Research Article

A Subclass of Analytic Functions Related to k -Uniformly Convex and Starlike Functions

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We investigate some subclasses of k -uniformly convex and k -uniformly starlike functions in open unit disc, which is generalization of class of convex and starlike functions. Some coefficient inequalities, a distortion theorem, the radii of close-to-convexity, and starlikeness and convexity for these classes of functions are studied. The behavior of these classes under a certain modified convolution operator is also discussed.

1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions f in open unit disc $\Delta = \{z : |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. Thus, any $f \in \mathcal{A}$ has the following Maclaurin's series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

A function f is said to be univalent if it never takes same value twice. By $\mathcal{S}$ we mean the subclass of $\mathcal{A}$ which is composed of univalent functions. By $\mathcal{ST}$ and $\mathcal{CV}$ we mean the well-known subclasses of $\mathcal{A}$ that are, respectively, starlike and convex.

In 1991, Goodman [1, 2] introduced the classes $\mathcal{UCV}$ and $\mathcal{UST}$ of uniformly convex and uniformly starlike functions, respectively. A function $f \in \mathcal{S}$ is uniformly convex if $f(z)$ maps every circular arc γ contained in Δ with center $\zeta \in \Delta$ onto a convex arc. The function $f \in \mathcal{S}$ is uniformly starlike if $f(z)$ maps every circular arc γ contained in Δ with center $\zeta \in \Delta$ onto a starlike arc with respect to $f(\zeta)$. A more useful representation of $\mathcal{UCV}$ and $\mathcal{UST}$ was given in [3–6] as

$$f \in \mathcal{UCV} \iff f \in \mathcal{A},$$

$$\operatorname{Re} \left(\frac{(zf'(z))'}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|,$$

$$z \in \Delta.$$

$$f \in \mathcal{UST} \iff f \in \mathcal{A},$$

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|,$$

$$z \in \Delta.$$

$$(2)$$

In 1999, for $k \geq 0$, Kanas and Wisniowska [7] introduced the class $k - \mathcal{UCV}$ and $k - \mathcal{UST}$ as

$$f \in k - \mathcal{UCV} \iff$$

$$zf' \in k - \mathcal{UST} \iff f \in \mathcal{A},$$

$$\operatorname{Re} \left(\frac{(zf'(z))'}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right|, \quad (3)$$

$$z \in \Delta.$$

Observe that $0 - \mathcal{UCV} \equiv \mathcal{CV}$, $0 - \mathcal{UST} \equiv \mathcal{ST}$ and $1 - \mathcal{UCV} \equiv \mathcal{UCV}$, $1 - \mathcal{UST} \equiv \mathcal{UST}$.

For fixed $k \geq 0$, these classes have a nice geometrical representation; for detail see [7–9].

A lot of authors obtain very useful properties of $\mathcal{UCV}$ and $\mathcal{UST}$ and their generalization in several direction; for example, see [1, 2, 7, 8, 10, 11] and reference cited therein.

For $(0 \leq \alpha < 1)$, in [4] (see also [12]), Ronning introduced the following two important subclasses $k - \mathcal{UST}(\alpha)$ and $k - \mathcal{UCV}(\alpha)$ as

$$f \in k - \mathcal{UST}(\alpha) \iff \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \Delta. \quad (4)$$

$$f \in k - \mathcal{UCV}(\alpha) \iff zf' \in k - \mathcal{UST}(\alpha).$$

Recently in [13] El-Ashwah et al. introduced two important subclass $k - \mathcal{UCV}(\alpha, \beta)$ and $k - \mathcal{UST}(\alpha, \beta)$ of k -uniformly convex starlike functions as

$$f \in k - \mathcal{UCV}(\alpha, \beta) \iff \operatorname{Re} \left\{ \frac{(zf'(z))'}{f'(z)} - \alpha \right\} > k \left| \frac{(zf'(z))'}{f'(z)} - \beta \right|, \quad z \in \Delta, \quad (5)$$

$$f \in k - \mathcal{UST}(\alpha, \beta) \iff \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > k \left| \frac{zf'(z)}{f(z)} - \beta \right|, \quad z \in \Delta,$$

where $(0 \leq \alpha < \beta \leq 1)$ and $k(1 - \beta) < 1 - \alpha$.

