Letter to the Editor

Comment on “On the Frame Properties of Degenerate System of Sines”

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The proof of Theorem 3.1 of the paper “On the Frame Properties of Degenerate System of Sines” (see (Bilalov and Guliyeva, 2012)) published earlier in this journal contains a gap; the reasoning given there to prove this theorem is not enough to state the validity of the mentioned theorem. To overcome this shortage we state the most general fact on the completeness of sine system which implies in particular the validity of this fact. It is shown in this note that the system $\{\omega(t)\varphi_n(t)\}$, where $\{\varphi_n(t)\}$ is an exponential or trigonometric (cosine or sine) systems, becomes complete in the corresponding Lebesgue space $L_p(-\pi, \pi)$ or $L_p(0, \pi)$, respectively, whenever $\{\omega(t)\varphi_n(t)\}$ belongs to the corresponding Lebesgue space for all indices n (under the evident natural condition $\text{mes}\{t : \omega(t) = 0\} = 0$). It is also shown that the same conclusion does not remain valid for, in general, any complete or complete orthonormal system $\{\varphi_n(t)\}$. Besides it, the largest class of functions $\omega(t)$ for which the system $\{\omega(t) \sin nt\}_{n \in \mathbb{N}}$ is complete in $L_p(0, \pi)$ space is determined.

1. Introduction

The basis properties (completeness, minimality, and Schauder basicity) of systems of the form $\{\omega(t)\varphi_n(t)\}$, where $\{\varphi_n(t)\}$ is an exponential or trigonometric (cosine or sine) system, are investigated in several papers (see, e.g., [1–15]). For example, Babenko gave an example $\{|t|^\alpha \cdot e^{int}\}_{n \in \mathbb{Z}}$, where $|\alpha| < 1/2$ and $\alpha \neq 0$, answering in the affirmative a question of Bari [16] on the existence of normalized basis for $L_2(-\pi, \pi)$ that is not a Riesz basis [2]. It is also known that, in general, the system $\{|t|^\alpha \cdot e^{int}\}_{n \in \mathbb{Z}}$ is a basis in $L_p(-\pi, \pi)$, $1 < p < \infty$ space if and only if $-1/p < \alpha < 1/q$, where $1/p + 1/q = 1$ (it is, e.g., a direct consequence of the fact that the weight $|t|^\alpha$ satisfies the Muckenhoupt condition if and only if $-1 < \alpha < p - 1$ (see, e.g., [6])).

Later, basicity properties (completeness, minimality, and Schauder basicity) of systems of more general form $\{\prod_{j=1}^r |t - t_j|^{\alpha_j} e^{int}\}_{n \in \mathbb{Z}}$ and $\{\prod_{j=1}^r |t - t_j|^{\alpha_j} \cos nt\}_{n \in \mathbb{Z}_+}$ and $\{\prod_{j=1}^r |t - t_j|^{\alpha_j} \sin nt\}_{n \in \mathbb{N}}$, where $r \geq 1$, have been investigated in several papers (see, e.g., [1, 3–15]).

The aim of this note is the determination of the largest class of functions $\omega(t)$ for which the system $\{\omega(t)\varphi_n(t)\}$, where

$\{\varphi_n(t)\}$ is an exponential or trigonometric (cosine or sine) system, becomes complete in the corresponding Lebesgue space $L_p(-\pi, \pi)$ or $L_p(0, \pi)$, respectively.

It is easy to see that the relation $\omega(t) \cos nt \in L_p(0, \pi)$, $n \in \mathbb{N}$ or $\omega(t)e^{int} \in L_p(-\pi, \pi)$, $n \in \mathbb{N}$ is possible if and only if $\omega(t) \in L_p(0, \pi)$ or $\omega(t) \in L_p(-\pi, \pi)$, respectively. This observation shows the validity of the following fact.

