Research Article

Hermite-Hadamard Type Inequalities Associated with Coordinated $((s, m), \text{QC})$ -Convex Functions

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In this paper, we introduce the definition of coordinated $((s, m), \text{QC})$ -convex function and establish some Hermite-Hadamard type integral inequalities for coordinated $((s, m), \text{QC})$ -convex functions.

1. Introduction

Let f be a convex function on $[a, b] \subset \mathbb{R}$. The following integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1)$$

is known in the literature as Hermite-Hadamard inequality.

The investigation on extended convex functions has become a hot research topic in recent years. The applications of various properties of extended convex functions in establishing and improving Hermite-Hadamard type inequalities especially have attracted the attention of many researchers. For convenience of our discussions in subsequent sections, let us introduce some relevant definitions and earlier results below.

We first recall the notion of quasi-convex functions which generalizes the notion of convex functions as follows.

Definition 1 (see [1, 2]). A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be quasi-convex (QC) on I , if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\} \quad (2)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

It is obvious that any convex function is a quasi-convex function but the reverse is not true. Because there exist quasi-convex functions which are not convex (see [3]).

Ion [3] and Alomari et al. [4] established the following Hermite-Hadamard type inequalities for quasi-convex functions, respectively.

Theorem 2. Let $f : I^o \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I^o , $a, b \in I^o$ with $a < b$. If $|f'|$ is quasi-convex function on $[a, b]$, then the following inequalities hold:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \\ & \quad \cdot \max\{|f'(a)|, |f'(b)|\}, \\ & \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{8} \left[\max\left\{ \left| f'\left(\frac{a+b}{2}\right) \right|, |f'(b)| \right\} \right. \\ & \quad \left. + \max\left\{ \left| f'\left(\frac{a+b}{2}\right) \right|, |f'(a)| \right\} \right]. \end{aligned} \quad (3)$$

Alomari et al. [5] presented the following Hermite-Hadamard type inequality for twice differentiable quasi-convex functions.

Theorem 3. Let $f: I^o \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I^o , $a, b \in I^o$ with $a < b$. If $|f''|$ is quasi-convex function on $[a, b]$, then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{12} \max \{ |f''(a)|, |f''(b)| \}. \quad (4)$$

In [6], Wu et al. proved the following Hermite-Hadamard type inequality for thrice differentiable quasi-convex functions.

Theorem 4. Let $f: I^o \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I^o , $a, b \in I^o$ with $a < b$. If $|f'''|^q$ is quasi-convex function on $[a, b]$, then for $q \geq 1$ one has

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \frac{b-a}{12} (f'(b) - f'(a)) \right| \leq \frac{(b-a)^3}{192} (\max \{ |f'''(a)|^q, |f'''(b)|^q \})^{1/q}. \quad (5)$$

In [1], the authors introduced the class of real functions of JQC type as follows.

Definition 5 (see [1]). A function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is Jensen- or J-quasi-convex (JQC) if

$$f\left(\frac{x+y}{2}\right) \leq \max \{f(x), f(y)\} \quad (6)$$

holds for all $x, y \in I$.

In [7], the m -convex function was defined as follows.

Definition 6 (see [7]). For $f: [0, b] \subseteq \mathbb{R}_0 = [0, +\infty) \rightarrow \mathbb{R}$ and $m \in (0, 1]$, if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \quad (7)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m -convex function on $[0, b]$.

In [8] the concept of s -convex functions was presented as follows.

Definition 7 (see [8]). Let $s \in (0, 1]$. A function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be s -convex (in the second sense) if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \quad (8)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

In [9] the concept of extended s -convex functions was introduced as follows.

Definition 8 (see [9]). For some $s \in [-1, 1]$, a function $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be extended s -convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \quad (9)$$

is valid for all $x, y \in I$ and $\lambda \in (0, 1)$.

Dragomir [10] and Dragomir and Pearce [11] considered the convexity of a function on the coordinates and put forward the following definition.

