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Research Article

Strong Convergence of New Two-Step Viscosity Iterative Approximation Methods for Set-Valued Nonexpansive Mappings in CAT(0) Spaces

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This paper is for the purpose of introducing and studying a class of new two-step viscosity iteration approximation methods for finding fixed points of set-valued nonexpansive mappings in CAT(0) spaces. By means of some properties and characteristic to CAT(0) space and using Cauchy-Schwarz inequality and Xu's inequality, strong convergence theorems of the new two-step viscosity iterative process for set-valued nonexpansive and contraction operators in complete CAT(0) spaces are provided. The results of this paper improve and extend the corresponding main theorems in the literature.

1. Introduction

In [1], the fixed point theory in CAT(0) spaces was first introduced and studied by Kirk. Further, Kirk [1] presented that each nonexpansive (single-valued) mapping on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. On the other hand, fixed point theory for set-valued mappings has been applied to applied sciences, game theory, and optimization theory. This promotes the rapid development of fixed point theory for single-valued (set-valued) operators in CAT(0) spaces, and it is natural and particularly meaningful to extensively study fixed point theory of set-valued operators. Particularly, some old relative works on Ishikawa iterations for multivalued mappings can be found in [2–4]. For more detail, we refer to [5–14] and the references therein.

Definition 1. Let $g : X \to X$ be a nonlinear operator on a metric space (X, d) and $G : E \to BC(X)$ be a set-valued operator, where $E \subset X$ is a nonempty subset and $BC(X)$ is the family of nonempty bounded closed subsets of X. Then

(i) g is said to be a *contraction*, if there exists a constant $\kappa \in [0, 1)$ such that

$$d(g(x), g(y)) \le \kappa d(x, y) \quad \forall x, y \in X. \tag{1}$$

Here, g is called *nonexpansive* when $\kappa = 1$ in (1).

(ii) G is said to be a nonexpansive, if

$$H(G(x), G(y)) \le d(x, y), \tag{2}$$

where $H(\cdot, \cdot)$ is Hausdorff distance on $BC(X)$, i.e.,

$$H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}, \quad \forall A, B \in BC(X). \tag{3}$$

Recently, Shi and Chen [5] first considered the following Moudafi's viscosity iteration for a nonexpansive mapping $g : E \to E$ with $\emptyset \ne Fix(g) = \{x \mid x = g(x)\}$ and a contraction mapping $f : E \to E$ in CAT(0) space X:

$$x_\alpha = \alpha f(x_\alpha) \oplus (1 - \alpha) g(x_\alpha), \tag{4}$$

and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) g(x_n), \quad n \ge 1, \tag{5}$$

where $\alpha, \alpha_n \in (0, 1)$ and x_1 is an any given element in a nonempty closed convex subset $E \subseteq X$. $x_\alpha \in E$ is called unique fixed point of contraction $x \mapsto \alpha f(x) \oplus (1 - \alpha) g(x)$. Shi and Chen [5] showed that $\{x_\alpha\}$ defined by (4) converges

strongly to $\tilde{x} \in Fix(g)$ as $\alpha \to 0^+$, where $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ in CAT(0) space (X, d) satisfies the following property $\mathscr{P}$: for all $x, u, y_1, y_2 \in X$,

$$d(x, m_1) d(x, y_1) \leq d(x, m_2) d(x, y_2) + d(x, u) d(y_1, y_2), \tag{6}$$

that is, an extra condition on the geometry of CAT(0) spaces is requested, where $m_i = P_{[x,y_i]} u$ for $i = 1, 2$. Further, the authors also found that the sequence $\{x_n\}$ generated by (5) converges strongly to $\tilde{x} \in Fix(g)$ under some suitable conditions about $\{\alpha_n\}$. Afterwards, based on the concept of quasilinearization introduced by Berg and Nikolaev [15], Wangkeeree and Preechasilp [6] explored strong convergence results of (4) and (5) in CAT(0) spaces without the property $\mathscr{P}$ and presented that the iterative processes (4) and (5) converges strongly to $\tilde{x} \in Fix(g)$ such that $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ is the unique solution of the following variational inequality:

$$\left\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{y\tilde{x}} \right\rangle \geq 0, \quad \forall y \in Fix(g). \tag{7}$$

In [16], Panyanak and Suantai extended (4) and (5) to T being a set-valued nonexpansive mapping from E to $BC(X)$. That is, for each $\alpha \in (0, 1)$, let a set-valued contraction G_α on E be defined by

$$G_\alpha(x) = \alpha f(x) \oplus (1 - \alpha) Tx, \quad \forall x \in E. \tag{8}$$