Proposition 1 (see [17]). *Let $\omega(t)$ be any measurable function on $(0, \pi)$ (on $(-\pi, \pi)$) such that the relations $\omega(t) \cos nt \in L_p(0, \pi)$, $n \in \mathbb{N}$ ($\omega(t)e^{int} \in L_p(-\pi, \pi)$, $n \in \mathbb{Z}$) and $\text{mes}\{t : \omega(t) = 0\} = 0$ hold. Then the system $\{\omega(t) \cos nt\}_{n \in \mathbb{N}}$ ($\{\omega(t)e^{int}\}_{n \in \mathbb{Z}}$) is complete in the space $L_p(0, \pi)$ ($L_p(-\pi, \pi)$).*

Indeed, if a function $f(t) \in L_q(0, \pi)$ ($f(t) \in L_q(-\pi, \pi)$) is orthogonal to the system $\{\omega(t) \cos nt\}_{n \in \mathbb{N}}$ ($\{\omega(t)e^{int}\}_{n \in \mathbb{Z}}$) then $\int_0^\pi f(t)\omega(t) \cos nt \, dt = 0$ for all $n \in \mathbb{N}$ ($\int_{-\pi}^\pi f(t)\omega(t)e^{int} \, dt = 0$ for all $n \in \mathbb{Z}$). By the above observation, we have $f(t)\omega(t) \in L_1(0, \pi)$ ($f(t)\omega(t) \in L_1(-\pi, \pi)$). This fact and the fact that the Fourier coefficients of a summable function with respect to the cosine (with respect to the exponential) system are

uniquely determined and imply that $f(t)\omega(t) = 0$ a.e. Using this equality and the condition $\text{mes}\{t : \omega(t) = 0\} = 0$, we arrive at the equality $f(t) = 0$ a.e., which shows the completeness of the mentioned system.

This proposition shows that the largest set of functions $\omega(t)$ for which the system $\{\omega(t)e^{int}\}_{n \in \mathbb{Z}}$ (or $\{\omega(t)\cos nt\}_{n \in \mathbb{N}}$) is complete in the space $L_p(-\pi, \pi)$ (or $L_p(0, \pi)$) is the set of all functions $\omega(t)$ from $L_p(-\pi, \pi)$ (or $L_p(0, \pi)$), for which $\text{mes}\{t : \omega(t) = 0\} = 0$.

Note that the observation similar to one that is given before the proposition is not valid for the sine system; more precisely, $\omega(t)\sin nt \in L_p(0, \pi)$, $n \in \mathbb{N}$ does not always imply $\omega(t) \in L_p(0, \pi)$. Therefore, the scheme used to prove the above proposition does not work in the case of the system $\{\omega(t)\sin nt\}_{n \in \mathbb{N}}$. This feature of sine system was overlooked in the proof of one of the results of the paper [1]; but the results of this note show that the statement of the mentioned result from [1] is true.

It should be mentioned that the proposition given above and the analogous result for sine system which will be proven in this note show that the system $\{\omega(t)\varphi_n(t)\}$, where $\{\varphi_n(t)\}$ is an exponential or trigonometric (cosine or sine) system, becomes complete in the corresponding Lebesgue space $L_p(-\pi, \pi)$ or $L_p(0, \pi)$, respectively, whenever $\text{mes}\{t : \omega(t) = 0\} = 0$ and $\{\omega(t)\varphi_n(t)\}$ belongs to the corresponding Lebesgue space for all indices n . This conclusion may give rise to the impression that if the function $\omega(t)$ is such that $\text{mes}\{t : \omega(t) = 0\} = 0$ and $\omega(t)\varphi_n(t) \in L_p(a, b)$ for all indices n , where $\{\varphi_n(t)\}$ is any complete system in $L_p(a, b)$, then the system $\{\omega(t)\varphi_n(t)\}$ is also complete in the space $L_p(a, b)$. The arguments given below show that, in general, this is not true.

2. Auxiliary Facts

Lemma 2. Let $\omega(t)$ be any measurable function on $(0, \pi)$. The relation $\omega(t)\sin nt \in L_p(0, \pi)$, $\forall n \in \mathbb{N}$ is possible if and only if $t(t - \pi)\omega(t) \in L_p(0, \pi)$.

Proof.

