Research Article

Lipschitz-Type and Bloch-Type Spaces of Pluriharmonic Mappings in a Hilbert Space

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We investigate some properties of pluriharmonic mappings in an infinite dimensional complex Hilbert space. Several characterizations for pluriharmonic mappings to be in Lipschitz-type and Bloch-type spaces are given, which are generalizations of the corresponding known ones for holomorphic functions with several complex variables.

1. Introduction

Let E be a complex Hilbert space of infinite dimension. Given a subdomain Ω of E , a function $h : \Omega \rightarrow \mathbb{C}$ is said to be *holomorphic* if it is Fréchet differentiable at each point $z \in \Omega$ or, equivalently, if $h(z) = \sum_{n=1}^{\infty} P_n(z)$ for all $z \in \Omega$, where P_n is an n -homogeneous polynomial.

A continuous complex-valued function f defined on Ω is said to be *pluriharmonic* if there are two holomorphic functions h and g on Ω such that $f = h + \bar{g}$. We denote the class of all pluriharmonic mappings on Ω by $\mathcal{P}(\Omega)$. Suppose that $(e_k)_{k \in \Gamma}$ is an orthonormal basis of E . Then every $z \in E$ can be written as $z = \sum_{k \in \Gamma} z_k e_k$ and $\bar{z} = \sum_{k \in \Gamma} \bar{z}_k \bar{e}_k$. For a pluriharmonic mapping $f \in \mathcal{P}(\Omega)$, we introduce the notion

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_k}, \dots \right)_{k \in \Gamma}, \\ \bar{\nabla} f &= \left(\frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{z}_k}, \dots \right)_{k \in \Gamma}. \end{aligned} \quad (1)$$

Let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous increasing function with $\omega(0) = 0$. We say that ω is a *majorant* if $\omega(t)/t$ is nonincreasing for $t > 0$. A function $f : \Omega \rightarrow \mathbb{C}$ is said to belong to *Lipschitz space* $\Lambda_\omega(\Omega)$ if there is a positive constant C such that

$$|f(z) - f(w)| \leq C\omega(|z - w|) \quad (2)$$

for all $z, w \in \Omega$ (cf. [1]).

Given a proper subdomain G of E and a majorant ω , we say that G is ω -*extension* if for each pair of points $z, w \in G$ can be joined by a rectifiable curve $\gamma \subset G$ satisfying

$$\int_\gamma \frac{\omega(d_G(z))}{d_G(z)} ds(z) < C\omega(|z - w|) \quad (3)$$

with some fixed positive constant $C = C(G, \omega)$, where ds stands for the arc length measure on γ and $d_G(z)$ denotes the distance from z to the boundary ∂G of G (cf. [2]).

In [1], Dyakonov characterized the holomorphic functions in $\Lambda_\omega(\mathbb{D})$ in terms of their modulus. Later, Pavlović [3] came up with a relatively simple proof of the results of Dyakonov. For the generalizations of this topic, we refer to [4–6]. In this paper, we consider the corresponding problem in the case of $\mathcal{P}(\Omega)$. Our first result is the following theorem which can be viewed as an extension of [6, Theorem 1] to the infinite dimensional setting.

Theorem 1. *Let ω be a majorant and Ω be a simply connected ω -extension subdomain of E . If $f = h + \bar{g} \in \mathcal{P}(\Omega)$, then the following statements are equivalent:*

- (a) $f \in \Lambda_\omega(\Omega)$.
- (b) $h \in \Lambda_\omega(\Omega)$ and $g \in \Lambda_\omega(\Omega)$.
- (c) $|h| \in \Lambda_\omega(\Omega)$ and $|g| \in \Lambda_\omega(\Omega)$.
- (d) $|h| \in \Lambda_\omega(\Omega, \partial\Omega)$ and $|g| \in \Lambda_\omega(\Omega, \partial\Omega)$.