By Nadler's [17] theorem, it is easy to know that there exists $x_\alpha \in E$ such that x_α is a fixed point of G_α, which does not have to be unique, and

$$x_\alpha \in \alpha f(x_\alpha) \oplus (1 - \alpha) Tx_\alpha, \tag{9}$$

i.e., for each x_α, there exists $y_\alpha \in Tx_\alpha$ such that

$$x_\alpha = \alpha f(x_\alpha) \oplus (1 - \alpha) y_\alpha. \tag{10}$$

Further, when the contraction constant coefficient of f is $k \in [0, 1/2)$ and $\{\alpha_n\} \subset (0, 1/(2 - k))$ satisfying some suitable conditions, Panyanak and Suantai [16] proved strong convergence of one-step viscosity approximation iteration defined by (10) or the following iterative process in CAT(0) spaces:

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n) y_n, \quad y_n \in M(x_n), \tag{11}$$

and $d(y_n, y_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, where M is a set-valued nonexpansive operator from E to $C(E)$, the family of nonempty compact subsets of E, $f : E \to E$ is a contraction, and $\{\alpha_n\} \subseteq (0, 1)$. Moreover, Chang et al. [7] affirmatively answered the open question proposed by Panyanak and Suantai [16, Question 3.6]: "If $k \in [0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfying the same conditions, does $\{x_n\}$ converge to $\tilde{x} = P_{F(M)} f(\tilde{x})$", where $F(M)$ denotes the set of all fixed points of M.

On the other hand, Piatek [18] introduced and studied the following two-step viscosity iteration in complete CAT(0) spaces with the nice projection property $\mathscr{N}$:

$$\begin{aligned} y_n &= \alpha_n f(x_n) \oplus (1 - \alpha_n) g(x_n), \\ x_{n+1} &= \beta_n x_n \oplus (1 - \beta_n) y_n, \quad \forall n \geq 1, \end{aligned} \tag{12}$$

where $x_1 \in E$ is an given element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$ satisfying some suitable conditions and the contraction coefficient of f is $k \in [0, 1/2)$.

Based on the ideas of Wangkeeree and Preechasilp [6] and Piatek [18] intensively, Kaewkhao et al. [19] omit the nice projection property $\mathscr{N}$. We note that the two-step viscosity iteration (12) is also considered and studied by Chang et al. [8] when the property $\mathscr{N}$ is not satisfied and $k \in [0, 1)$, which is due to the open questions in [19], where the property $\mathscr{N}$ depends on whether its metric projection onto a geodesic segment preserves points on each geodesic segment, that is, for every geodesic segment $\chi \subset X$ and $x, y \in X$, $m \in [x, y]$ implies $P_\chi m \in [P_\chi x, P_\chi y]$, where P_χ denotes the metric projection from X onto χ. For more works on the convergence analysis of (viscosity) iteration approximation method for (split) fixed point problems, one can refer to [20–27].

Motivated and inspired mainly by Panyanak and Suantai [16] and Piatek [18] and so on, we consider the following two-step viscosity iteration for set-valued nonexpansive operator $T : E \to C(E)$:

$$\begin{aligned} x_{n+1} &= \beta_n x_n \oplus (1 - \beta_n) y_n, \\ y_n &= \alpha_n f(x_n) \oplus (1 - \alpha_n) z_n, \quad \forall n \geq 1, \end{aligned} \tag{13}$$

where E is a nonempty closed convex subset of complete CAT(0) space (X, d), $x_1 \in E$ is an given element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$, $f : E \to E$ is a contraction mapping, and $z_n \in T(x_n)$ satisfying $d(z_n, z_{n+1}) \leq d(x_n, x_{n+1})$ for any $n \in \mathbb{N}$.

By using the method due to Chang et al. [7, 8], the purpose of this paper is to prove some strong convergence theorems of the viscosity iteration procedure (13) in complete CAT(0) spaces. Hence, the results of Chang et al. [7, 8] and many others in the literature can be special cases of main results in this paper.