Let f_j ($j = 1, 2, \dots$) be defined by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0; \quad j \in N; \quad (6)$$

then the modified Hadmard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \quad (7)$$

We denote $\mathcal{T}$ by subclass of $\mathcal{S}$ consisting of functions having all negative coefficients in their Maclaurin's series expansions, so any $f \in \mathcal{T}$ has a series of the form:

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \Delta. \quad (8)$$

Let $\mathcal{V}_\eta$ be the class of functions $f \in \mathcal{S}$ given in (1) for which $\arg(a_n) = \pi + (n-1)\eta$, $n \geq 2$. Note that $\mathcal{V}_0 = \mathcal{T}$ [11].

In recent years, more and more researchers are interested in the above defined classes (see [9, 11, 14–22]).

In this paper, by taking inspiration from the above cited paper, we introduce some new subclasses of analytic functions and obtain some interesting results.

Definition 1. For $(0 \leq \alpha < \beta \leq 1)$, $0 \leq \delta < 1$, $k(1 - \beta) < 1 - \alpha$, and $0 \leq \lambda < 1$, a function $f \in \mathcal{S}$ is in class $k - \mathcal{U}(\alpha, \beta, \lambda, \delta)$ if and only if

$$\operatorname{Re} \left\{ \frac{(1 - \delta)zf' + \delta(zf' + (1 + 2\lambda)z^2f'' + \lambda z^3f''')}{(1 - \delta)f + \delta(zf' + \lambda z^2f'')} - \alpha \right\} \geq k \left| \frac{(1 - \delta)zf' + \delta(zf' + (1 + 2\lambda)z^2f'' + \lambda z^3f''')}{(1 - \delta)f + \delta(zf' + \lambda z^2f'')} - \beta \right|, \quad z \in \Delta. \quad (9)$$

Also

$$k - \mathcal{VU}_\eta(\alpha, \beta, \lambda, \delta) = k - \mathcal{U}(\alpha, \beta, \lambda, \delta) \cap \mathcal{V}_\eta. \quad (10)$$

It is worth mentioning that, for special values of parameters, these classes were extensively studied by many authors; here we mention few of them.

- (1) $k - \mathcal{U}(\alpha, \beta, \lambda, 1) = k - \mathcal{U}(\lambda, \beta, \alpha)$ [21].
- (2) $k - \mathcal{VU}_0(\alpha, \beta, \lambda, 1) = k - \mathcal{VU}_\eta(\lambda, \beta, \alpha)$ [21].
- (3) $0 - \mathcal{VU}_0(\alpha, 1, 0, 1) = \mathcal{CV}(\alpha)$ [11].
- (4) $k - \mathcal{VU}_0(\alpha, 1, 0, 1) = k - \mathcal{UCV}(\alpha)$ [23].
- (5) $1 - \mathcal{U}(\alpha, 1, 0, 1) = \mathcal{UST}(\alpha)$ [4].
- (6) $k - \mathcal{U}(\alpha, \beta, 0, 0) = k - \mathcal{UST}(\alpha, \beta)$ [13].

Throughout the paper $-1 \leq \alpha \leq \beta \leq 1$, $0 \leq \lambda < 1$, $1 - \alpha > k(1 - \beta)$, and $z \in \Delta$, unless otherwise stated.

