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Research Article

Herz-Type Hardy Spaces Associated with Operators

Yan Chai, Yaoyao Han, and Kai Zhao

School of Mathematics and Statistics, Qingdao University, Qingdao 266071, China

Correspondence should be addressed to Kai Zhao; zhkzhc@aliyun.com

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Suppose L is a nonnegative, self-adjoint differential operator. In this paper, we introduce the Herz-type Hardy spaces associated with operator L. Then, similar to the atomic and molecular decompositions of classical Herz-type Hardy spaces and the Hardy space associated with operators, we prove the atomic and molecular decompositions of the Herz-type Hardy spaces associated with operator L. As applications, the boundedness of some singular integral operators on Herz-type Hardy spaces associated with operators is obtained.

1. Introduction

As we know, the theory of function spaces constitutes an important part of harmonic analysis and partial differential equations. Some results of the classical Hardy spaces can be found in [1–6], etc. Since there are some important situations in which the theory of classical Hardy spaces is not applicable, many authors begin to study Hardy spaces that are adapted to the differential operator L. For example, Auscher, Duong, and McIntosh [7], then Duong and Yan [8, 9], introduced the Hardy and BMO spaces adapted to the operator L which satisfies the Gaussian heat kernel upper bounds. Yang and his cooperators discussed new Orlicz-Hardy spaces associated with operators [10–13]. For more results, we refer to [14–19] and the references therein.

It is known that many classical function spaces and the Hardy type spaces associated with operators have the atomic decompositions and the molecular decompositions, and the atomic and molecular decompositions of function spaces make the linear operators acting on spaces very simple; see [20–29], etc. In fact, the characterizations of spaces of functions or distributions, including the atomic and molecular characterizations, have many important applications in harmonic analysis. In recent years, it has been proved that many results in the classical theory of Hardy spaces and singular integrals can transplant to the function spaces associated with operators, such as [30–36].

Suppose L is a nonnegative, self-adjoint differential operator, and L has H_∞-calculus on $L^2(\mathbb{R}^n)$. The kernel $p_t(x,y)$ of e^{-tL} satisfies the Gaussian upper bound on $\mathbb{R}^n\times\mathbb{R}^n$. Motivated by [17, 20, 37], etc, in this paper, we use the area integral function S_L associated with the operator L to define the Herz-type Hardy space $H\dot{K}_{q,L}^{\alpha,p}(\mathbb{R}^n)$. In order to obtain the atomic and molecular decompositions of the Herz-type Hardy space, the (α,q,M,L)-atom and the (α,q,M,L,ϵ)-molecule are introduced. By the method of the atomic and molecular decompositions of classical Herz-type Hardy spaces and the Hardy space $H_L^1(\mathbb{R}^n)$ associated with operators, we characterize $H\dot{K}_{q,L}^{\alpha,p}(\mathbb{R}^n)$ spaces for atoms and molecules; that is, we prove the atomic and molecular decompositions of Herz-type Hardy spaces associated with the operator L. Finally, as applications, we prove some singular integral operators are bounded from $H\dot{K}_{q,L}^{\alpha,p}(\mathbb{R}^n)$ to Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and also bounded on $H\dot{K}_{q,L}^{\alpha,p}(\mathbb{R}^n)$.

Throughout the paper, we always use the letter C to denote a positive constant, which may change from one to another and only depends on main parameters. We also use χ_E to denote the characteristic function of E which is the subset of $\mathbb{R}^n$.

2. Preliminaries

For convenience, we recall the definitions of Herz and Herz-type Hardy spaces on $\mathbb{R}^n$. For details, we refer to [37, 38], etc.

Let $k\in\mathbb{Z}$, $B_k=\{x\in\mathbb{R}^n : |x|\le 2^k\}$, $C_k=B_k\backslash B_{k-1}$ and $\chi_k=\chi_{C_k}$.

Definition 1 (see [38]). Let $0 < \alpha < \infty$, $0 < p \leq \infty$, $0 < q \leq \infty$.