2. Preliminaries

Throughout of this paper, let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map $\xi : \mathbb{R} \supseteq [0, l] \to X$ such that $\xi(0) = x, \xi(l) = y$, and $d(\xi(s), \xi(t)) = |s - t|$ for each $s, t \in [0, l]$. In particular, ξ is a isometry and $d(x, y) = l$. The image of ξ is called a geodesic segment (or metric) joining x and y if unique is bespoke by $[x, y]$. The space (X, d) is called a geodesic space when every two points in X are joined by a geodesic, and X is called uniquely geodesic if there is exactly one geodesic joining x and y for any $x, y \in X$. A subset E of X is said to be convex if E includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(p, q, r)$ in a geodesic space (X, d) consists of three

points p, q, r in X (the vertices of $\triangle$) and a choice of three geodesic segments $[p, q], [q, r], [r, p]$ (the edge of $\triangle$) joining them. A comparison triangle for geodesic triangle $\triangle(p, q, r)$ in X is a triangle $\overline{\triangle}(\overline{p}, \overline{q}, \overline{r})$ in the Euclidean plane $\mathbb{R}^2$ such that

$$\begin{aligned} d_{\mathbb{R}^2}(\overline{p}, \overline{q}) &= d(p, q), \\ d_{\mathbb{R}^2}(\overline{q}, \overline{r}) &= d(q, r), \\ d_{\mathbb{R}^2}(\overline{r}, \overline{p}) &= d(r, p). \end{aligned} \tag{14}$$

A point $\overline{u} \in [\overline{p}, \overline{q}]$ is said to be a comparison point for $u \in [p, q]$ if $d(p, u) = d_{\mathbb{R}^2}(\overline{p}, \overline{u})$. Similarly, we can give the definitions of comparison points on $[\overline{q}, \overline{r}]$ and $[\overline{r}, \overline{p}]$.

Definition 2. Suppose that $\triangle$ is a geodesic triangle in (X, d) and $\overline{\triangle}$ is a comparison triangle for $\triangle$. A geodesic space is said to be a CAT(0) space, if all geodesic triangles of appropriate size satisfy the following comparison axiom (i.e., CAT(0) inequality):

$$d(u, v) \le d_{\mathbb{R}^2}(\overline{u}, \overline{v}), \quad \forall u, v \in \triangle, \overline{u}, \overline{v} \in \overline{\triangle}. \tag{15}$$

Complete CAT(0) spaces are often called Hadamard spaces (see [28]). For other equivalent definitions and basic properties of CAT(0) spaces, one can refer to [29]. It is well known that every CAT(0) space is uniquely geodesic and any complete, simply connected Riemannina manifold having nonpositive sectional curvature is a CAT(0) space. Other examples for CAT(0) spaces include pre-Hilbert spaces [29], $\mathbb{R}$−trees [9], Euclidean buildings [30], and complex Hilbert ball with a hyperbolic metric [31] as special case.

Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d). By Proposition 2.4 of [29], it follows that, for all $x \in X$, there exists a unique point $x_0 \in E$ such that

$$d(x, x_0) = \inf\{d(x, y) : y \in E\}. \tag{16}$$

Here, x_0 is said to be *unique nearest point* of x in E.

Assume that (X, d) is a CAT(0) space. For all $x, y \in X$ and $t \in [0, 1]$, by Lemma 2.1 of Phompongsa and Panyanak [10], there exists a unique point $z \in [x, y]$ such that

$$\begin{aligned} d(x, z) &= (1 - t) d(x, y) \\ \text{and } d(y, z) &= t d(x, y). \end{aligned} \tag{17}$$

We shall denote by $tx \oplus (1 - t)y$ the unique point z satisfying (17). Now, we give some results about CAT(0) spaces for the proof of our main results.

Lemma 3 ([1, 10]). *Let (X, d) be a* CAT(0) *space. Then for each $x, y, z \in X$ and $\alpha \in [0, 1]$,*

(i) $d(\alpha x \oplus (1 - \alpha)y, z) \le \alpha d(x, z) + (1 - \alpha)d(y, z)$.

(ii) $d^2(\alpha x \oplus (1 - \alpha)y, z) \le \alpha d^2(x, z) + (1 - \alpha)d^2(y, z) - \alpha(1 - \alpha)d^2(x, y)$.

(iii) $d(\alpha x \oplus (1 - \alpha)z, \alpha y \oplus (1 - \alpha)z) \le \alpha d(x, y)$.