Definition 9 (see [10, 11]). A function $f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be convex on the coordinates on Δ with $a < b$ and $c < d$ if the partial functions

$$\begin{aligned} f_y: [a, b] &\rightarrow \mathbb{R}, \\ f_y(u) &= f_y(u, y), \\ f_x: [c, d] &\rightarrow \mathbb{R}, \\ f_x(v) &= f_x(x, v) \end{aligned} \quad (10)$$

are convex when defined for all $x \in (a, b)$, $y \in (c, d)$.

It should be noted that a formal definition for coordinated convex functions is stated as follows.

Definition 10. A function $f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be convex on the coordinates on Δ with $a < b$ and $c < d$ if the following inequality

$$\begin{aligned} &f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ &\leq t\lambda f(x, y) + t(1-\lambda)f(x, w) + (1-t)\lambda f(z, y) \\ &\quad + (1-t)(1-\lambda)f(z, w) \end{aligned} \quad (11)$$

holds for all $t, \lambda \in [0, 1]$, $(x, y), (z, w) \in \Delta$.

In [12] the concept of coordinated (s, QC) -convex functions was presented as follows.

Definition 11 (see [12]). For some $s \in [-1, 1]$, a function $f: \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is called coordinated (s, QC) -convex on Δ with $a < b$ and $c < d$, if the following inequality

$$\begin{aligned} &f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ &\leq t^s \max \{f(x, y), f(x, w)\} \\ &\quad + (1-t)^s \max \{f(z, y), f(z, w)\} \end{aligned} \quad (12)$$

is valid for all $t \in (0, 1)$, $\lambda \in [0, 1]$, and $(x, y), (z, w) \in \Delta$.

Remark 12. In (12), if $\lambda = 1/2$ and $t = \lambda = 1/2$, then $(Js, \text{JQC}) \subseteq (s, \text{JQC}) \subseteq (s, \text{QC})$.

In [13], the following Hermite-Hadamard type inequality for m -convex functions was proved.

Theorem 13 (see [13]). Let $f : \mathbb{R}_0 = [0, +\infty) \rightarrow \mathbb{R}$ be m -convex and $m \in (0, 1]$. If $f \in L_1([a, b])$ for $0 \leq a < b < \infty$, then

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx \\ & \leq \min \left\{ \frac{f(a) + mf(b/m)}{2}, \frac{mf(a/m) + f(b)}{2} \right\}. \end{aligned} \quad (13)$$

Dragomir [10] and Dragomir and Pearce [11] established the following inequality for coordinated convex functions.

Theorem 14 (see [10, 11]). Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be convex on the coordinates on Δ with $a < b$ and $c < d$. Then, one has the inequalities

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx \right. \\ & \quad \left. + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \leq \frac{1}{(b-a)(d-c)} \\ & \quad \cdot \int_a^b \int_c^d f(x, y) dy dx \\ & \leq \frac{1}{4} \left[\frac{1}{b-a} \left(\int_a^b f(x, c) dx + \int_a^b f(x, d) dx \right) \right. \\ & \quad \left. + \frac{1}{d-c} \left(\int_c^d f(a, y) dy + \int_c^d f(b, y) dy \right) \right] \\ & \leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}. \end{aligned} \quad (14)$$

Besides the results mentioned above, the inequalities of Hermite-Hadamard type involving harmonically s -convex functions, extended s -convex functions, (α, m) -geometrically convex functions, and so on can be found in [14–17]. It deserves to be noticed that, in a recent paper [18], Krnić et al. use some operator techniques with interpolation of convex functions to prove the Hermite-Hadamard type inequalities, which provides a new approach to study this type of inequalities.

In this paper, we will establish some integral inequalities of Hermite-Hadamard type for the coordinated $((s, m), \text{QC})$ -convex functions.

2. Definitions and Lemmas

Firstly, we introduce the definition of coordinated $((s, m), \text{QC})$ -convex function on the rectangle of $\mathbb{R}_0 \times \mathbb{R}$ ($\mathbb{R}_0 = [0, +\infty)$).