Here $\Lambda_\omega(G, \partial G)$ denotes the class of continuous functions on $G \cup \partial G$ which satisfy (2) with some positive constant C , whenever $z \in G$ and $w \in \partial G$.

Let $\mathbb{B}_E$ be the unit ball of E . For each $f \in \mathcal{P}(\mathbb{B}_E)$, we denote

$$\Lambda_f(z) = \max_{\theta \in \partial \mathbb{B}_E} |\nabla f(z) \cdot \theta + \bar{\nabla} f(z) \cdot \bar{\theta}|. \quad (4)$$

Following [4], the ω - α -Bloch space $\mathcal{B}_\omega^\alpha$ of $\mathcal{P}(\mathbb{B}_E)$ consists of all functions $f \in \mathcal{P}(\mathbb{B}_E)$ such that

$$\|f\|_{\omega, \alpha} = \sup_{z \in \mathbb{B}_E} \omega((1 - |z|^2)^\alpha) \Lambda_f(z) < \infty, \quad (5)$$

and the little ω - α -Bloch space $\mathcal{B}_{\omega, 0}^\alpha$ consists of the functions $f \in \mathcal{P}(\mathbb{B}_E)$ such that

$$\lim_{|z| \rightarrow 1^-} \sup_{z \in \mathbb{B}_E} \omega((1 - |z|^2)^\alpha) \Lambda_f(z) = 0. \quad (6)$$

In particular, when f is holomorphic and $\omega(t) = t$, the space $\mathcal{B}_\omega^\alpha$ is $\mathcal{B}^\alpha$ of $H(\mathbb{B}_E)$ which has been studied in [7].

Let $h : \mathbb{B}_E \rightarrow \mathbb{C}$ be continuous. If there exists a constant $C > 0$ such that

$$(1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2} \frac{|h(z) - h(w)|}{|z - w|} \leq C, \quad (7)$$

for any $z, w \in \mathbb{B}_E$, $z \neq w$, then we say that h satisfies *weighted Lipschitz condition* (cf. [8]).

In the theory of function spaces, the relationship between Bloch spaces and weighted Lipschitz functions has attracted much attention. In 1986, Holland and Walsh established a standard criterion for analytic Bloch space in the unit disc $\mathbb{D}$ in terms of weighted Lipschitz functions. Since then, a series of work has been carried out to characterize Bloch, α -Bloch, little α -Bloch, and Besov spaces of holomorphic and harmonic functions along this line. For instance, Ren and Tu [9] extended Holland and Walsh's criterion to the Bloch space in the unit ball of C^n , Li and Wulan [10] and Zhao [11] characterized holomorphic α -Bloch space in terms of $(1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} |f(z) - f(w)|/|z - w|$. For the related results of harmonic functions, we refer to [4, 12–14] and the references therein.

The second purpose of this paper is to consider the corresponding problems for pluriharmonic mappings in an infinite dimensional complex Hilbert space E . In Section 2, we collect some known results that will be needed in the sequel. Our main results and their proofs are presented in Sections 3 and 4.

Throughout this paper, constants are denoted by C , and they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there exists a positive constant C such that $B/C \leq A \leq CB$.

2. Preliminaries

We need the following preliminary material (see [7, 8] for the details).

For $a \in \mathbb{B}_E$, the involution $\varphi_a : \mathbb{B}_E \rightarrow \mathbb{B}_E$ is defined as

$$\varphi_a(z) = (s_a Q_a + P_a)(m_a(z)), \quad (8)$$

where $s_a = \sqrt{1 - |a|^2}$ and $m_a : \mathbb{B}_E \rightarrow \mathbb{B}_E$ is the analytic map

$$m_a(z) = \frac{a - z}{1 - \langle z, a \rangle}, \quad (9)$$

$P_a : E \rightarrow E$ is the orthogonal projection along the one-dimensional subspace spanned by a , more precisely,

$$P_a(z) = \frac{\langle z, a \rangle}{\langle a, a \rangle} a \quad (10)$$

and Q_a is the orthogonal complement, and $Q_a = Id - P_a$. The automorphisms of the unit ball $\mathbb{B}_E$ turn to be compositions of such analogous involutions with unitary transformations of E .