Lemma 4 ([11]). *Suppose that (X, d) is a* CAT(0) *space. Then for all $x, y \in X$ and $\alpha, \beta \in [0, 1]$,*

$$d(\alpha x \oplus (1 - \alpha) y, \beta x \oplus (1 - \beta) y) \le |\alpha - \beta| d(x, y). \tag{18}$$

Lemma 5 ([12]). *Assume that $\{x_n\}$ and $\{y_n\}$ are two bounded sequences in a* CAT(0) *space (X, d) and $\{\beta_n\}$ is a sequence in $[0, 1]$ with $0 < \liminf_n \beta_n \le \limsup_n \beta_n < 1$. If*

$$\begin{aligned} x_{n+1} &= \beta_n x_n \oplus (1 - \beta_n) y_n, \quad \forall n \in \mathbb{N}, \\ \limsup_{n\to\infty} &(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \le 0, \end{aligned} \tag{19}$$

then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 6 ([32]). *Suppose that nonnegative real numbers sequence $\{u_n\}$ is defined by*

$$u_{n+1} \le (1 - \alpha_n) u_n + \alpha_n \beta_n, \quad \forall n \ge 1, \tag{20}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\beta_n\} \subset \mathbb{R}$ are two sequences satisfying

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\limsup_{n\to\infty} \beta_n \le 0$ or $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n\to\infty}\{u_n\} = 0$.

Lemma 7 ([13]). *Assume that E is a closed convex subset of a complete* CAT(0) *space (X, d). If a set-valued nonexpansive operator $T : E \to BC(X)$ satisfies endpoint condition $\mathbb{C}$, i.e., $F(T) \ne \emptyset$ and $T(x) = \{x\}$ for any $x \in F(T)$ (see [33]), then $F(T)$ is closed and convex.*

In [15], Berg and Nikolaev introduced the concept of quasilinearization. Now we denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$, which is a vector. Define the quasilinearization by a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ as follows:

$$\begin{aligned} &\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \\ &\quad = \frac{1}{2}\left[d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right], \\ &\qquad \forall a, b, c, d \in X. \end{aligned} \tag{21}$$

One can easily know that

$$\begin{aligned} \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle &= \left\langle \overrightarrow{cd}, \overrightarrow{ab} \right\rangle, \\ \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle &= -\left\langle \overrightarrow{ba}, \overrightarrow{cd} \right\rangle, \\ \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle + \left\langle \overrightarrow{ad}, \overrightarrow{bc} \right\rangle &= \left\langle \overrightarrow{ac}, \overrightarrow{bd} \right\rangle, \\ \left\langle \overrightarrow{ax}, \overrightarrow{cd} \right\rangle + \left\langle \overrightarrow{xb}, \overrightarrow{cd} \right\rangle &= \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \end{aligned} \tag{22}$$

for every $a, b, c, d, x \in X$. We say that a geodesic metric space (X, d) satisfies the Cauchy-Schwarz inequality if

$$\left|\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle\right| \le d(a, b) d(c, d), \quad \forall a, b, c, d \in X. \tag{23}$$

From [15, Corollary 3], it is known that a geodesic space (X, d) is a CAT(0) space if and only if X satisfies the Cauchy-Schwarz inequality. Further, we give the following other properties of quasilinearization.

Lemma 8 ([14]). *Assume that E is a nonempty closed convex subset of a complete* CAT(0) *space (X, d). Then for $u \in X$ and $x \in E$,*

$$x = P_E u \quad \text{if and only if} \left\langle \overrightarrow{xu}, \overrightarrow{yx} \right\rangle \geq 0, \ \forall y \in E. \tag{24}$$

Lemma 9 ([6]). *For two points u and v in a* CAT(0) *space (X, d) and any $\alpha \in [0, 1]$, letting $u_\alpha = \alpha u \oplus (1-\alpha)v$, then for all $x, y \in X$, the following results hold:*

(i) $\langle \overrightarrow{u_\alpha x}, \overrightarrow{u_\alpha y} \rangle \leq \alpha \langle \overrightarrow{ux}, \overrightarrow{u_\alpha y} \rangle + (1-\alpha) \langle \overrightarrow{vx}, \overrightarrow{u_\alpha y} \rangle$;

(ii) $\langle \overrightarrow{u_\alpha x}, \overrightarrow{uy} \rangle \leq \alpha \langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-\alpha) \langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ and $\langle \overrightarrow{u_\alpha x}, \overrightarrow{vy} \rangle \leq \alpha \langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-\alpha) \langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Definition 10. A continuous linear functional μ is said to be Banach limit on ℓ_∞ if

$$\begin{gathered} \|\mu\| = \mu(1, 1, \cdots) = 1 \\ \text{and } \mu_n(u_n) = \mu_n(u_{n+1}), \quad \forall \{u_n\} \in \ell_\infty. \end{gathered} \tag{25}$$

Lemma 11 ([34]). *Suppose that, for real number α and all Banach limits μ, $(u_1, u_2, \cdots) \in \ell_\infty$ satisfies*

$$\begin{gathered} \mu_n(u_n) \leq \alpha, \\ \limsup_n (u_{n+1} - u_n) \leq 0. \end{gathered} \tag{26}$$

Then $\limsup_n u_n \leq \alpha$.