Definition 15. For some $s \in [-1, 1]$ and $m \in (0, 1]$, a function $f : \Delta = [0, b] \times [c, d] \subseteq \mathbb{R}_0 \times \mathbb{R} \rightarrow \mathbb{R}$ is called coordinated $((s, m), \text{QC})$ -convex on Δ with $0 < b$ and $c < d$, if the following inequality

$$\begin{aligned} & f(tx + m(1-t)z, \lambda y + (1-\lambda)w) \\ & \leq t^s \max\{f(x, y), f(x, w)\} \\ & \quad + m(1-t)^s \max\{f(z, y), f(z, w)\} \end{aligned} \quad (15)$$

is valid for all $t \in (0, 1)$, $\lambda \in [0, 1]$, and $(x, y), (z, w) \in \Delta$.

To establish new Hermite-Hadamard type inequalities for coordinated $((s, m), \text{QC})$ -convex functions, we need the following lemma.

Lemma 16. Let $A, B, C, D \geq 0$, $m \in (0, 1]$, and $s \in (-1, 1]$; then

$$\begin{aligned} F(s, m; A, B, C, D) & \triangleq \int_0^1 \max\{t^s A + m(1-t)^s B, t^s C + m(1-t)^s D\} dt \\ & = \begin{cases} \frac{\max\{A + mB, C + mD\}}{s+1}, & (A-C)(B-D) \geq 0, \text{ or } s = 0, \\ \frac{(1-\mu^{s+1})A + m(1-\mu)^{s+1}B + \mu^{s+1}C + m(1-(1-\mu)^{s+1})D}{s+1}, & A > C, B < D, 0 < s \leq 1, \\ \frac{\mu^{s+1}A + m(1-(1-\mu)^{s+1})B + (1-\mu^{s+1})C + m(1-\mu)^{s+1}D}{s+1}, & A > C, B < D, -1 < s < 0, \\ \frac{\varphi^{s+1}A + m(1-(1-\varphi)^{s+1})B + (1-\varphi^{s+1})C + m(1-\varphi)^{s+1}D}{s+1}, & A < C, B > D, 0 < s \leq 1, \\ \frac{(1-\varphi^{s+1})A + m(1-\varphi)^{s+1}B + \varphi^{s+1}C + m(1-(1-\varphi)^{s+1})D}{s+1}, & A < C, B > D, -1 < s < 0, \end{cases} \end{aligned} \quad (16)$$

where

$$\mu = \frac{[m(D-B)]^{1/s}}{(A-C)^{1/s} + [m(D-B)]^{1/s}} \quad \text{for } A > C, B < D, \quad (17)$$

$$\varphi = \frac{[m(B-D)]^{1/s}}{(C-A)^{1/s} + [m(B-D)]^{1/s}} \quad \text{for } A < C, B > D.$$

Proof. For $(A-C)(B-D) \geq 0$, or $s = 0$, then

$$F(s, m; A, B, C, D) = \frac{1}{s+1} \max\{A + mB, C + mD\}. \quad (18)$$

(1) If $A > C, B < D$ and $0 < s \leq 1$, we have

$$\begin{aligned} F(s, m; A, B, C, D) &= \int_{\mu}^1 [t^s A + m(1-t)^s B] dt \\ &+ \int_0^{\mu} [t^s C + m(1-t)^s D] dt \\ &= \frac{1}{s+1} \left[(1-\mu^{s+1}) A + m(1-\mu)^{s+1} B + \mu^{s+1} C \right. \\ &\left. + m(1-(1-\mu)^{s+1}) D \right]. \end{aligned} \quad (19)$$

(2) If $A > C, B < D$ and $-1 < s < 0$, we have

$$\begin{aligned} F(s, m; A, B, C, D) &= \int_0^{\mu} [t^s A + m(1-t)^s B] dt \\ &+ \int_{\mu}^1 [t^s C + m(1-t)^s D] dt = \frac{1}{s+1} \left[\mu^{s+1} A \right. \\ &+ m(1-(1-\mu)^{s+1}) B + (1-\mu^{s+1}) C \\ &\left. + m(1-\mu)^{s+1} D \right]. \end{aligned} \quad (20)$$