(1) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L_{\mathrm{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}, \quad (1)$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}. \quad (2)$$

(2) The nonhomogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L_{\mathrm{loc}}^q(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}, \quad (3)$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \|f\chi_{B_0}\|_{L^q} + \sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p}. \quad (4)$$

Let Gf be the grand maximal function of f defined as $Gf(x) = \sup_{\varphi \in A_N} |\varphi_\nabla^*(f)(x)|$, where $A_N = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leq N} |x^\alpha D^\beta \varphi(x)| \leq 1\}$, $N > n+1$.

Definition 2 (see [38]). Let $0 < \alpha < \infty$, $0 < p < \infty$, $0 < q < \infty$.

(1) The homogeneous Herz-Hardy space $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : Gf \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \right\}, \quad (5)$$

Moreover,

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \|Gf\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}. \quad (6)$$

(2) The nonhomogeneous Herz-Hardy space $HK_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$HK_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : Gf \in K_q^{\alpha,p}(\mathbb{R}^n) \right\}. \quad (7)$$

Moreover,

$$\|f\|_{HK_q^{\alpha,p}(\mathbb{R}^n)} = \|Gf\|_{K_q^{\alpha,p}(\mathbb{R}^n)}. \quad (8)$$

Now, we introduce the Herz-type Hardy spaces associated with operators.

Suppose that the differential operator L satisfies the following two assumptions.

Assumption (A1). L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$, and has a bounded H_∞-functional calculus in $L^2(\mathbb{R}^n)$.

Assumption (A2). Each of the heat semigroup e^{-tL} generated by L has the kernel $p_t(x,y)$ which satisfies the following Gaussian upper bounds; i.e., there exist constants $C, c > 0$ such that

$$|p_t(x,y)| \leq \frac{C}{t^{n/2}} \exp\left(-\frac{|x-y|^2}{ct} \right). \quad (9)$$

Obviously, the typical second-order elliptic or subelliptic differential operators are to satisfy these assumptions (see for instance, [39]).

Now we introduce the following lemmas which will be used in this paper.

Lemma 3 (see [40, 41]). *Let L be a nonnegative self-adjoint operator satisfying Assumptions (A1) and (A2). For every $j = 0, 1, 2, \ldots$, there exist two positive constants C_j, c_j such that the kernel $p_{t,j}(x,y)$ of the operator $(t^2L)^j e^{-t^2L}$ satisfies*

$$|p_{t,j}(x,y)| \leq \frac{C_j}{(4\pi t)^n} \exp\left(-\frac{|x-y|^2}{c_j t^2} \right), \quad (10)$$

for all $t > 0$ and almost every $x, y \in \mathbb{R}^n$.

Lemma 4 (see [19]). *Let $\varphi \in C_0^\infty(\mathbb{R})$ be even and* $\operatorname{supp}\varphi \subseteq [-c_0^{-1}, c_0^{-1}]$. *Suppose Φ denotes the Fourier transform of φ. Then for each $j = 0, 1, 2, \ldots$, kernel $K_{(t^2L)^j\Phi(t\sqrt{L})}(x,y)$ of $(t^2L)^j\Phi(t\sqrt{L})$ satisfies*

$$\operatorname{supp} K_{(t^2L)^j\Phi(t\sqrt{L})} \subseteq \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \leq t\} \quad (11)$$

and

$$\left| K_{(t^2L)^j\Phi(t\sqrt{L})}(x,y) \right| \leq Ct^{-n}, \quad (12)$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$.

Lemma 5 (see [19]). *For $s > 0$, we define*

$$\mathbb{F}(s) := \left\{ \psi : \mathbb{C} \longrightarrow \mathbb{C} \text{ measurable} : \quad |\psi(z)| \leq C \frac{|z|^s}{(1+|z|^{2s})} \right\}. \quad (13)$$

Then for any nonzero function $\psi \in \mathbb{F}(s)$, $\kappa = \{\int_0^\infty |\psi(t)|^2 dt/t\}^{1/2} < \infty$.