Lemma 12 ([16]). *Assume that (X, d) is a complete* CAT(0) *space, $E \subset X$ is a nonempty closed convex subset, $T : E \to C(E)$ is a set-valued nonexpansive operator satisfying endpoint condition $\mathbb{C}$, and $f : E \to E$ is a contraction with $k \in [0, 1)$. Then we have following results:*

(i) $\{x_\alpha\}$ generated by (10) converges strongly to $\tilde{x}$ as $\alpha \to 0^+$, where $\tilde{x} = P_{F(T)} f(\tilde{x})$.

(ii) In addition, if $\{x_n\} \subset E$ is a bounded sequence such that $\lim_{n\to\infty} dist(x_n, T(x_n)) = 0$, where $dist(a, B)$ is the distance from a point $a \in X$ to the set $B \in C(X)$, then for all Banach limits μ,

$$d^2(f(\tilde{x}), \tilde{x}) \leq \mu_n d^2(f(\tilde{x}), x_n). \tag{27}$$

3. Main Results

Employing the preliminaries in the previous section, now we will study the strong convergence of the new two-step viscosity iteration (13) for set-valued nonexpansive operators in complete CAT(0) spaces.

Theorem 13. *Assume that (X, d) is a complete* CAT(0) *space, $E \subset X$ is a nonempty closed convex subset, $T : E \to C(E)$ is a set-valued nonexpansive operator satisfying endpoint condition $\mathbb{C}$, and $f : E \to E$ is contraction with $k \in [0, 1)$. If sequences $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$ satisfy*

(L_1) $\lim_{n\to\infty} \alpha_n = 0$, (L_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (L_3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, then the sequence $\{x_n\}$ generated by (13) converges strongly to $\tilde{x}$, where

$$\begin{gathered} \tilde{x} = P_{F(T)} f(\tilde{x}), \\ \left\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \right\rangle \geq 0, \quad \forall x \in F(T). \end{gathered} \tag{28}$$

Proof. The proof shall be divided into the following four steps.

Step 1. We first prove that sequences $\{x_n\}$, $\{f(x_n)\}$, $\{y_n\}$, and $\{z_n\}$ are bounded. In fact, setting $p \in F(T)$, then from Lemma 3, we know

$$\begin{aligned} d(y_n, p) &\leq \alpha_n d(f(x_n), p) \\ &\quad + (1-\alpha_n) dist(z_n, T(p)) \\ &\leq \alpha_n d(f(x_n), p) \\ &\quad + (1-\alpha_n) H(T(x_n), T(p)) \\ &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(x_n, p) \\ &\leq \alpha_n d(f(x_n), f(p)) + \alpha_n d(f(p), p) \\ &\quad + (1-\alpha_n) d(x_n, p) \\ &\leq [1 - \alpha_n(1-k)] d(x_n, p) \\ &\quad + \alpha_n d(f(p), p), \end{aligned} \tag{29}$$

and

$$\begin{aligned} d(x_{n+1}, p) &\leq \beta_n d(x_n, p) + (1-\beta_n) d(y_n, p) \\ &\leq [1 - \alpha_n(1-k)(1-\beta_n)] d(x_n, p) \\ &\quad + \alpha_n(1-k)(1-\beta_n) \frac{d(f(p), p)}{1-k} \\ &\leq \max\left\{ d(x_n, p), \frac{d(f(p), p)}{1-k} \right\}. \end{aligned} \tag{30}$$

Thus, we obtain

$$d(x_n, p) \leq \max\left\{ d(x_1, p), \frac{d(f(p), p)}{1-k} \right\}. \tag{31}$$

Hence, $\{x_n\}$ is bounded, so is $\{f(x_n)\}$. By (29), it is easy to know that $\{y_n\}$ is bounded. Since $d(z_n, p) \leq H(T(x_n, T(p)) \leq d(x_n, p)$, one can easily know that the sequence $\{z_n\}$ is also bounded.