2. Main Results

Theorem 2. A function $f(z)$ given by (1) is in class $k - \mathcal{U}(\alpha, \beta, \lambda, \delta)$ if

$$\sum_{n=2}^{\infty} [\Pi_n(1 + k) - (k\beta + \alpha)\Omega_n] |a_n| \leq 1 - \alpha - k(1 - \beta), \quad (11)$$

where

$$\begin{aligned} \Pi_n &= (n + \delta n(n-1)(1 + \lambda n)), \\ \Omega_n &= 1 - \delta + \delta n(1 + \lambda(n-1)). \end{aligned} \quad (12)$$

Proof. It is sufficient to prove that inequality (9) holds. As we know

$$\begin{aligned} \operatorname{Re}(w) &> k|w - \beta| + \alpha \\ \text{iff } \operatorname{Re}((1 + ke^{i\theta})w - \beta ke^{i\theta}) &\geq \alpha; \end{aligned} \quad (13)$$

then inequality (9) can be written as

$$\operatorname{Re} \left(\left(1 + ke^{i\theta} \right) \cdot \frac{(1-\delta)zf' + \delta(zf' + (1+2\lambda)z^2f'' + \lambda z^3f''')}{(1-\delta)f + \delta(zf' + \lambda z^2f'')} - \beta ke^{i\theta} \right) \geq \alpha. \quad (14)$$

This is,

$$\operatorname{Re} \left(\frac{A(z)}{B(z)} \right) \geq \alpha, \quad (15)$$

where

$$\begin{aligned} A(z) &= (1 + ke^{i\theta}) (zf' + \delta((1+2\lambda)z^2f'' + \lambda z^3f''')) \\ &\quad - \beta ke^{i\theta} ((1-\delta)f + \delta(zf' + \lambda z^2f'')), \end{aligned} \quad (16)$$

$$B(z) = (1-\delta)f + \delta(zf' + \lambda z^2f'');$$

then we have

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0. \quad (17)$$

Now

$$\begin{aligned} |A(z) + (1-\alpha)B(z)| &= \left| (1-\beta)ke^{i\theta} + (2-\alpha)z \right. \\ &\quad \left. - \sum_{n=2}^{\infty} [(\beta\Omega_n - \Pi_n)ke^{i\theta} - (1-\alpha)\Omega_n - \Pi_n] a_n z^n \right| \\ &\geq -((1-\beta)k + (2-\alpha))|z| \\ &\quad - \sum_{n=2}^{\infty} [(\beta\Omega_n - \Pi_n)k + (1-\alpha)\Omega_n + \Pi_n] |a_n| |z|^n, \end{aligned} \quad (18)$$

and also

$$\begin{aligned} |A(z) - (1+\alpha)B(z)| &= \left| ((1-\beta)ke^{i\theta} - \alpha)z \right. \\ &\quad \left. + \sum_{n=2}^{\infty} [(\Pi_n - \beta\Omega_n)ke^{i\theta} - (1+\alpha)\Omega_n + \Pi_n] a_n z^n \right| \\ &\leq ((1-\beta)k + \alpha)|z| \\ &\quad + \sum_{n=2}^{\infty} [(\Pi_n - \beta\Omega_n)k - (1+\alpha)\Omega_n + \Pi_n] |a_n| |z|^n. \end{aligned} \quad (19)$$

From (18) and (19), we have

$$\begin{aligned} |A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| &\geq 2[(1-\alpha) - k(1-\beta)]|z| \\ &\quad - 2 \sum_{n=2}^{\infty} [(\Pi_n - \beta\Omega_n)k + (\Pi_n - \alpha\Omega_n)] |a_n| |z|^n \\ &= 2 \left[[(1-\alpha) - k(1-\beta)]|z| \right. \\ &\quad \left. - \sum_{n=2}^{\infty} [\Pi_n(1+k) - (k\beta + \alpha)\Omega_n] |a_n| |z|^n \right]. \end{aligned} \quad (20)$$

The last expression is bounded below by 0 if

$$\begin{aligned} \sum_{n=2}^{\infty} [\Pi_n(1+k) - (k\beta + \alpha)\Omega_n] |a_n| &\leq 1 - \alpha - k(1-\beta), \end{aligned} \quad (21)$$

which completes the proof. $\square$

In the next theorem, we prove that condition (11) is also necessary for function $f \in k - \mathcal{W}_\eta(\alpha, \beta, \lambda, \delta)$.