Lemma 6 (see [19]). *Let $\psi \in \mathbb{F}(s)$. Then, for any $f \in L^2(\mathbb{R}^n)$,*

$$\left\{ \int_0^\infty \left\| \psi\left(t\sqrt{L}\right) f \right\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right\}^{1/2} = \kappa \|f\|_{L^2(\mathbb{R}^n)}. \quad (14)$$

Definition 7 (see [8]). For any $f \in L^1(\mathbb{R}^n)$, the area integral function $S_L(f)$ associated with operators L is defined by

$$S_L(f)(x) = \left(\iint_{|x-y|<t} |Q_{t^2} f(y)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}, \quad (15)$$

where $Q_{t^2} f(x) = t^2 L e^{-t^2L} f(x)$, and L satisfies Assumptions (A1) and (A2).

Thus, for any integer $m > 0$, q_{t^m}, the kernel of Q_{t^m} satisfies $|q_{t^m}| \le Ct^{-n}s(|x-y|/t)$, where s is a positive decreasing function, satisfying $\lim_{r\to\infty} r^{n+\varepsilon}s(r) = 0$, for any $\varepsilon > 0$. Therefore, for convenience, in the following, we always set $m = 2$ in (15).

By the definition (15) and Assumptions (A1) and (A2), it is easy to check that for any $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$; there exist $C_1, C_2 > 0$ such that (or see, for example, [11]):

$$C_1 \|f\|_p \le \|S_L(f)\|_p \le C_2 \|f\|_p. \tag{16}$$

Definition 8. Suppose $0 < \alpha < \infty$, $0 < p < \infty$, $1 < q < \infty$. Let L satisfy Assumptions (A1) and (A2).

(1) The homogeneous Herz-type Hardy space $H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)$ associated with the operator L is defined by

$$H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : S_L(f) \in \dot{K}^{\alpha,p}_{q}(\mathbb{R}^n) \right\}. \tag{17}$$

The norm of f in $H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)$ is

$$\|f\|_{H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)} = \|S_L(f)\|_{\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)}. \tag{18}$$

(2) The nonhomogeneous Herz-type Hardy space $HK^{\alpha,p}_{q,L}(\mathbb{R}^n)$ associated with the operator L is defined by

$$HK^{\alpha,p}_{q,L}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : S_L(f) \in K^{\alpha,p}_{q}(\mathbb{R}^n) \right\}. \tag{19}$$

The norm of f in $HK^{\alpha,p}_{q,L}(\mathbb{R}^n)$ is

$$\|f\|_{HK^{\alpha,p}_{q,L}(\mathbb{R}^n)} = \|S_L(f)\|_{K^{\alpha,p}_{q}(\mathbb{R}^n)}. \tag{20}$$

Since $\dot{K}^{0,p}_{p}(\mathbb{R}^n) = K^{0,p}_{p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, the Herz-type Hardy spaces associated with the operator introduced in Definition 8 are really the expansion of Hardy space associated with operators in [8].

3. The Decompositions of Herz-Type Hardy Spaces

In this section, we give the atomic decomposition and molecular decomposition of Herz-type Hardy spaces associated with the operator L, respectively, which are the main results in this paper.

3.1. Atomic Decomposition of the Herz-Type Hardy Space. For the purpose of the atomic decomposition of the Herz-type Hardy space associated with operator, we first introduce the (α, q, M, L)-atom.

Definition 9. Let $1 < q < \infty$, $0 < \alpha < \infty$, $M \ge 1$. Set $D(L) = \{u \in L^2(\mathbb{R}^n) : Lu \in L^2(\mathbb{R}^n)\}$, where L satisfies Assumptions (A1) and (A2).