Step 2. We present that $\lim_{n\to\infty} d(x_n, y_n) = 0$, $\lim_{n\to\infty} dist(x_n, T(x_n)) = 0$, $\lim_{n\to\infty} d(x_n, z_n) = 0$, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$,

$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$, and $\lim_{n\to\infty} dist(z_n, T(z_n)) = 0$. Indeed, by applying Lemmas 3 and 4, we have

$$\begin{aligned} d(y_n, y_{n+1}) &\le d(\alpha_n f(x_n) \\ &\oplus (1-\alpha_n) z_n, \alpha_{n+1} f(x_{n+1}) \oplus (1-\alpha_{n+1}) z_{n+1}) \\ &\le d(\alpha_n f(x_n) \oplus (1-\alpha_n) z_n, \alpha_n f(x_n) \\ &\oplus (1-\alpha_n) z_{n+1}) + d(\alpha_n f(x_n) \\ &\oplus (1-\alpha_n) z_{n+1}, \alpha_n f(x_{n+1}) \oplus (1-\alpha_n) z_{n+1}) \\ &+ d(\alpha_n f(x_{n+1}) \oplus (1-\alpha_n) z_{n+1}, \alpha_{n+1} f(x_{n+1}) \\ &\oplus (1-\alpha_{n+1}) z_{n+1}) \le (1-\alpha_n) d(z_n, z_{n+1}) \\ &+ \alpha_n d(f(x_n), f(x_{n+1})) + |\alpha_n \\ &- \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) \le [1 \\ &- \alpha_n (1-k)] d(x_n, x_{n+1}) + |\alpha_n \\ &- \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}), \end{aligned} \tag{32}$$

and so

$$\begin{aligned} &d(y_n, y_{n+1}) - d(x_n, x_{n+1}) \\ &\quad \le |\alpha_n - \alpha_{n+1}| d(f(x_{n+1}), z_{n+1}) \\ &\qquad - (1-k)\alpha_n d(x_n, x_{n+1}). \end{aligned} \tag{33}$$

From $\lim_{n\to\infty}\alpha_n = 0$ and the boundedness of $\{x_n\}$, $\{f(x_n)\}$, and $\{z_n\}$, we know

$$\limsup_{n\to\infty} [d(y_{n+1}, y_n) - d(x_{n+1}, x_n)] \le 0. \tag{34}$$

It follows from Lemma 5 that

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{35}$$

Thus,

$$\begin{aligned} dist(x_n, T(x_n)) &\le d(x_n, z_n) \\ &\le d(x_n, y_n) + \alpha_n d(f(x_n), z_n) \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{36}$$

By (36), now we know that

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{37}$$

Moreover,

$$d(x_n, x_{n+1}) = (1-\beta_n) d(x_n, y_n) \longrightarrow 0 \tag{38}$$

and

$$d(z_n, z_{n+1}) \le d(x_n, x_{n+1}) \longrightarrow 0 \tag{39}$$

as $n \to \infty$. By (36) and (37), we get

$$\begin{aligned} dist(z_n, T(z_n)) &\le d(z_n, x_n) + dist(x_n, T(x_n)) \\ &\quad + H(T(x_n), T(z_n)) \\ &\le 2d(x_n, z_n) + dist(x_n, T(x_n)) \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{40}$$

Step 3. Now, we show that

$$\limsup_{n\to\infty} \left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_n)\right) \le 0, \tag{41}$$

with $\tilde{x} = P_{F(T)} f(\tilde{x})$ satisfying

$$\left\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \right\rangle \ge 0, \quad \forall x \in F(T). \tag{42}$$

Above all, since $T(x)$ is compact for any $x \in E$, then $T(x) \in BC(X)$. It follows from Lemma 7 that $F(T)$ is closed and convex, which implies that $P_{F(T)}u$ is well defined for any $u \in X$. By Lemma 12 (i), we know that $\{x_\alpha\}$ generated by (10) converges strongly to $\tilde{x} = P_{F(T)} f(\tilde{x})$ as $\alpha \to 0^+$. Then by Lemma 8, we know that $\tilde{x}$ is the unique solution of the following variational inequality:

$$\left\langle \overrightarrow{\tilde{x} f(\tilde{x})}, \overrightarrow{x\tilde{x}} \right\rangle \ge 0, \quad \forall x \in F(T). \tag{43}$$

Next, since $\{z_n\}$ is bounded and $\lim_{n\to\infty} dist(z_n, T(z_n)) = 0$, it follows from Lemma 12 (ii) that for all Banach limits μ,

$$d^2(f(\tilde{x}), \tilde{x}) \le \mu_n d^2(f(\tilde{x}), z_n), \tag{44}$$

and so

$$\mu_n \left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_n)\right) \le 0. \tag{45}$$

Further, $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$ implies that

$$\begin{aligned} &\limsup_{n\to\infty} \left[\left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_{n+1})\right)\right. \\ &\quad \left. - \left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_n)\right)\right] = 0. \end{aligned} \tag{46}$$