Necessity. Let the relation $\omega(t)\sin nt \in L_p(0, \pi)$, $n \in \mathbb{N}$ hold. Write the function $\omega(t)\sin t$ in the following form:

$$\omega(t)\sin t = t(t - \pi)\omega(t) \cdot \frac{\sin t}{t(t - \pi)}. \quad (1)$$

Consider an auxiliary function

$$\Phi(t) = \begin{cases} \frac{\sin t}{t(t - \pi)}, & \text{if } t \in (0, \pi); \\ -\frac{1}{\pi}, & \text{if } t = 0, \pi. \end{cases} \quad (2)$$

It is evident that the function $\Phi(t)$ is continuous and never vanishes at the segment $[0, \pi]$. Therefore, there is a positive number m such that $|\Phi(t)| > m$ for all $t \in [0, \pi]$. Using these inequalities in (1), we obtain the following estimation:

$$|\omega(t)\sin t| \geq m \cdot |t(t - \pi)\omega(t)|, \quad \forall t \in [0, \pi]. \quad (3)$$

This estimation implies $t(t - \pi)\omega(t) \in L_p(0, \pi)$ since $\omega(t)\sin t \in L_p(0, \pi)$ by the condition of the Lemma. The necessity part of the lemma is proven.

Sufficiency. Assume that $t(t - \pi)\omega(t) \in L_p(0, \pi)$. Take an arbitrary natural number n . Write the function $\omega(t)\sin nt$ in the following form:

$$\omega(t)\sin nt = t(t - \pi)\omega(t) \cdot \frac{\sin nt}{t(t - \pi)}. \quad (4)$$

Consider an auxiliary function

$$\Phi(t) = \begin{cases} \frac{\sin nt}{t(t - \pi)}, & \text{if } t \in (0, \pi); \\ -\frac{n}{\pi}, & \text{if } t = 0; \\ \frac{n \cdot (-1)^n}{\pi}, & \text{if } t = \pi. \end{cases} \quad (5)$$

It is evident that the function $\Phi(t)$ is continuous on $[0, \pi]$. Therefore, there is a number M such that $|\Phi(t)| < M$ for all $t \in [0, \pi]$. Using these inequalities in (4), we obtain the following estimation:

$$|\omega(t)\sin nt| \leq M \cdot |t(t - \pi)\omega(t)|, \quad \forall t \in [0, \pi]. \quad (6)$$

Since $t(t - \pi)\omega(t) \in L_p(0, \pi)$, the last estimation implies $\omega(t)\sin nt \in L_p(0, \pi)$. The lemma is proven. $\square$

Taking into account that the proof of the necessity part of this lemma relies only on the relation $\omega(t)\sin t \in L_p(0, \pi)$ and using the sufficiency part of the same lemma we obtain the validity of the following.

Lemma 3. Let $\omega(t)$ be any measurable function on $(0, \pi)$. The relation $\omega(t)\sin nt \in L_p(0, \pi)$, $\forall n \in \mathbb{N}$ is possible if and only if $\omega(t)\sin t \in L_p(0, \pi)$.

3. Main Results

The main aim of this note is to prove the following.

Theorem 4. Let $\omega(t)$ be any measurable function on $(0, \pi)$ such that

$$\begin{aligned} (1) \quad & \text{mes}\{t : \omega(t) = 0\} = 0, \\ (2) \quad & \omega(t)\sin nt \in L_p(0, \pi), \quad \forall n \in \mathbb{N}. \end{aligned} \quad (7)$$

Then the system $\{\omega(t)\sin nt\}_{n \in \mathbb{N}}$ is complete in $L_p(0, \pi)$ space.

Proof. Let the function $f(t) \in L_q(0, \pi)$ be orthogonal to the system $\{\omega(t)\sin nt\}_{n \in \mathbb{N}}$:

$$\int_0^\pi f(t)\omega(t)\sin nt \, dt = 0, \quad \forall n \in \mathbb{N}. \quad (8)$$

Then the equalities

$$\int_0^\pi f(t)\omega(t)\sin(n+1)t \, dt = 0, \quad (9)$$

$$\int_0^\pi f(t)\omega(t)\sin(n-1)t \, dt = 0 \quad (10)$$

hold for all $n = 0, 1, 2, \dots$. Subtracting (10) from (9) we obtain

$$\int_0^\pi f(t) \omega(t) \sin t \cos nt \, dt = 0, \quad (11)$$

for all $n = 0, 1, 2, \dots$. Note that $\omega(t) \sin t \in L_p(0, \pi)$ (by the condition of the theorem). Therefore $f(t)\omega(t) \sin t \in L_1(0, \pi)$ and the fact that the Fourier coefficients of a summable function with respect to the cosine system is unique (along with (11)) imply that $f(t)\omega(t) \sin t = 0$ a.e. on $(0, \pi)$. The last relation implies $f(t) = 0$ a.e. on $(0, \pi)$ since $\text{mes}\{t : \omega(t) = 0\} = 0$.