(1) A function $a(x) \in L^2(\mathbb{R}^n)$ is said to be an (α, q, M, L)-atom, if there exists $b \in D(L^M)$, such that

(i) $a = L^M b$;

(ii) $\operatorname{supp} L^j b \subset B(0,r)$, $j = 0, 1, \cdots, M$;

(iii) $\|(r^2L)^j b\|_{L^q(\mathbb{R}^n)} \le r^{2M}|B|^{-\alpha/n}$, $j = 0, 1, \cdots, M$;

$B = B(0,r) = \{x \in \mathbb{R}^n : |x| \le r\}$, $r > 0$.

(2) Function $a(x) \in L^2(\mathbb{R}^n)$ is said to be a restrictive (α, q, M, L)-atom, if there exists $b \in D(L^M)$, satisfying (i), (ii), (iii), and $B(0,r) = \{x \in \mathbb{R}^n : |x| \le r\}$, $r \ge 1$.

The main result of this subsection is the following atomic decomposition of the Herz-type Hardy spaces associated with the operator L. The part of the idea is from [1].

Theorem 10. *Let $0 < p < \infty$, $1 < q < \infty$, $0 < \alpha < n(1-1/q)+1$. Suppose L satisfies Assumptions (A1) and (A2). Then, $f \in H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)$ if and only if there exist a family of (α, q, M, L)-atoms $\{a_k\}$ and a sequence of numbers $\{\lambda_k\}$ such that f can be represented in the following form:*

$$f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x), \tag{21}$$

and the sum converges in the sense of L^2-norm, $(\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p} < \infty$. Moreover,

$$\|f\|_{H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)} \sim \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p}, \tag{22}$$

where the infimum is taken over all of the decompositions of f.

Proof. First, we prove the theorem for $q = 2$.

Necessity. Let φ and Φ be the same as those in Lemma 4. Set $\Psi(x) = x^{2M}\Phi(x)$. Then by L^2-functional calculus (see for example, [42]), for every $f \in H\dot{K}^{\alpha,p}_{2,L}(\mathbb{R}^n)$, there is

$$f(x) = C_\Psi \int_0^\infty \Psi\left(t\sqrt{L}\right) t^2 L e^{-t^2 L} f(x) \frac{dt}{t}. \tag{23}$$

Set $\Omega_k = \{x \in \mathbb{R}^n : S_L(f)(x) > 2^k\}$, $k \in \mathbb{Z}$. $\mathcal{D}$ denotes the collection of all dyadic cubes in $\mathbb{R}^n$. Let $D_k = \{Q \in \mathcal{D} : |Q \cap \Omega_k| > |Q|/2, |Q \cap \Omega_{k+1}| \le |Q|/2\}$. Then, for any $Q \in \mathcal{D}$, there exists only one $k \in \mathbb{Z}$ such that $Q \in D_k$. Let $D_k^l = \{Q_k^l \in D_k : Q \in D_k, Q \cap Q_k^l \ne \emptyset,\ Q \subset Q_k^l\}$; i.e., D_k^l denote the collection of maximal dyadic cubes in D_k. Set

$$\widehat{Q} = \left\{ (y,t) : y \in Q, \frac{l(Q)}{2} < t < l(Q) \right\}, \tag{24}$$

where $l(Q)$ is the side length of Q.

Then, by (23), we have that

$$\begin{aligned} f(x) &= \sum_k \sum_l C_\Psi \iint_{\widehat{Q}_k^l} \Psi\left(t\sqrt{L}\right)(x,y)\, t^2 L e^{-t^2 L} f(y) \frac{dy\,dt}{t} \\ &:= \sum_{k,l} \lambda_k^l a_k^l(x), \end{aligned} \tag{25}$$

where $a_k^l = L^M b_k^l$ and

$$b_k^l(x) = \frac{C_\Psi}{\lambda_k^l} \iint_{\widehat{Q}_k^l} t^{2M} \Phi(t\sqrt{L})(x,y)\, t^2 L e^{-t^2 L} f(y) \frac{dydt}{t}, \tag{26}$$

$$\lambda_k^l = \left|Q_k^l\right|^{\alpha/n} \left(\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dydt}{t} \right)^{1/2}.$$

We will prove that, up to a normalization by a multiplicative constant, every $a_k^l(x)$ is an (α, q, M, L)-atom.