(3) If $A < C, B > D$ and $0 < s \leq 1$, we have

$$\begin{aligned} F(s, m; A, B, C, D) &= \int_0^{\varphi} [t^s A + m(1-t)^s B] dt \\ &+ \int_{\varphi}^1 [t^s C + m(1-t)^s D] dt = \frac{1}{s+1} \left[\varphi^{s+1} A \right. \\ &+ m(1-(1-\varphi)^{s+1}) B + (1-\varphi^{s+1}) C \\ &\left. + m(1-\varphi)^{s+1} D \right]. \end{aligned} \quad (21)$$

(4) If $A < C, B > D$ and $-1 < s < 0$, we have

$$\begin{aligned} F(s, m; A, B, C, D) &= \int_{\varphi}^1 [t^s A + m(1-t)^s B] dt \\ &+ \int_0^{\varphi} [t^s C + m(1-t)^s D] dt \end{aligned}$$

$$\begin{aligned} &= \frac{1}{s+1} \left[(1-\varphi^{s+1}) A + m(1-\varphi)^{s+1} B + \varphi^{s+1} C \right. \\ &\left. + m(1-(1-\varphi)^{s+1}) D \right]. \end{aligned} \quad (22)$$

Lemma 16 is thus proved. $\square$

3. Some Integral Inequalities of Hermite-Hadamard Type

In this section, we establish some integral inequalities of Hermite-Hadamard type for coordinated $((s, m), \text{QC})$ -convex functions on the plane $\mathbb{R}_0 \times \mathbb{R}$ ($\mathbb{R}_0 = [0, +\infty)$).

Theorem 17. Let $m \in (0, 1]$ and $s \in [-1, 1]$, and let the function $f : \mathbb{R}_0 \times \mathbb{R} \rightarrow \mathbb{R}_0$ be nonnegative. Suppose that f is coordinated $((s, m), \text{QC})$ -convex on $[0, b^*/m^2] \times [c, d]$ with $0 \leq a < b \leq b^*, c < d$. If $f \in L_1([a, b] \times [c, d])$, then

$$\begin{aligned} \frac{1}{2^{3-2s}} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2^{3-s}} \left\{ \frac{1}{2(b-a)} \right. \\ &\cdot \int_a^b \left[f\left(x, \frac{c+d}{2}\right) + mf\left(\frac{x}{m}, \frac{c+d}{2}\right) \right] dx + \frac{1}{2(d-c)} \\ &\cdot \int_c^d \left[f\left(\frac{a+b}{2}, y\right) + mf\left(\frac{a+b}{2m}, y\right) \right] dy \Big\} \\ &\leq \frac{1}{(b-a)(d-c)} \\ &\cdot \int_c^d \int_a^b \frac{f(x, y) + 2mf(x/m, y) + m^2 f(x/m^2, y)}{4} dx dy. \end{aligned} \quad (23)$$

Proof. From the coordinated $((s, m), \text{QC})$ -convexity of f , we obtain

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &= f\left(\frac{ta + (1-t)b + (1-t)a + tb}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{1}{2^s} \left[f\left(ta + (1-t)b, \frac{c+d}{2}\right) \right. \\ &\left. + mf\left(\frac{(1-t)a + tb}{m}, \frac{c+d}{2}\right) \right] \end{aligned} \quad (24)$$

for all $t \in (0, 1)$.

Integrating with respect to t over $[0, 1]$ and putting $x = ta + (1-t)b$ for $t \in (0, 1)$, we have

$$\begin{aligned} \frac{1}{2^{3-2s}} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2^{3-s}} \\ &\cdot \int_0^1 \left[f\left(ta + (1-t)b, \frac{c+d}{2}\right) \right. \\ &\left. + mf\left(\frac{(1-t)a + tb}{m}, \frac{c+d}{2}\right) \right] dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^{3-s}(b-a)} \int_a^b \left[f\left(x, \frac{c+d}{2}\right) \right. \\
&\quad \left. + mf\left(\frac{x}{m}, \frac{c+d}{2}\right) \right] dx.
\end{aligned} \tag{25}$$