Theorem 3. Let $f(z)$ be given by (1) and in V_η ; then $f \in k - \mathcal{W}_\eta(\alpha, \beta, \lambda, \delta)$ if and only if

$$\begin{aligned} \sum_{n=2}^{\infty} [\Pi_n(1+k) - (k\beta + \alpha)\Omega_n] |a_n| &\leq 1 - \alpha - k(1-\beta). \end{aligned} \quad (22)$$

Proof. From Theorem 2, we need only to show that $f \in k - \mathcal{W}_\eta(\alpha, \beta, \lambda, \delta)$ satisfies inequality (22). If $f \in k - \mathcal{W}_\eta(\alpha, \beta, \lambda, \delta)$, then by definition, we have

$$\begin{aligned} \operatorname{Re} \left(\frac{(1-\alpha) + \sum_{n=2}^{\infty} (\Pi_n - \alpha\Omega_n) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \Omega_n a_n z^{n-1}} \right) &\geq k \left| \frac{(1-\beta) + \sum_{n=2}^{\infty} (\Pi_n - \beta\Omega_n) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \Omega_n a_n z^{n-1}} \right|. \end{aligned} \quad (23)$$

Since f is function of form (1) with the argument property given in class V_η and letting $z = re^{i\theta}$ in the above inequality, we have

$$\begin{aligned} \frac{(1-\alpha) - \sum_{n=2}^{\infty} (\Pi_n - \alpha\Omega_n) |a_n| r^{n-1}}{1 - \sum_{n=2}^{\infty} \Omega_n |a_n| r^{n-1}} &\geq k \left(\frac{(1-\beta) - \sum_{n=2}^{\infty} (\Pi_n - \beta\Omega_n) |a_n| r^{n-1}}{1 - \sum_{n=2}^{\infty} \Omega_n |a_n| r^{n-1}} \right) \end{aligned} \quad (24)$$

for $r \rightarrow 1$, and (24) leads to require inequality

$$\begin{aligned} \sum_{n=2}^{\infty} [\Pi_n(1+k) - (k\beta + \alpha)\Omega_n] |a_n| &\leq 1 - \alpha - k(1-\beta). \end{aligned} \quad (25)$$

The function

$$f_{n,\eta}(z) = z - \frac{(1-\alpha-k(1-\beta))e^{i(1-n)\eta}}{\Pi_n(1+k)-(k\beta+\alpha)\Omega_n}z^n, \quad (26)$$

$$0 \leq \eta \leq 2\pi, \quad n \geq 2$$

is extremal function. $\square$

Corollary 4. Let $f(z)$ given in (1) be in class $k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$. Then

$$|a_n| \leq \frac{1-\alpha-k(1-\beta)}{\Pi_n(1+k)-(k\beta+\alpha)\Omega_n}, \quad n \geq 2. \quad (27)$$

Inequality (27) is attained for the function given in (26).

Theorem 5. Let the function $f(z)$ given in (1) be in class $k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$. Then for $|z| < r = 1$

$$|f(z)| \geq r - \frac{1-\alpha-k(1-\beta)}{\Pi_2(1+k)-(k\beta+\alpha)\Omega_2}r^2, \quad (28)$$

$$|f(z)| \leq r + \frac{1-\alpha-k(1-\beta)}{\Pi_2(1+k)-(k\beta+\alpha)\Omega_2}r^2.$$

The results in (28) are attained for the function given in (26) for $z = \pm r$.