Obviously, by Lemma 4, we conclude that $\operatorname{supp}(L^j b_k^l) \subset 3Q_k^l$, $j = 0, 1, \cdots, M$.

For $h(x) \in L^2(\mathbb{R}^n)$, and $\|h\|_{L^2(\mathbb{R}^n)} \le 1$, then, by Hölder inequality together with Lemma 6, we have that

$$\begin{aligned}
&\left\| \left(\left(l(Q_k^l) \right)^2 L \right)^j b_k^l \right\|_{L^2(\mathbb{R}^n)} \\
&= \sup_{\|h\|_{L^2(\mathbb{R}^n)} \le 1} \left| \int_{\mathbb{R}^n} \left(\left(l(Q_k^l) \right)^2 L \right)^j b_k^l(x) h(x)\, dx \right| \\
&= \sup_{\|h\|_{L^2(\mathbb{R}^n)} \le 1} \frac{C_\Psi}{\lambda_k^l} \left| \iiint_{\widehat{Q}_k^l} t^{2M} \left(\left(l(Q_k^l) \right)^2 L \right)^j \Phi(t\sqrt{L}) \right. \\
&\quad \left. \cdot (x,y)\, t^2 L e^{-t^2 L} f(y) \frac{dydt}{t} h(x)\, dx \right| \\
&= \sup_{\|h\|_{L^2(\mathbb{R}^n)} \le 1} \frac{C_\Psi}{\lambda_k^l} \left| \iint_{\widehat{Q}_k^l} t^{2M} \left(\left(l(Q_k^l) \right)^2 L \right)^j \Phi(t\sqrt{L}) \right. \\
&\quad \left. \cdot h(y)\, t^2 L e^{-t^2 L} f(y) \frac{dydt}{t} \right| \le C \sup_{\|h\|_{L^2(\mathbb{R}^n)} \le 1} \frac{1}{\lambda_k^l} \\
&\quad \cdot l(Q_k^l)^{2M} \left(\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dydt}{t} \right)^{1/2} \\
&\quad \cdot \left(\int_{\mathbb{R}^{n+1}} \left| (t^2 L)^j \Phi(t\sqrt{L}) \right|^2 h(y) \frac{dydt}{t} \right)^{1/2} = C \\
&\quad \cdot \sup_{\|h\|_{L^2(\mathbb{R}^n)} \le 1} \frac{l(Q_k^l)^{2M}}{\lambda_k^l} \\
&\quad \cdot \left(\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dydt}{t} \right)^{1/2} \|h\|_{L^2(\mathbb{R}^n)} \\
&= C l(Q_k^l)^{2M} \left|Q_k^l\right|^{-\alpha/n}.
\end{aligned} \tag{27}$$

Furthermore, we prove the following estimate:

$$\left(\sum_{k,l} \left|\lambda_k^l\right|^p \right)^{1/p} \le C \|f\|_{H\dot{K}_{2,L}^{\alpha,p}(\mathbb{R}^n)}. \tag{28}$$

Noting the definition of λ_k^l in (26) and Definition 8 for $H\dot{K}_{2,L}^{\alpha,p}(\mathbb{R}^n)$, it means that we should establish the following inequality:

$$\begin{aligned}
&\sum_{k,l} \left|Q_k^l\right|^{\alpha p/n} \left(\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dydt}{t} \right)^{p/2} \\
&\quad \le C \sum_{k,l} \left|Q_k^l\right|^{\alpha p/n} \left(\int_{\mathbb{R}^n} \left| S_L(f) \chi_{l,k} \right|^2 dx \right)^{p/2},
\end{aligned} \tag{29}$$

where $\chi_{l,k} = \chi_{E_k^l}$ is the characteristic function of $E_k^l = Q_k^l \setminus Q_{k-1}^l$.