As in the finite dimensional case, the *pseudohyperbolic* and *hyperbolic* metrics on $\mathbb{B}_E$ are, respectively, defined by

$$\begin{aligned} \rho_E(z, w) &= |\varphi_z(w)| = |\varphi_w(z)|, \\ \beta_E(z, w) &= \frac{1}{2} \ln \frac{1 + \rho_E(z, w)}{1 - \rho_E(z, w)}, \end{aligned} \quad (11)$$

$z, w \in \mathbb{B}_E$.

It is known (see [7]) that

$$|\varphi_z(w)|^2 = 1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w \rangle|^2} \quad (12)$$

$$= \frac{-2\Re \langle z, w \rangle + |\langle z, w \rangle|^2 + |z|^2 + |w|^2 - |z|^2 |w|^2}{|1 - \langle z, w \rangle|^2} \quad (13)$$

$$\leq \frac{|z - w|^2}{|1 - \langle z, w \rangle|^2}. \quad (14)$$

For each $a \in \mathbb{B}_E$ and $r \in (0, 1)$, we define the *pseudohyperbolic ball* with center a and radius r as

$$E(a, r) = \{z \in \mathbb{B}_E : \rho_E(a, z) < r\}. \quad (15)$$

A simple computation gives that $E(a, r)$ is a Euclidean ball with center and radius given by

$$\begin{aligned} &\frac{1 - r^2}{1 - r^2 |z|^2} z, \\ &\frac{1 - |z|^2}{1 - r^2 |z|^2} r. \end{aligned} \quad (16)$$

The following lemma will be needed in the sequel. See [15] for the analogue of this result in several complex variables.

Lemma 2. *Let $r \in (0, 1)$ and $w \in E(z, r)$. Then $1 - |z|^2 \asymp 1 - |w|^2 \asymp |1 - \langle z, w \rangle|$.*

Proof. From (15), we have $|\varphi_z(w)| < r$. It follows from (12) that

$$\begin{aligned} \frac{1-r^2}{4} &< \frac{(1-|z|^2)(1-|w|^2)}{4|1-\langle z, w \rangle|^2} \\ &\leq \frac{(1-|z|^2)(1-|w|^2)}{4(1-|z|)^2} \leq \frac{(1-|w|^2)}{(1-|z|^2)}. \end{aligned} \quad (17)$$

Similarly, we can obtain that $(1-r^2)/4 < (1-|z|^2)/(1-|w|^2)$. Combining these two inequalities with (12), we have

$$1-|z|^2 \asymp 1-|w|^2 \asymp |1-\langle z, w \rangle|. \quad (18)$$

□

The following lemma comes from [4].

Lemma 3. Let $\omega(t)$ be a majorant and $u \in (0, 1]$ and $v \in (1, \infty)$. Then, for $t \in (0, \infty)$,

$$\begin{aligned} \omega(ut) &\geq u\omega(t), \\ \omega(vt) &\leq v\omega(t). \end{aligned} \quad (19)$$

A combination of Lemmas 2 and 3 yields the following.

Lemma 4. Let $r \in (0, 1)$ and $u, v \in E(z, r)$. Then $\omega(1-|u|^2) \asymp \omega(1-|v|^2)$.

3. Lipschitz Spaces

We begin this section with some lemmas which will be used in the proof of Theorem 1.