By Lemma 11, we have

$$\limsup_{n\to\infty} \left(d^2(f(\tilde{x}), \tilde{x}) - d^2(f(\tilde{x}), z_n)\right) \le 0. \tag{47}$$

Step 4. $\lim_{n\to\infty} x_n = \tilde{x}$ will be verified. In fact, by Lemma 3 and (13), now we know

$$\begin{aligned} d^2(x_{n+1}, \tilde{x}) &\le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) d^2(y_n, \tilde{x}) \\ &\quad - \beta_n (1-\beta_n) d^2(x_n, y_n) \\ &\le \beta_n d^2(x_n, \tilde{x}) + (1-\beta_n) d^2(y_n, \tilde{x}), \end{aligned} \tag{48}$$

and

$$\begin{aligned} d^2(y_n,\tilde{x}) &\le \alpha_n d^2(f(x_n),\tilde{x}) + (1-\alpha_n) d^2(z_n,\tilde{x}) \\ &\quad - \alpha_n(1-\alpha_n) d^2(f(x_n),z_n) \\ &\le (1-\alpha_n) H^2(T(x_n),T(\tilde{x})) \\ &\quad + \alpha_n^2 d^2(f(x_n),z_n) \\ &\quad + \alpha_n\left(d^2(f(x_n),\tilde{x}) - d^2(f(x_n),z_n)\right) \\ &\le (1-\alpha_n) d^2(x_n,\tilde{x}) + \alpha_n^2 d^2(f(x_n),z_n) \\ &\quad + \alpha_n\left(d^2(f(x_n),\tilde{x}) - d^2(f(x_n),z_n)\right). \end{aligned} \tag{49}$$

It follows from (21), Cauchy-Schwarz inequality, and Lemma 9 that

$$\begin{aligned} &\alpha_n\left(d^2(f(x_n),\tilde{x}) - d^2(f(x_n),z_n)\right) \\ &\quad \le 2\alpha_n\left(d(f(x_n),f(\tilde{x}))\, d(z_n,\tilde{x}) \right. \\ &\quad \left. + \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_n\tilde{x}}\right\rangle - d^2(z_n,\tilde{x})\right) \\ &\quad \le 2\alpha_n\left(kd(x_n,\tilde{x})\, d(z_n,\tilde{x}) + \left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_n\tilde{x}}\right\rangle \right. \\ &\quad \left. - d^2(z_n,\tilde{x})\right) \le \alpha_n k\left(d^2(x_n,\tilde{x}) + d^2(z_n,\tilde{x})\right) \\ &\quad + 2\alpha_n\left\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_n\tilde{x}}\right\rangle - 2\alpha_n d^2(z_n,\tilde{x}) \\ &\quad \le \alpha_n k d^2(x_n,\tilde{x}) + \alpha_n\left(d^2(f(\tilde{x}),\tilde{x}) \right. \\ &\quad \left. - d^2(f(\tilde{x}),z_n)\right). \end{aligned} \tag{50}$$

From (50) and (49), we know

$$\begin{aligned} d^2(y_n,\tilde{x}) &\le (1-\alpha_n(1-k)) d^2(x_n,\tilde{x}) \\ &\quad + \alpha_n\left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) \\ &\quad + \alpha_n^2 d^2(f(x_n),z_n). \end{aligned} \tag{51}$$

Combining (51) and (48), we get

$$\begin{aligned} d^2(x_{n+1},\tilde{x}) &\le \beta_n d^2(x_n,\tilde{x}) + (1-\beta_n) \\ &\quad \cdot \left((1-\alpha_n(1-k)) d^2(x_n,\tilde{x}) \right. \\ &\quad + \alpha_n\left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) \\ &\quad \left. + \alpha_n^2 d^2(f(x_n),z_n)\right) \le (1 \\ &\quad - (1-k)\alpha_n(1-\beta_n)) d^2(x_n,\tilde{x}) + \alpha_n(1 \\ &\quad - \beta_n)\left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) + (1-\beta_n) \\ &\quad \cdot \alpha_n^2 d^2(f(x_n),z_n) \le (1 \\ &\quad - (1-k)\alpha_n(1-\beta_n)) d^2(x_n,\tilde{x}) + \alpha_n(1 \\ &\quad - \beta_n)\left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) \\ &\quad + \alpha_n^2 d^2(f(x_n),z_n), \end{aligned} \tag{52}$$

i.e.,

$$u_{n+1} \le \left(1-\alpha_n'\right) u_n + \alpha_n'\beta_n', \quad \forall n \ge 1, \tag{53}$$

where $u_n = d^2(x_n,\tilde{x})$, $\alpha_n' = (1-k)\alpha_n(1-\beta_n)$, and

$$\beta_n' = \frac{(1-\beta_n)\left(d^2(f(\tilde{x}),\tilde{x}) - d^2(f(\tilde{x}),z_n)\right) + \alpha_n d^2(f(x_n),z_n)}{(1-k)(1-\beta_n)}. \tag{54}$$

