar{x}) - V(J(\bar{x}) - ry^*; s) \]
\[ + V(J(\bar{x}), s) \]
\[ \leq \inf_{z \in S} V(J(\bar{x}) - ry^*; z) - V(J(\bar{x}) - ry^*; s) \]
\[ + V(J(\bar{x}), s) \leq V(J(\bar{x}), s); \]
that is,
\[ 2 \langle J(\bar{x}) - \rho y^* - J\bar{x}; s - \bar{x} \rangle \leq V(J(\bar{x}), s). \] (79)

Therefore, from (76) and (79) we have in both cases
\[ 2 \langle J(\bar{x}) - \rho y^* - J\bar{x}; s - \bar{x} \rangle \leq V(J(\bar{x}), s), \quad \forall s \in S. \] (80)

Hence
\[ 2 \langle J(\bar{x}) - \rho y^* - J\bar{x}; s - \bar{x} \rangle - V(J(\bar{x}), s) \leq 0, \]
\[ \forall s \in S. \] (81)

On the other hand we have the decomposition
\[ 2 \langle J(\bar{x}) - \rho y^* - J\bar{x}; s - \bar{x} \rangle - V(J(\bar{x}), s) \]
\[ = 2 \langle J(\bar{x}) - \rho y^*; s \rangle - 2 \langle J(\bar{x}) - \rho y^*; \bar{x} \rangle + 2 \| \bar{x} \|^2 \]
\[ - 2 \langle J\bar{x}; s \rangle - \| \bar{x} \|^2 - 2 \langle J\bar{x}; s \rangle + \| s \|^2 \]
\[ = \| J(\bar{x}) - \rho y^* \|^2 - 2 \langle J(\bar{x}) - \rho y^*; \bar{x} \rangle + \| \bar{x} \|^2 \]
\[ - \| J(\bar{x}) - \rho y^* \|^2 - 2 \langle J(\bar{x}) - \rho y^*; s \rangle + \| s \|^2 \]
\[ = V(J(\bar{x}) - \rho y^*; \bar{x}) - V(J(\bar{x}) - \rho y^*; s). \]

Consequently, we have
\[ V(J(\bar{x}) - \rho y^*; \bar{x}) - V(J(\bar{x}) - \rho y^*; s) \leq 0, \]
\[ \text{for any } s \in S, \] (83)

that is,
\[ V(J(\bar{x}) - \rho y^*; \bar{x}) = \inf_{s \in S} V(J(\bar{x}) - \rho y^*; s); \] (84)

which means that \( \bar{x} \in \pi_s(J(\bar{x}) - \rho y^*) \) and hence the proof is complete. \( \square \)

Now, we state and prove our main theorem for (NQVP(C, T)).

**Theorem 20.** Let \( X \) be a 2-uniformly smooth Banach space. Let \( C : X \rightrightarrows X \) be a set-valued mapping with closed nonempty values and \( T : X \rightarrow X^* \). Let \( \{x_n\}_n \) be a sequence generated by Algorithm 17. Assume that

1. The solution set of (NQVP(C, T)) is nonempty;
2. \( T \) is bounded by some constant \( L > 0 \);
3. \( T \) is \( J \)-Lipschitz, with constant \( \beta > 0 \); that is,
\[ \| T(x_1) - T(x_2) \| \leq \beta \| J(x_1) - J(x_2) \|, \]
\[ \forall x_1, x_2 \in X; \] (85)

4. \( T \) is \( J \)-strongly monotone with constant \( \alpha > 0 \); that is,
\[ \langle J^*(T(x_1) - T(x_2)); J(x_1) - J(x_2) \rangle \]
\[ \geq \alpha \| J(x_1) - J(x_2) \|^2, \quad \forall x_1, x_2 \in X; \] (86)

5. The values of \( C \) satisfy for some \( r \in (0, \infty) \):
\[ \bar{u} \in \pi_{\pi_{C, T}}(J(\bar{u}) + ru^*), \quad \forall u^* \in X^* \] (87)

for any unit vector \( u^* \) in \( X^* \) and any \( \bar{u} \) solution of (NQVP(C, T));

6. There exists some constant \( k \in (0, 1) \) and \( \xi > 0 \) such that
\[ \| J(\pi_{C, T}(x_1^*)) - J(\pi_{C, T}(x_2^*)) \| \]
\[ \leq \xi \| x_1^* - x_2^* \|^2 + k \| J(x_1) - J(x_2) \|, \]
\[ \forall x_1, x_2 \in X; \] (88)

7. The positive constants \( \alpha \) and \( \beta \) satisfy the inequality \( \alpha > \beta \sqrt{1 - (1 - k)^2/c\xi^2}; \)

8. The parameter \( \rho \) in Algorithm 17 satisfies
\[ \frac{\alpha}{\beta^2} - \varepsilon < \rho < \min \left\{ \frac{\mu - \delta_0}{L \beta^2 + \varepsilon}, \frac{\alpha}{\beta^2} \right\}, \]
\[ \varepsilon := \sqrt{\alpha^2 - \beta^2 (1 - (1 - k)^2/c\xi^2) \beta^2} \] (89)

Then, the sequence \( \{x_n\}_n \) generated by Algorithm 17 converges to a solution of (NQVP(C, T)).