It is sufficient to show that

$$\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dydt}{t} \le C \int_{E_k^l} \left| S_L(f)(x) \right|^2 dx. \tag{30}$$

In fact, if $(y,t) \in \widehat{Q}_k^l$, then $y \in Q_k^l$, $l(Q_k^l)/2 < t < l(Q_k^l)$. Let $\chi(x,y,t)$ denote the characteristic function of $\{(x,y,t) : x \in E_k^l, |x-y| < t\}$. Thus, by the definition of D_k, we obtain that

$$\int_{\mathbb{R}^n} \chi(x,y,t)\, dx \ge C t^n. \tag{31}$$

Therefore,

$$\begin{aligned}
&\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dydt}{t} \\
&\quad \le C \iint_{\widehat{Q}_k^l} \int_{E_k^l} \chi(x,y,t) \left| t^2 L e^{-t^2 L} f(y) \right|^2 dx \frac{dydt}{t^{n+1}} \\
&\quad \le C \int_{E_k^l} \int_{\mathbb{R}^{n+1}} \chi(x,y,t) \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dydt}{t^{n+1}} dx \\
&\quad \le C \int_{E_k^l} \left| S_L(f)(x) \right|^2 dx.
\end{aligned} \tag{32}$$

Hence, the necessity is proved.

Sufficiency. Let $f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x)$, where every a_k is an (α, q, M, L)-atom. We will prove the sufficiency for two situations: $0 < p \le 1$ and $1 < p < \infty$.

If $0 < p \le 1$, then, to prove $f \in H\dot{K}_{2,L}^{\alpha,p}(\mathbb{R}^n)$, it is only need to show that, for any (α, q, M, L)-atom a, there exits a constant $C > 0$ independent of a such that

$$\left\| S_L(a) \right\|_{\dot{K}_2^{\alpha,p}(\mathbb{R}^n)} \le C. \tag{33}$$

In fact, then, we have

$$\begin{aligned}
\|f\|_{H\dot{K}_{2,L}^{\alpha,p}(\mathbb{R}^n)}^p &= \left\| S_L(f) \right\|_{\dot{K}_2^{\alpha,p}(\mathbb{R}^n)}^p \\
&\le \sum_{k=-\infty}^{\infty} |\lambda_k|^p \left\| S_L(a_k) \right\|_{\dot{K}_2^{\alpha,p}(\mathbb{R}^n)}^p \\
&\le C \sum_{k=-\infty}^{\infty} |\lambda_k|^p.
\end{aligned} \tag{34}$$

Suppose that a is an (α, q, M, L)-atom with $a = L^M b$ and for any m $(m = 0, 1, 2, \dots, M)$ $\operatorname{supp} L^m b \subset B_{k_0} = B(0, 2^{k_0})$, $k_0 \in \mathbb{Z}^+$. Then

$$\begin{aligned}\|S_L(a)\|^p_{\dot K_2^{\alpha,p}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \|S_L(a)\chi_k\|^p_{L^2(\mathbb{R}^n)} \\ &= \sum_{k=-\infty}^{k_0+1} |B_k|^{\alpha p/n} \|S_L(a)\chi_k\|^p_{L^2(\mathbb{R}^n)} \\ &\quad + \sum_{k=k_0+2}^{\infty} |B_k|^{\alpha p/n} \|S_L(a)\chi_k\|^p_{L^2(\mathbb{R}^n)} \\ &:= I_1 + I_2.\end{aligned} \tag{35}$$