Since $\max\{u, v\} \leq u + v$ for $u, v \geq 0$, by $(c+d)/2 = (\lambda c + (1-\lambda)d)/2 + ((1-\lambda)c + \lambda d)/2$ for $\lambda \in [0, 1]$ and the coordinated $((s, m), \text{QC})$ -convexity of f , we deduce that

$$\begin{aligned}
&\frac{1}{2^{3-s}(b-a)} \int_a^b \left[f\left(x, \frac{c+d}{2}\right) + mf\left(\frac{x}{m}, \frac{c+d}{2}\right) \right] dx \leq \frac{1}{8(b-a)} \\
&\cdot \int_0^1 \int_a^b \left[\max\{f(x, \lambda c + (1-\lambda)d), f(x, (1-\lambda)c + \lambda d)\} \right. \\
&\quad \left. + 2m \max\left\{f\left(\frac{x}{m}, \lambda c + (1-\lambda)d\right), f\left(\frac{x}{m}, (1-\lambda)c + \lambda d\right)\right\} \right. \\
&\quad \left. + m^2 \max\left\{f\left(\frac{x}{m^2}, \lambda c + (1-\lambda)d\right), f\left(\frac{x}{m^2}, (1-\lambda)c + \lambda d\right)\right\} \right] dx d\lambda \leq \frac{1}{8(b-a)} \\
&\cdot \int_0^1 \int_a^b \left\{ f(x, \lambda c + (1-\lambda)d) + f(x, (1-\lambda)c + \lambda d) + 2m \left[f\left(\frac{x}{m}, \lambda c + (1-\lambda)d\right) + f\left(\frac{x}{m}, (1-\lambda)c + \lambda d\right) \right] \right. \\
&\quad \left. + m^2 \left[f\left(\frac{x}{m^2}, \lambda c + (1-\lambda)d\right) + f\left(\frac{x}{m^2}, (1-\lambda)c + \lambda d\right) \right] \right\} dx d\lambda.
\end{aligned} \tag{26}$$

Now we perform the change of variable $y = \lambda c + (1-\lambda)d$ in the above integrals; we obtain

$$\begin{aligned}
&\frac{1}{2^{3-s}(b-a)} \int_a^b \left[f\left(x, \frac{c+d}{2}\right) + mf\left(\frac{x}{m}, \frac{c+d}{2}\right) \right] dx \\
&\leq \frac{1}{4(b-a)(d-c)} \int_c^d \int_a^b \left[f(x, y) + 2mf\left(\frac{x}{m}, y\right) \right. \\
&\quad \left. + m^2 f\left(\frac{x}{m^2}, y\right) \right] dx dy.
\end{aligned} \tag{27}$$

From inequalities (25) and (27), it follows that

$$\begin{aligned}
&\frac{1}{2^{3-2s}} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2^{3-s}(b-a)} \\
&\cdot \int_a^b \left[f\left(x, \frac{c+d}{2}\right) + mf\left(\frac{x}{m}, \frac{c+d}{2}\right) \right] dx \\
&\leq \frac{1}{(b-a)(d-c)} \\
&\cdot \int_c^d \int_a^b \frac{f(x, y) + 2mf(x/m, y) + m^2 f(x/m^2, y)}{4} dx dy.
\end{aligned} \tag{28}$$

Similarly, then one obtains

$$\begin{aligned}
&\frac{1}{2^{3-2s}} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2^{3-s}(d-c)} \\
&\cdot \int_c^d \left[f\left(\frac{a+b}{2}, y\right) + mf\left(\frac{a+b}{m}, y\right) \right] dy
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{(b-a)(d-c)} \\
&\cdot \int_c^d \int_a^b \frac{f(x, y) + 2mf(x/m, y) + m^2 f(x/m^2, y)}{4} dx dy.
\end{aligned} \tag{29}$$

Using inequalities (28) and (29), we get (23). Theorem 17 is proved. $\square$

Corollary 18. Under the conditions of Theorem 17, if $m = 1$, then

$$\begin{aligned}
&\frac{1}{2^{3-2s}} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2^{3-s}} \left[\frac{1}{b-a} \right. \\
&\cdot \int_a^b f\left(x, \frac{c+d}{2}\right) + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \Big] \\
&\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy.
\end{aligned} \tag{30}$$