3.1. Several Lemmas

Lemma 5. Let f be a real pluriharmonic function of $\mathbb{B}_E$ with $|f| < 1$. Then, for each $\theta \in \partial\mathbb{B}_E$,

$$|\nabla f(z) \cdot \theta| \leq \frac{4}{\pi} \frac{1-|f(z)|^2}{1-|z|^2}. \quad (20)$$

Proof. For a fixed $\theta \in \partial\mathbb{B}_E$, let $F(\zeta) = f(\theta\zeta)$ in the unit disc $\mathbb{D}$. Then F is a harmonic function on $\mathbb{D}$ with $|F(\zeta)| < 1$. It follows from [16] that

$$|F_\zeta| \leq \frac{4}{\pi} \frac{1-|F(\zeta)|^2}{1-|\zeta|^2}, \quad (21)$$

which implies that

$$|\nabla f(z) \cdot \theta| \leq \frac{4}{\pi} \frac{1-|f(z)|^2}{1-|z|^2}, \quad (22)$$

where $z = \theta\zeta$. This completes the proof. □

Lemma 6. Let ω be a majorant and Ω be a ω -extension domain of E . If $g = u + iv$ be a holomorphic function on Ω and $u \in \Lambda_\omega(\Omega)$, then $g \in \Lambda_\omega(\Omega)$.

Proof. Fixing a point $z \in \Omega$ and considering the function U , defined on $\mathbb{B}_E$ by

$$U(w) = \frac{u(z + d(z)w)}{M_z}, \quad w \in \mathbb{B}_E, \quad (23)$$

here

$$\begin{aligned} d(z) &= d_\Omega(z), \\ M_z &= \sup \{|u(\zeta)| : |\zeta - z| < d(z)\}. \end{aligned} \quad (24)$$

Since U is pluriharmonic in $\mathbb{B}_E$ and $|U(w)| < 1$, by Lemma 5, we have that, for each $\theta \in \partial\mathbb{B}_E$,

$$|\nabla U(0) \cdot \theta| \leq \frac{4}{\pi} (1-|U(0)|^2) \leq \frac{8}{\pi} (1-|U(0)|), \quad (25)$$

which in turn gives

$$d(z) |\nabla u(z) \cdot \theta| \leq \frac{8}{\pi} (M_z - |u(z)|). \quad (26)$$

Hence

$$d(z) |\nabla g(z) \cdot \theta| \leq \frac{16}{\pi} (M_z - |u(z)|). \quad (27)$$

By the assumption $u \in \Lambda_\omega(\Omega)$, we have, for each $\zeta \in \mathbb{B}(z, d(z))$,

$$|u(\zeta)| - |u(z)| \leq |u(\zeta) - u(z)| \leq C\omega(d(z)), \quad (28)$$

which implies that

$$M_z - |u(z)| \leq C\omega(d(z)). \quad (29)$$

Thus, for any $\theta \in \partial\mathbb{B}_E$, we have

$$d(z) |\nabla g(z) \cdot \theta| \leq C\omega(d(z)), \quad z \in \Omega. \quad (30)$$

For a pair of points $z_1, z_2 \in \Omega$, we let γ be a rectifiable curve which joins z_1 and z_2 satisfying (2). Integrating (30) along γ leads to

$$|g(z_1) - g(z_2)| \leq C \int_\gamma \frac{\omega(d(z))}{d(z)} ds(z). \quad (31)$$

Combining (31) with (3), we have

$$|g(z_1) - g(z_2)| \leq C\omega(|z_1 - z_2|), \quad (32)$$

which completes the proof. □

As an application of Lemma 6, we can obtain the following.

Lemma 7. Let ω be a majorant and let $f = h + \bar{g}$ be a pluriharmonic mapping on a simply connected ω -extension domain Ω . Then $f \in \Lambda_\omega(\Omega)$ if and only if both $g, h \in \Lambda_\omega(\Omega)$.