Proof. As we know from Theorem 3

$$\begin{aligned} & [\Pi_2(1+k)-(k\beta+\alpha)\Omega_2] \sum_{n=2}^{\infty} |a_n| \\ & \leq \sum_{n=2}^{\infty} [\Pi_n(1+k)-(k\beta+\alpha)\Omega_n] |a_n| \\ & \leq 1-\alpha-k(1-\beta). \end{aligned} \quad (29)$$

As

$$\begin{aligned} |f(z)| & \geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \geq r - r^2 \sum_{n=2}^{\infty} |a_n| \\ & \geq r - \frac{1-\alpha-k(1-\beta)}{\Pi_2(1+k)-(k\beta+\alpha)\Omega_2}r^2, \end{aligned} \quad (30)$$

similarly

$$\begin{aligned} |f(z)| & \leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n \leq r + r^2 \sum_{n=2}^{\infty} |a_n| \\ & \leq r + \frac{1-\alpha-k(1-\beta)}{\Pi_2(1+k)-(k\beta+\alpha)\Omega_2}r^2. \end{aligned} \quad (31)$$

This completes the proof. $\square$

Theorem 6. Let the function $f(z)$ given in (1) be in class $k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$. Then for $|z| < r = 1$

$$\begin{aligned} 1 - \frac{2(1-\alpha-k(1-\beta))}{\Pi_2(1+k)-(k\beta+\alpha)\Omega_2}r & \leq |f(z)| \\ & \leq 1 + \frac{2(1-\alpha-k(1-\beta))}{\Pi_2(1+k)-(k\beta+\alpha)\Omega_2}r. \end{aligned} \quad (32)$$

Proof. For $f(z)$ given by (1), we have

$$\begin{aligned} |f'(z)| & \geq 1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \geq 1 - r \sum_{n=2}^{\infty} n |a_n|, \\ |f'(z)| & \leq 1 + \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} n |a_n|. \end{aligned} \quad (33)$$

In view of Theorem 3,

$$\begin{aligned} & \frac{[\Pi_2(1+k)-(k\beta+\alpha)\Omega_2]}{2} \sum_{n=2}^{\infty} n |a_n| \\ & \leq \sum_{n=2}^{\infty} [\Pi_n(1+k)-(k\beta+\alpha)\Omega_n] |a_n| \\ & \leq 1-\alpha-k(1-\beta), \end{aligned} \quad (34)$$

or, equivalently,

$$\sum_{n=2}^{\infty} n |a_n| \leq \frac{2(1-\alpha-k(1-\beta))}{[\Pi_2(1+k)-(k\beta+\alpha)\Omega_2]}. \quad (35)$$

A substitution from (35) into (33) yields inequality (32), which is required. $\square$

Theorem 7. Let $f \in k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$ with argument property as in class V_η . Define $f_j(z) = z$ and

$$f_{n,\eta} = z - \frac{1-\alpha-k(1-\beta)e^{i(1-n)\eta}}{[\Pi_n(1+k)-(k\beta+\alpha)\Omega_n]}z^n, \quad (36)$$

where $0 \leq \eta \leq 2\pi, n \geq 2$.

Then function $f(z)$ is in class $k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$ if and only if it can be expressed as

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_{n,\eta}, \quad (37)$$

where $\mu_n \geq 0$ ($n \geq 1$) and $\sum_{n=1}^{\infty} \mu_n = 1$.

Proof. Assume that

$$f(z) = \mu_1 f_1(z)$$

$$\begin{aligned} & + \sum_{n=2}^{\infty} \mu_n \left[z - \frac{1-\alpha-k(1-\beta)e^{i(1-n)\eta}}{\Pi_n(1+k)-(k\beta+\alpha)\Omega_n}z^n \right] \\ & = \sum_{n=1}^{\infty} \mu_n z \\ & - \sum_{n=2}^{\infty} \left[\frac{1-\alpha-k(1-\beta)e^{i(1-n)\eta}}{\Pi_n(1+k)-(k\beta+\alpha)\Omega_n} \right] \mu_n z^n. \end{aligned} \quad (38)$$