The theorem is proven. $\square$

It should be mentioned that the proposition given above and the analogous result for sine system show that the system $\{\omega(t)\varphi_n(t)\}$, where $\{\varphi_n(t)\}$ is an exponential or trigonometric (cosine or sine) system, becomes complete in the corresponding Lebesgue space $L_p(-\pi, \pi)$ or $L_p(0, \pi)$, respectively, whenever $\text{mes}\{t : \omega(t) = 0\} = 0$ and $\{\omega(t)\varphi_n(t)\}$ belongs to the corresponding Lebesgue space for all indices n . This conclusion may give rise to the impression that if the function $\omega(t)$ is such that $\text{mes}\{t : \omega(t) = 0\} = 0$ and $\omega(t)\varphi_n(t) \in L_p(a, b)$ for all indices n , where $\{\varphi_n(t)\}$ is any complete system in $L_p(a, b)$, then the system $\{\omega(t)\varphi_n(t)\}$ is also complete in the space $L_p(a, b)$. The arguments given below show that, in general, this is not true.

Theorem 5. *There is a complete system $\{\varphi_n(t)\}$ in $L_p(-\pi, \pi)$ space and a measurable function $\omega(t)$ such that $\text{mes}\{t : \omega(t) = 0\} = 0$ and $\omega(t)\varphi_n(t) \in L_p(a, b)$ for all indices n but the system $\{\omega(t)\varphi_n(t)\}$ is not complete in $L_p(-\pi, \pi)$.*

Proof. Define the function $\varphi_n(t)$ as follows:

$$\varphi_n(t) = te^{int} \quad (12)$$

for all $n \in \mathbb{Z} \setminus \{0\}$; in other words, let $\{\varphi_n(t)\}$ be the system $\{te^{int}\}_{n \in \mathbb{Z} \setminus \{0\}}$. Take $\omega(t) \equiv 1/t$. Then $\text{mes}\{t : \omega(t) = 0\} = 0$ and $\{\varphi_n(t)\}$ is complete in $L_2(-\pi, \pi)$ (see, e.g., [9] or [10]). But, in this case, the system $\{\omega(t)\varphi_n(t)\}$ coincides with $\{e^{int}\}_{n \in \mathbb{Z} \setminus \{0\}}$ which, obviously, is not complete in $L_2(-\pi, \pi)$. The theorem is proven. $\square$

Applying the Schmidt's orthogonalization process to the system given in Theorem 5 we obtain immediately the validity of the following fact.

Theorem 6. *There is a complete orthonormal system $\{\varphi_n(t)\}$ in $L_p(-\pi, \pi)$ space and a measurable function $\omega(t)$ such that $\text{mes}\{t : \omega(t) = 0\} = 0$ and $\omega(t)\varphi_n(t) \in L_p(-\pi, \pi)$ for all indices n , but the system $\{\omega(t)\varphi_n(t)\}$ is not complete in $L_p(-\pi, \pi)$.*

Using Lemmas 2 and 3 from the previous section, we obtain the following descriptions of the largest set of functions $\omega(t)$ for which the system $\{\omega(t) \sin nt\}_{n \in \mathbb{N}}$ is complete in $L_p(0, \pi)$.

Theorem 7. *The system $\{\omega(t) \sin nt\}_{n \in \mathbb{N}}$ is complete in $L_p(0, \pi)$ if and only if $t(t - \pi)\omega(t) \in L_p(0, \pi)$ and $\text{mes}\{t : \omega(t) = 0\} = 0$.*

Theorem 8. *The system $\{\omega(t) \sin nt\}_{n \in \mathbb{N}}$ is complete in $L_p(0, \pi)$ if and only if $\omega(t) \sin t \in L_p(0, \pi)$ and $\text{mes}\{t : \omega(t) = 0\} = 0$.*

Competing Interests

The authors state that there are no competing interests related to this paper.

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