Proof. We only need to prove necessity since the sufficiency is obvious. Let $f = h + \bar{g} = u + iv$, where u, v are real. As $f, \bar{f} \in \Lambda_\omega(\Omega)$ and

$$\begin{aligned} u &= \frac{f + \bar{f}}{2} = \frac{h + g + \overline{h + g}}{2}, \\ v &= \frac{f - \bar{f}}{2i} = \frac{(h - g) - \overline{(h - g)}}{2i}, \end{aligned} \quad (33)$$

by Lemma 6, we see that $g, h \in \Lambda_\omega(\Omega)$. $\square$

The following lemma is an analogue of [5, Theorem 1] for holomorphic functions in an infinite dimensional Hilbert space E . Since the proof is almost the same as the one in [5], we leave it to the readers.

Lemma 8. *Let ω be a majorant and Ω be a simply connected ω -extension domain of E . If h is a holomorphic function on Ω , then the following statements are equivalent:*

- (i) $h \in \Lambda_\omega(\Omega)$.
- (ii) $|h| \in \Lambda_\omega(\Omega)$.
- (iii) $|h| \in \Lambda_\omega(\Omega, \partial\Omega)$,

where $\Lambda_\omega(\Omega, \partial\Omega)$ denotes the class of continuous functions on $\Omega \cup \partial\Omega$ which satisfy (2) with some positive constant C , whenever $z \in \Omega$ and $w \in \partial\Omega$.

3.2. The Proof of Theorem 1

Proof. The proof follows from Lemmas 5–8. $\square$

4. Bloch Spaces

In this section, we show some characterizations of the spaces $\mathcal{B}_\omega^\alpha$ and $\mathcal{B}_{\omega,0}^\alpha$ in terms of $|f(z) - f(w)|/|z - w|$ on the unit ball of $\mathbb{B}_E$. We first extend [4, Theorem 3] to the setting of $\mathcal{P}(\mathbb{B}_E)$ as follows.

Theorem 9. *Let $f \in \mathcal{P}(\mathbb{B}_E)$, $0 \leq \beta < 1$, and $\beta \leq \alpha < 1 + \beta$. Then $f \in \mathcal{B}_\omega^\alpha$ if and only if*

$$\begin{aligned} L &= \sup_{z, w \in \mathbb{B}_E, z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \\ &\quad \cdot \frac{|f(z) - f(w)|}{|z - w|} < \infty. \end{aligned} \quad (34)$$

Proof. For a fixed point z in (34), we can find $w \in \mathbb{B}_E$ which satisfies that $w = z + r\theta_0$ and

$$\Lambda_f(z) = |\nabla f(z) \cdot \theta_0 + \bar{\nabla} f(z) \cdot \bar{\theta}_0|, \quad (35)$$

where $r > 0$, and $\theta_0 \in \partial\mathbb{B}_E$. Consequently, we have

$$\begin{aligned} &\omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) |f(z) - f(w)| \\ &= \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \\ &\quad \cdot \left| \int_{\overline{zw}} f_\zeta(\zeta) d\zeta + f_{\bar{\zeta}}(\zeta) d\bar{\zeta} \right|. \end{aligned} \quad (36)$$

By letting $r \rightarrow 0$, we deduce that

$$\omega \left((1 - |z|^2)^\alpha \right) \Lambda_f(z) \leq L. \quad (37)$$

Conversely, we assume that $f \in \mathcal{B}_\omega^\alpha$. For $z, w \in \mathbb{B}_E$,

$$\begin{aligned} |f(z) - f(w)| &= \left| \int_{\overline{zw}} f_\zeta(\zeta) d\zeta + f_{\bar{\zeta}}(\zeta) d\bar{\zeta} \right| \\ &\leq |z - w| \int_0^1 \Lambda_f(w + t(z - w)) dt \\ &\leq C |z - w| \|f\|_{\omega, \alpha} \int_0^1 \frac{dt}{\omega \left((1 - |tz + (1 - t)w|^2)^\alpha \right)}. \end{aligned} \quad (38)$$