Theorem 19. Suppose that $f: \mathbb{R}_0 \times \mathbb{R} \rightarrow \mathbb{R}_0$ is a coordinated $((s, m), \text{QC})$ -convex on $[0, b^*/m^2] \times [c, d]$ with $0 \leq a < b \leq b^*$, $c < d$, and for some $m \in (0, 1]$ and $s \in (-1, 1]$. If $f \in L_1([a, b] \times [c, d])$, then

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \leq \frac{1}{2} \left\{ \frac{1}{2^s(b-a)} \right. \\
&\cdot \int_a^b \left[\max\{f(x, c), f(x, d)\} + m \right. \\
&\quad \left. \cdot \max\left\{f\left(\frac{x}{m}, c\right), f\left(\frac{x}{m}, d\right)\right\} \right] dx
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(s+1)(d-c)} \int_c^d \left[f(a, y) + mf\left(\frac{b}{m}, y\right) \right] dy \Big\} \\
& \leq \frac{1}{2^s(s+1)} \min \left\{ F\left(s, m; f(a, c), f\left(\frac{b}{m}, c\right), \right. \right. \\
& \quad \left. \left. f(a, d), f\left(\frac{b}{m}, d\right)\right) + mF\left(s, m; f\left(\frac{a}{m}, c\right), \right. \right. \\
& \quad \left. \left. f\left(\frac{b}{m^2}, c\right), f\left(\frac{a}{m}, d\right), f\left(\frac{b}{m^2}, d\right)\right), \right. \\
& \quad \max\{f(a, c), f(a, d)\} + m \\
& \quad \cdot \max\left\{f\left(\frac{a}{m}, c\right), f\left(\frac{a}{m}, d\right)\right\} + m \\
& \quad \cdot \max\left\{f\left(\frac{b}{m}, c\right), f\left(\frac{b}{m}, d\right)\right\} + m^2 \\
& \quad \cdot \max\left\{f\left(\frac{b}{m^2}, c\right), f\left(\frac{b}{m^2}, d\right)\right\} \Big\}, \tag{31}
\end{aligned}$$

where $F(s, m; A, B, C, D)$ is defined by (16).

Proof. Letting $y = \lambda c + (1 - \lambda)d$ for $0 \leq \lambda \leq 1$, from the coordinated $((s, m), \text{QC})$ -convexity of f , we have

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy = \frac{1}{b-a} \\
& \cdot \int_0^1 \int_a^b f(x, \lambda c + (1 - \lambda)d) dx d\lambda \leq \frac{1}{2^s(b-a)} \\
& \cdot \int_0^1 \int_a^b \left[\max\{f(x, c), f(x, d)\} \right. \\
& \quad \left. + m \max\left\{f\left(\frac{x}{m}, c\right), f\left(\frac{x}{m}, d\right)\right\} \right] dx d\lambda \\
& = \frac{1}{2^s(b-a)} \int_a^b \left[\max\{f(x, c), f(x, d)\} + m \right. \\
& \quad \left. \cdot \max\left\{f\left(\frac{x}{m}, c\right), f\left(\frac{x}{m}, d\right)\right\} \right] dx. \tag{32}
\end{aligned}$$

For all $(x, y_0) \in [a, b] \times \{c, d\}$, by making the change of variable $x = ta + (1 - t)b$ for all $t \in (0, 1)$ and using the coordinated $((s, m), \text{QC})$ -convexity of f , we deduce that

$$\begin{aligned}
& f(x, y_0) \leq t^s f(a, y_0) + m(1 - t)^s f\left(\frac{b}{m}, y_0\right) \\
& f\left(\frac{x}{m}, y_0\right) \leq t^s f\left(\frac{a}{m}, y_0\right) + m(1 - t)^s f\left(\frac{b}{m^2}, y_0\right). \tag{33}
\end{aligned}$$