Then it follows that

$$\begin{aligned} & \sum_{n=2}^{\infty} \left| \frac{1 - \alpha - k(1 - \beta) e^{i(1-n)\eta}}{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n} \right| \\ & \cdot \mu_n [\Pi_n(1+k) - (k\beta + \alpha)\Omega_n] \\ & = \sum_{n=2}^{\infty} \mu_n [1 - \alpha - k(1 - \beta)] \leq (1 - \mu_1) \\ & \cdot [1 - \alpha - k(1 - \beta)] \leq 1 - \alpha - k(1 - \beta), \end{aligned} \quad (39)$$

by Theorem 3, $f \in k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$. Conversely, assume that the function $f(z)$ defined by (1) belongs to class $k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$, and then

$$|a_n| \leq \frac{1 - \alpha - k(1 - \beta)}{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}, \quad n \geq 2. \quad (40)$$

Set

$$\mu_n = \frac{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}{1 - \alpha - k(1 - \beta)} |a_n|, \quad n \geq 2, \quad (41)$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$, $n \geq 2$. Then $f(z) = \sum_{n=1}^{\infty} \mu_n f_{n,\eta}$ and this completes the proof. $\square$

Theorem 8. Let $f \in k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$. Then $f(z)$ is close to convex of order σ ($0 \leq \sigma < 1$) in the disc $|z| < r_1$, where

$$r_1 = \inf \left[\frac{(1 - \sigma)(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)}{n(1 - \alpha - k(1 - \beta))} \right]^{1/(n-1)}, \quad (42)$$

$n \geq 2$.

Proof. As $f \in V_\eta$, where f is close to convex of order σ , we have

$$|f'(z) - 1| < 1 - \sigma, \quad (43)$$

as

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n |a_n| |z|^{n-1}, \quad (44)$$

this expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n}{1 - \sigma} |a_n| |z|^{n-1} < 1. \quad (45)$$

By the fact that $f \in k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$ if and only if

$$\sum_{n=2}^{\infty} \left[\frac{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}{(1 - \alpha - k(1 - \beta))} \right] a_n \leq 1; \quad (46)$$

inequality (43) is true if

$$\frac{n}{1 - \sigma} z^{n-1} \leq \frac{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}{(1 - \alpha - k(1 - \beta))}; \quad (47)$$

or, equivalently,

$$|z|^{n-1} = \left[\frac{(1 - \sigma)(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)}{n(1 - \alpha - k(1 - \beta))} \right]. \quad (48)$$

$\square$

Theorem 9. Let $f \in k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$. Then $f(z)$ is close to convex of order σ ($0 \leq \sigma < 1$) in the disc $|z| < r_2$, where

$$r_2 = \inf \left[\frac{(1 - \sigma)(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)}{(n - \sigma)(1 - \alpha - k(1 - \beta))} \right]^{1/(n-1)}, \quad (49)$$

$n \geq 2$.

Proof. As $f \in V_\eta$ and f is starlike of order σ , then we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \sigma, \quad (50)$$

as

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} |a_n| |z|^{n-1}}. \quad (51)$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n - \sigma}{1 - \sigma} |a_n| |z|^{n-1} < 1. \quad (52)$$

Using the fact that $f \in k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$ if and only if

$$\sum_{n=2}^{\infty} \left[\frac{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}{(1 - \alpha - k(1 - \beta))} \right] a_n \leq 1 \quad (53)$$

(50) is true if

$$\frac{n - \sigma}{1 - \sigma} |z|^{n-1} < \frac{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}{(1 - \alpha - k(1 - \beta))}. \quad (54)$$

Or equivalently

$$|z|^{n-1} = \frac{(1 - \sigma)(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)}{(n - \sigma)(1 - \alpha - k(1 - \beta))}, \quad (55)$$

which is required. $\square$

Theorem 10. Let $f \in k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$. Then $f(z)$ is convex of order σ ($0 \leq \sigma < 1$) in the disc $|z| < r_3$, where

$$r_3 = \inf \left[\frac{(1 - \sigma)[\Pi_n(1+k) - (k\beta + \alpha)\Omega_n]}{n(n - \sigma)(1 - \alpha - k(1 - \beta))} \right]^{1/(n-1)}, \quad (56)$$

$n \geq 2$.