Thus, from the conditions (L_1)-(L_3) and the inequality (41), it follows that $\alpha_n' \in (0,1)$, and

$$\begin{gathered} \sum_{n=1}^{\infty} \alpha_n' = \infty, \\ \limsup_{n\to\infty} \beta_n' \le 0. \end{gathered} \tag{55}$$

Hence, it follows from Lemma 6 that $u_n \to 0$. This implies that the proof is completed. □

If $T \equiv g$ is a nonexpansive single-valued operator with $Fix(g) \neq \emptyset$, then from Theorem 13, one can easy to obtain the following result.

Corollary 14. *Suppose that f, E, and (X,d) are the same as in Theorem 13, and the conditions (L_1)-(L_3) in Theorem 13 are satisfied. If $g : E \to E$ is a nonexpansive single-valued operator with $Fix(g) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (12) converges strongly to $\tilde{x} = P_{Fix(g)} f(\tilde{x})$ with*

$$\left\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{y\tilde{x}}\right\rangle \ge 0, \quad \forall y \in Fix(g). \tag{56}$$

Remark 15. Corollary 14 is the corresponding results of Theorem 3.1 in [8].

If $f \equiv I$, the identity operator, then by Theorem 13, now we directly have the following theorem.

Theorem 16. *Assume that T, E, and (X,d) are the same as in Theorem 13, and the conditions (L_1)-(L_3) in Theorem 13 hold. Then for any given $u, x_1 \in E$, sequence $\{x_n\}$ generated by*

$$\begin{gathered} y_n = \alpha_n u \oplus (1-\alpha_n) z_n, \\ d(z_n,z_{n+1}) \le d(x_n,x_{n+1}), \\ z_n \in T(x_n), \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) y_n, \quad \forall n \ge 1, \end{gathered} \tag{57}$$

converges strongly to the unique nearest point $\tilde{x}$ of u in $F(T)$, i.e., $\tilde{x} = P_{F(T)}u$, where $\tilde{x}$ also satisfies

$$\left\langle \overrightarrow{\tilde{x}u}, \overrightarrow{v\tilde{x}} \right\rangle \geq 0, \quad \forall v \in F(T). \tag{58}$$

Remark 17. Theorems 13 and 16 also extend and improve the corresponding results of Chang et al. [7], Piatek [18], Kaewkhao et al. [19], Panyanak and Suantai [16], and many others in the literature.

4. Concluding Remarks

The purpose of this paper is to introduce and study the following new two-step viscosity iterative approximation for finding fixed points of a set-valued nonlinear mapping $G : D \to C(D)$ and a contraction mapping $g : D \to D$:

$$\begin{aligned} u_{n+1} &= \beta_n u_n \oplus (1-\beta_n) v_n, \\ v_n &= \alpha_n g(u_n) \oplus (1-\alpha_n) w_n, \quad \forall n \geq 1, \end{aligned} \tag{59}$$

where D is a nonempty closed convex subset of a metric space $\mathbb{E}$, $u_1 \in D$ is an any given element and $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$, and $w_n \in G(u_n)$ satisfying $d(w_n, w_{n+1}) \leq d(u_n, u_{n+1})$ for any $n \in \mathbb{N}$.

By using the method due to Chang et al. [7, 8], Cauchy-Schwarz inequality, and Xu's inequality, we exposed strong convergence theorems of the new two-step viscosity iteration approximation (59) in complete CAT(0) spaces. The main theorems of this paper extend and improve the corresponding results of Chang et al. [7, 8], Piatek [18], Kaewkhao et al. [19], Panyanak and Suantai [16], and many others in the literature.

However, when g is a set-value contraction operator or is also nonexpansive in (59), whether can our main results be obtained? Furthermore, can our results be obtained when the iterations (13) (i.e., (10)), (12), and (57) become three-step iterations as in [35] or operator T is total asymptotically nonexpansive single-valued (set-valued) operator? These are still **open questions** to be worth further studying.

Conflicts of Interest

The authors declare that there are not any conflicts of interest regarding the publication of this paper.

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