From (33) and Lemma 16, we obtain

$$\begin{aligned}
& \frac{1}{b-a} \int_a^b \max\{f(x, c), f(x, d)\} dx \\
& \leq \int_0^1 \max\left\{t^s f(a, c) \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + m(1 - t)^s f\left(\frac{b}{m}, c\right), t^s f(a, d) \right. \\
& \left. + m(1 - t)^s f\left(\frac{b}{m}, d\right)\right\} dt = \frac{1}{s+1} F\left(s, m; f(a, \right. \\
& \quad \left. c), f\left(\frac{b}{m}, c\right), f(a, d), f\left(\frac{b}{m}, d\right)\right), \\
& \frac{1}{b-a} \int_a^b \max\left\{f\left(\frac{x}{m}, c\right), mf\left(\frac{x}{m}, d\right)\right\} dx \\
& \leq \int_0^1 \max\left\{t^s f\left(\frac{a}{m}, c\right) \right. \\
& \quad \left. + m(1 - t)^s f\left(\frac{b}{m^2}, c\right), t^s f\left(\frac{a}{m}, d\right) \right. \\
& \quad \left. + m(1 - t)^s f\left(\frac{b}{m^2}, d\right)\right\} dt = \frac{1}{s+1} F\left(s, m; \right. \\
& \quad \left. f\left(\frac{a}{m}, c\right), f\left(\frac{b}{m^2}, c\right), f\left(\frac{a}{m}, d\right), f\left(\frac{b}{m^2}, d\right)\right). \tag{34}
\end{aligned}$$

Using (32) and (34), it follows that

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \leq \frac{1}{2^s(b-a)} \\
& \cdot \int_a^b \left[\max\{f(x, c), f(x, d)\} + m \right. \\
& \quad \left. \cdot \max\left\{f\left(\frac{x}{m}, c\right), f\left(\frac{x}{m}, d\right)\right\} \right] dx \\
& \leq \frac{1}{2^s(s+1)} \left[F\left(s, m; f(a, c), f\left(\frac{b}{m}, c\right), f(a, d), \right. \right. \\
& \quad \left. \left. f\left(\frac{b}{m}, d\right)\right) + mF\left(s, m; f\left(\frac{a}{m}, c\right), f\left(\frac{b}{m^2}, c\right), \right. \right. \\
& \quad \left. \left. f\left(\frac{a}{m}, d\right), f\left(\frac{b}{m^2}, d\right)\right) \right]. \tag{35}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy = \frac{1}{d-c} \\
& \cdot \int_c^d \int_0^1 f(ta + (1 - t)b, y) dt dy \leq \frac{1}{d-c} \\
& \cdot \int_c^d \int_0^1 \left[t^s f(a, y) + m(1 - t)^s f\left(\frac{b}{m}, y\right) \right] dt dy \\
& = \frac{1}{(s+1)(d-c)} \int_c^d \left[f(a, y) + mf\left(\frac{b}{m}, y\right) \right] dy \\
& = \frac{1}{2^s(s+1)} \left[\max\{f(a, c), f(a, d)\} + m \right.
\end{aligned}$$

$$\begin{aligned}
& \cdot \max \left\{ f \left(\frac{a}{m}, c \right), f \left(\frac{a}{m}, d \right) \right\} + m \\
& \cdot \max \left\{ f \left(\frac{b}{m}, c \right), mf \left(\frac{b}{m}, d \right) \right\} + m^2 \\
& \cdot \max \left\{ f \left(\frac{b}{m^2}, c \right), f \left(\frac{b}{m^2}, d \right) \right\} \Big].
\end{aligned} \quad (36)$$

Utilizing inequalities (35) and (36), we obtain the inequality (31) asserted by Theorem 19. $\square$

Corollary 20. Under the conditions of Theorem 19, if $m = 1$, then

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \\
& \leq \frac{1}{2} \left\{ \frac{1}{2^{s-1}(b-a)} \int_a^b \max \{f(x, c), f(x, d)\} dx \right. \\
& \quad \left. + \frac{1}{(s+1)(d-c)} \int_c^d [f(a, y) + f(b, y)] dy \right\} \\
& \leq \frac{1}{2^{s-1}(s+1)} \\
& \cdot \min \{F(s, 1; f(a, c), f(b, c), f(a, d), f(b, d)), \\
& \quad \max \{f(a, c), f(a, d)\} + \max \{f(b, c), f(b, d)\}\}.
\end{aligned} \quad (37)$$