Proof. Using the fact that f is convex if and only if zf' is starlike, following the lines of Theorem 9, we have the required results. $\square$

Theorem 11. Let $f_j(z)$ ($j = 1, 2, \dots$) given by (6) be in class $k - \mathcal{V}\mathcal{U}_\eta(\alpha, \beta, \lambda, \delta)$. Then $(f_1 * f_2) \in k - \mathcal{V}\mathcal{U}_\eta(\phi_1, \lambda, \delta)$, for

$$\phi_1 = \frac{(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)^2 - (\Pi_n(1+k) - k\beta\Omega_n)(1 - \alpha - k(1 - \beta))^2}{(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)^2 - \Omega_n(1 - \alpha - k(1 - \beta))^2}. \quad (57)$$

Proof. We need to prove the largest ϕ_1 such that

$$\frac{(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)}{(1 - \phi_1 - k(1 - \beta))} a_{n,1} a_{n,2} \leq 1. \quad (58)$$

From Theorem 3, we have

$$\sum_{n=2}^{\infty} \left[\frac{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}{(1 - \alpha - k(1 - \beta))} \right] a_{n,1} \leq 1, \quad (59)$$

$$\sum_{n=2}^{\infty} \left[\frac{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}{(1 - \alpha - k(1 - \beta))} \right] a_{n,2} \leq 1.$$

By Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \left[\frac{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}{(1 - \alpha - k(1 - \beta))} \right] \sqrt{a_{n,1} a_{n,2}} \leq 1. \quad (60)$$

Thus, it is sufficient to show

$$\left[\frac{\Pi_n(1+k) - (k\beta + \phi_1)\Omega_n}{(1 - \phi_1 - k(1 - \beta))} \right] a_{n,1} a_{n,2} \leq \left[\frac{\Pi_n(1+k) - (k\beta + \alpha)\Omega_n}{(1 - \alpha - k(1 - \beta))} \right] \sqrt{a_{n,1} a_{n,2}}, \quad n \geq 2. \quad (61)$$

For $n \geq 2$

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)(1 - \phi_1 - k(1 - \beta))}{(\Pi_n(1+k) - (k\beta + \phi_1)\Omega_n)(1 - \alpha - k(1 - \beta))}. \quad (62)$$

Note that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(1 - \alpha - k(1 - \beta))}{(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)}. \quad (63)$$

We need to show

$$\frac{(1 - \alpha - k(1 - \beta))}{(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)} \leq \frac{(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)(1 - \phi_1 - k(1 - \beta))}{(\Pi_n(1+k) - (k\beta + \phi_1)\Omega_n)(1 - \alpha - k(1 - \beta))}, \quad (64)$$

or equivalently

$$\phi_1 \leq \frac{(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)^2 - (\Pi_n(1+k) - k\beta\Omega_n)(1 - \alpha - k(1 - \beta))^2}{(\Pi_n(1+k) - (k\beta + \alpha)\Omega_n)^2 - \Omega_n(1 - \alpha - k(1 - \beta))^2} = \omega(n). \quad (65)$$

$\omega(n)$ is an increasing function for $n \geq 2$. For $n = 2$ in (65),

$$\phi_1 \leq \omega(2) = \frac{(\Pi_2(1+k) - (k\beta + \alpha)\Omega_2)^2 - (\Pi_2(1+k) - k\beta\Omega_2)(1 - \alpha - k(1 - \beta))^2}{(\Pi_2(1+k) - (k\beta + \alpha)\Omega_2)^2 - \Omega_2(1 - \alpha - k(1 - \beta))^2}, \quad (66)$$

which proves main assertion of Theorem 11. $\square$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors jointly work on the results, and they read and approved the final manuscript.

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