Since, for $z, w \in \mathbb{B}_E$ and $t \in [0, 1]$,

$$\begin{aligned} (1 - |tz + (1 - t)w|^2)^\alpha &\geq (1 - |tz + (1 - t)w|)^\alpha \\ &\geq (t(1 - |z|) + (1 - t)(1 - |w|))^\alpha \\ &\geq \left(t \left(\frac{1 - |z|^2}{2} \right) + (1 - t) \left(\frac{1 - |w|^2}{2} \right) \right)^\alpha \\ &\geq \left(\frac{t}{2} \right)^\beta \left(\frac{1 - t}{2} \right)^{\alpha-\beta} (1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta}, \end{aligned} \quad (39)$$

we get

$$\begin{aligned} \frac{|f(z) - f(w)|}{|z - w|} &\leq C \int_0^1 \frac{dt}{\omega \left((1 - |tz + (1 - t)w|^2)^\alpha \right)} \\ &\leq C \int_0^1 \frac{dt}{\omega \left((t/2)^\beta ((1 - t)/2)^{\alpha-\beta} (1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right)} \\ &\leq \frac{C}{\omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right)} \int_0^1 \frac{ds}{t^\beta (1 - t)^{\alpha-\beta}} \\ &\leq \frac{C}{\omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right)}, \end{aligned} \quad (40)$$

where the last integral converges since $\alpha < 1 + \beta$. Thus

$$\omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \frac{|f(z) - f(w)|}{|z - w|} < \infty. \quad (41)$$

This completes the proof of Theorem 9. $\square$

Theorem 10. *Let $f \in \mathcal{P}(\mathbb{B}_E)$, $0 \leq \beta < 1$, and $k\omega\beta \leq \alpha < 1 + \beta$. Then $f \in \mathcal{B}_{\omega,0}^\alpha$ if and only if*

$$\begin{aligned} &\lim_{|z| \rightarrow 1^-} \sup_{w \in \mathbb{B}_E, z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \\ &\quad \cdot \frac{|f(z) - f(w)|}{|z - w|} = 0. \end{aligned} \quad (42)$$

Proof.

Sufficiency. Assume that (42) holds. Then, for any $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\sup_{w \in \mathbb{B}_E, z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \frac{|f(z) - f(w)|}{|z - w|} < \epsilon \quad (43)$$

whenever $\delta < |z| < 1$. It follows by an argument similar to that in the proof of Theorem 9 that we have

$$\begin{aligned} & \omega \left((1 - |z|^2)^\alpha \right) \Lambda_f(z) \\ & < C \sup_{w \in \mathbb{B}_E, z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \\ & \cdot \frac{|f(z) - f(w)|}{|z - w|} \leq C\epsilon, \end{aligned} \quad (44)$$

whenever $\delta < |z| < 1$. Hence

$$\lim_{|z| \rightarrow 1^-} \omega \left((1 - |z|^2)^\alpha \right) \Lambda_f(z) = 0. \quad (45)$$

Necessity. For $t \in (0, 1)$, let $f_t(z) = f(tz)$. By the proof of Theorem 9, we have

$$\begin{aligned} & \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \\ & \cdot \frac{|(f - f_t)(z) - (f - f_t)(w)|}{|z - w|} \leq C \|f - f_t\|_{\omega, \alpha}, \\ & \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \frac{|f_t(z) - f_t(w)|}{|z - w|} \\ & < \frac{\omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right)}{\omega \left((1 - |tz|^2)^\beta (1 - |tw|^2)^{\alpha-\beta} \right)} \\ & \cdot \omega \left((1 - |tz|^2)^\beta (1 - |tw|^2)^{\alpha-\beta} \right) \frac{|f(tz) - f(tw)|}{t|z - w|} \\ & \leq \frac{C \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right)}{\omega \left((1 - |tz|^2)^\beta (1 - |tw|^2)^{\alpha-\beta} \right)} \|f\|_{\omega, \alpha} \end{aligned} \quad (46)$$