**Proof.** Let \( \bar{x} \in C(\bar{x}) \) be a solution of (NQVP(C, T)), that is, \( -T(\bar{x}) \in N^\circ(C(\bar{x}); \bar{x}) \). Then by the characterisation of the \( V \)-proximal normal cone in Proposition 18, there exists \( \lambda > 0 \) such that \( \bar{x} \in \pi_{C,T}(J(\bar{x}) - \lambda T(\bar{x})) \). Using Lemma 19 we obtain \( \bar{x} \in \pi_{C,T}(J(\bar{x}) - \tau T(\bar{x})), \) for any \( \tau \in [0, \lambda] \). By assumption (5) we may assume that \( \lambda \leq r/L \) and so we get \( \rho \leq r/L \). Hence \( \bar{x} \in \pi_{C,T}(J(\bar{x})) \). Since \( X \) is 2-uniformly smooth we have \( X^* \) is 2-uniformly convex; that is,
\[ \delta_{X^*}(e) \geq 2c^{-1} e^2, \] (90)
Thus, we get

\[ V_\ast(J^*x^*, y^*) \geq 8C^2\delta x_0 \left( \frac{\|x^* - y^*\|}{4C} \right) \]

(91)

\[ \geq c^{-1}\|x^* - y^*\|^2, \quad \forall x^*, y^* \in X^*. \]

Thus we can write

\[ \|\rho [T(x_n) - T(\overline{x})] - (J(x_n) - J(\overline{x}))\|^2 \]

\[ \leq c [V_\ast(\rho I^* (T(x_n) - T(\overline{x})); J(x_n) - J(\overline{x}))]. \]

Therefore,

\[ \|J(z_{n+1}) - J(\overline{x})\|^2 \]

\[ = \|J(x_n) - \rho I^* (T(x_n) - T(\overline{x})); J(x_n) - J(\overline{x}))\| \]

\[ \leq c [V_\ast(\rho I^* (T(x_n) - T(\overline{x})); J(x_n) - J(\overline{x}))] \]

\[ \leq c [\rho^2 \|T(x_n) - T(\overline{x})\|^2 + \|J(x_n) - J(\overline{x})\|^2] \]

\[ - 2c\rho \langle J^* (T(x_n) - T(\overline{x})); J(x_n) - J(\overline{x}) \rangle. \]

Using the $J$-Lipschitz continuity of $T$ with ratio $\beta$ we have

\[ \|T(x_n) - T(\overline{x})\|^2 \leq \beta^2 \|J(x_n) - J(\overline{x})\|^2 \]

(94)

and by the $J$-strong monotonicity of $T$ with ratio $\alpha$ we have

\[ \langle J^* (T(x_n) - T(\overline{x})); J(x_n) - J(\overline{x}) \rangle \]

\[ \geq \alpha \|J(x_n) - J(\overline{x})\|^2. \]

(95)

Thus, we get

\[ \|J(z_{n+1}) - J(\overline{x})\|^2 \]

\[ \leq c [\rho^2 \beta^2 \|J(x_n) - J(\overline{x})\|^2 + \|J(x_n) - J(\overline{x})\|^2] \]

\[ - 2c\rho \alpha \|J(x_n) - J(\overline{x})\|^2 \]

\[ \leq c (1 + \rho^2 \beta^2 - 2\rho\alpha) \|J(x_n) - J(\overline{x})\|^2 \]

(96)

\[ \leq c \sqrt{c (1 + \rho^2 \beta^2 - 2\rho\alpha)} \|J(x_n) - J(\overline{x})\|. \]

(97)

On the other hand we have by the 6th assumption

\[ \|J(x_{n+1}) - J(\overline{x})\| \]

\[ \leq \|J(\pi_{C(x_n)} (J(z_{n+1}))) - J(\pi_{C(\overline{x})} (J(\overline{x})))\| \]

\[ \leq \xi \|J(z_{n+1}) - J(\overline{x})\| + k \|J(x_n) - J(\overline{x})\|. \]

(98)

Thus

\[ \|J(x_{n+1}) - J(\overline{x})\| \]

\[ \leq (k + \xi \sqrt{c (1 + \rho^2 \beta^2 - 2\rho\alpha)}) \|J(x_n) - J(\overline{x})\|. \]

(99)

Our assumptions and the choice of $\rho$ ensure that $(k + \xi \sqrt{c (1 + \rho^2 \beta^2 - 2\rho\alpha)}) < 1$ and hence $\|J(x_n) - J(\overline{x})\| \to 0$ which means that $x_n \to \overline{x}$ by the uniform continuity of $J^*$ and thus completing the proof.

Remark 21. A simple inspection of the proof of the previous theorem shows that the result is valid in the case when $T$ is taken a general set-valued mapping instead of a single-valued operator defined from $X$ to $X^*$ and of course the assumptions on $T$ should be adapted naturally for the set-valued case.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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