Corollary 21. Under the conditions of Theorems 17 to 19, if $s = m = 1$, then

$$\begin{aligned}
& \frac{1}{2} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{4} \left[\frac{1}{b-a} \int_a^b f \left(x, \frac{c+d}{2} \right) \right. \\
& \quad \left. + \frac{1}{d-c} \int_c^d f \left(\frac{a+b}{2}, y \right) dy \right] \leq \frac{1}{(b-a)(d-c)} \\
& \cdot \int_c^d \int_a^b f(x, y) dx dy \leq \frac{1}{2} \left\{ \frac{1}{2(b-a)} \right. \\
& \cdot \int_a^b \max \{f(x, c), f(x, d)\} dx + \frac{1}{2(d-c)} \\
& \cdot \int_c^d [f(a, y) + f(b, y)] dy \Big\} \leq \frac{1}{2} \\
& \cdot \min \{F(1, 1; f(a, c), f(b, c), f(a, d), f(b, d)), \\
& \quad \max \{f(a, c), f(a, d)\} + \max \{f(b, c), f(b, d)\}\}.
\end{aligned} \quad (38)$$

Theorem 22. Suppose that $f : \mathbb{R}_0 \times \mathbb{R} \rightarrow \mathbb{R}_0$ is a coordinated $((-1, m), \text{QC})$ -convex on $[0, b^*/m] \times [c, d]$ with

$0 \leq a < b \leq b^*, c < d$, and for some $m \in (0, 1], r, \ell > 0$. If $f \in L_1([a, b] \times [c, d])$, then

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \frac{(b-x)^r (x-a)^\ell}{(b-a)^{r+\ell}} f(x, y) dx dy \\
& \leq B(r, \ell + 1) \max \{f(a, c), f(a, d)\} \\
& \quad + mB(r + 1, \ell) \max \left\{ f \left(\frac{b}{m}, c \right), f \left(\frac{b}{m}, d \right) \right\},
\end{aligned} \quad (39)$$

where $B(\alpha, \beta)$ is the classical Beta function which is defined by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt, \quad \alpha, \beta > 0. \quad (40)$$

Proof. Letting $x = ta + (1-t)b$ and $y = \lambda c + (1-\lambda)d$ for all $t \in (0, 1)$ and $\lambda \in [0, 1]$, from the coordinated $((-1, m), \text{QC})$ -convexity of f , we have

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \frac{(b-x)^r (x-a)^\ell}{(b-a)^{r+\ell}} f(x, y) dx dy \\
& = \int_0^1 \int_0^1 t^r (1-t)^\ell \\
& \cdot f(ta + (1-t)b, \lambda c + (1-\lambda)d) dt d\lambda \\
& \leq \int_0^1 \left[t^{r-1} (1-t)^\ell \max \{f(a, c), f(a, d)\} \right. \\
& \quad \left. + mt^r (1-t)^{\ell-1} \right. \\
& \cdot \max \left\{ f \left(\frac{b}{m}, c \right), f \left(\frac{b}{m}, d \right) \right\} \Big] dt = B(r, \ell \\
& \quad + 1) \max \{f(a, c), f(a, d)\} + mB(r + 1, \ell) \\
& \cdot \max \left\{ f \left(\frac{b}{m}, c \right), f \left(\frac{b}{m}, d \right) \right\},
\end{aligned} \quad (41)$$

which is the desired inequality (39) stated in Theorem 22. $\square$

Corollary 23. Under the conditions of Theorem 22, if $r = \ell = 1$, then

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x, y) dx dy \\
& \leq \frac{1}{2} \left[\max \{f(a, c), f(a, d)\} \right. \\
& \quad \left. + m \max \left\{ f \left(\frac{b}{m}, c \right), f \left(\frac{b}{m}, d \right) \right\} \right].
\end{aligned} \quad (42)$$

Corollary 24. Under the conditions of Theorem 22, if $r = \ell = m = 1$, then

$$\begin{aligned}
& \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \frac{(b-x)(x-a)}{(b-a)^2} f(x, y) dx dy \\
& \leq \frac{1}{2} \left[\max \{f(a, c), f(a, d)\} \right. \\
& \quad \left. + \max \{f(b, c), f(b, d)\} \right].
\end{aligned} \quad (43)$$

Disclosure

All authors read and approved the final manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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