for all $z, w \in \mathbb{B}_E$. By the triangle inequality, we have

$$\begin{aligned} & \sup_{z, w \in \mathbb{B}_E, z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \\ & \cdot \frac{|f(z) - f(w)|}{|z - w|} \leq C \|f - f_t\|_{\omega, \alpha} \\ & + \frac{C \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right)}{\omega \left((1 - |tz|^2)^\beta (1 - |tw|^2)^{\alpha-\beta} \right)} \|f\|_{\omega, \alpha}. \end{aligned} \quad (47)$$

In the above inequality, first by letting $|z| \rightarrow 1^-$ and then letting $t \rightarrow 1^-$, we obtain the desired result. $\square$

In the following, by adding the restriction $w \in E(z, r)$, we generalize [10, Theorems 1 and 2] to the following forms.

Theorem 11. Let $r \in (0, 1)$, $f \in \mathcal{P}(\mathbb{B}_E)$, and $0 < \beta \leq \alpha$. Then $f \in \mathcal{B}_\omega^\alpha$ if and only if

$$\begin{aligned} K &= \sup_{w \in E(z, r), z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \\ & \cdot \frac{|f(z) - f(w)|}{|z - w|} < \infty. \end{aligned} \quad (48)$$

Proof. We only need to prove the necessity since the sufficiency easily follows from the proof of Theorem 9. Assume that $f \in \mathcal{B}_\omega^\alpha$. Then for any $w \in E(z, r)$, $z \neq w$, we have

$$\begin{aligned} |f(z) - f(w)| &= \left| \int_{z\bar{w}} f_\zeta(\zeta) d\zeta + f_{\bar{z}}(\zeta) d\bar{\zeta} \right| \\ &\leq C |z - w| \int_0^1 \frac{ds}{\omega \left((1 - |sz + (1-s)w|^2)^\alpha \right)}. \end{aligned} \quad (49)$$

Since, for $s \in [0, 1]$,

$$\begin{aligned} 1 - |sz + (1-s)w|^2 &\geq 1 - |sz + (1-s)w| \\ &\geq s(1 - |z|) + (1-s)(1 - |w|) \\ &\geq s \left(\frac{1 - |z|^2}{2} \right) \\ &\quad + (1-s) \left(\frac{1 - |w|^2}{2} \right) \\ &\geq \frac{1}{2} (1 - |z|^2)^s (1 - |w|^2)^{1-s}, \end{aligned} \quad (50)$$

by Lemma 2, we obtain that

$$\begin{aligned} & \frac{|f(z) - f(w)|}{|z - w|} \\ & \leq C \int_0^1 \frac{ds}{\omega \left((1/2)^\alpha (1 - |z|^2)^{\alpha s} (1 - |w|^2)^{\alpha(1-s)} \right)} \\ & \leq \frac{C}{\omega \left((1 - |z|^2)^\alpha \right)} \\ & \leq \frac{C}{\omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right)}. \end{aligned} \quad (51)$$

Thus,

$$\begin{aligned} & \sup_{w \in E(z, r), z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \\ & \cdot \frac{|f(z) - f(w)|}{|z - w|} < \infty. \end{aligned} \quad (52)$$

This completes the proof of Theorem 11. $\square$

Similarly, we can obtain the following.

Theorem 12. Let $r \in (0, 1)$, $f \in H(\mathbb{B}_E)$, and $0 < \beta \leq \alpha$. Then $f \in \mathcal{B}_{\omega,0}^\alpha$ if and only if

$$\lim_{|z| \rightarrow 1^-} \sup_{w \in E(z,r), z \neq w} \omega \left((1 - |z|^2)^\beta (1 - |w|^2)^{\alpha-\beta} \right) \cdot \frac{|f(z) - f(w)|}{|z - w|} = 0. \quad (53)$$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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