For I_1, L^2 boundedness of S_L and the size condition of atom tell us

$$\begin{aligned}I_1 &\le C \sum_{k=-\infty}^{k_0+1} |B_k|^{\alpha p/n} \|S_L(a)\chi_k\|^p_{L^2(\mathbb{R}^n)} \\ &\le C \sum_{k=-\infty}^{k_0+1} |B_k|^{\alpha p/n} \|a\|^p_{L^2(\mathbb{R}^n)} \\ &\le C \sum_{k=-\infty}^{k_0+1} |B_k|^{\alpha p/n} |B_{k_0}|^{-\alpha p/n} \le C \sum_{k=-\infty}^{k_0+1} 2^{(k-k_0)\alpha p} \le C.\end{aligned} \tag{36}$$

In order to estimate I_2, we write

$$\begin{aligned}S_L(a)^2(x) &= \int_0^\infty \int_{|x-y|<t} \left| t^2 L e^{-t^2 L} a(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \\ &= \left(\int_0^{2^{k_0}} + \int_{2^{k_0}}^\infty \right) \int_{|x-y|<t} \left| t^2 L e^{-t^2 L} a(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \\ &:= I_{2,1} + I_{2,2}.\end{aligned} \tag{37}$$

For $I_{2,1}$, noting that $k > k_0 + 1$ and $|x - y| < t < 2^{k_0}$, if $x \in C_k$, $z \in B_{k_0}$, then $|y - z| \ge |x - z| - |x - y| > 2^{k-1} - 2^{k_0} \ge C2^k$. Thus, by Lemma 3, we can have that

$$\begin{aligned}&I_{2,1} \\ &\le C \int_0^{2^{k_0}} \int_{|x-y|<t} \left(\int_{\mathbb{R}^n} \left| \frac{t}{(t + |y - z|)^{n+1}} \right| |a(z)|\, dz \right)^2 dy \frac{dt}{t^{n+1}} \\ &\le C \frac{\|a\|^2_{L^1(\mathbb{R}^n)}}{(2^k)^{2(n+1)}} \int_0^{2^{k_0}} \int_{|x-y|<t} t^2 dy \frac{dt}{t^{n+1}} \le C \frac{\|a\|^2_{L^2(\mathbb{R}^n)} |B_{k_0}|}{(2^k)^{2(n+1)}} 2^{2k_0} \\ &= C |B_{k_0}|^{-2\alpha/n} |B_{k_0}| |B_k|^{-2} 2^{-2k} 2^{2k_0}.\end{aligned} \tag{38}$$

For $I_{2,2}$, noting that $|x - y| < t$, if $x \in C_k$, $z \in B_{k_0}$, then $|x - z| \le |x - y| + |y - z| < t + |y - z|$. So that $t + |y - z| > C2^k$ holds true. Therefore, we can obtain that

$$\begin{aligned}I_{2,2} &= \int_{2^{k_0}}^\infty \int_{|x-y|<t} \left| t^2 L e^{-t^2 L} (L^M b) \right|^2 \frac{dy\,dt}{t^{n+1}} \\ &= \int_{2^{k_0}}^\infty \int_{|x-y|<t} \left| (t^2 L)^{M+1} e^{-t^2 L} b(y) \right|^2 \frac{dy\,dt}{t^{n+4M+1}} \\ &\le C \int_{2^{k_0}}^\infty \int_{|x-y|<t} \\ &\cdot \left(\int_{\mathbb{R}^n} \left| \frac{t}{(t + |y - z|)^{n+1}} \right| |b(z)|\, dz \right)^2 dy \frac{dt}{t^{n+4M+1}} \\ &\le C \frac{\|b\|^2_{L^1(\mathbb{R}^n)}}{(2^k)^{2(n+1)}} \int_{2^{k_0}}^\infty \frac{dt}{t^{4M-1}} = C |B_{k_0}|^{-2\alpha/n} |B_{k_0}| |B_k|^{-2} \\ &\cdot 2^{-k} 2^{2k_0}.\end{aligned} \tag{39}$$

Hence, combining (38) and (39), one can have

$$\begin{aligned}I_2 &= \sum_{k=k_0+2}^{\infty} |B_k|^{\alpha p/n} \|S_L(a)\chi_k\|^p_{L^2(\mathbb{R}^n)} \\ &\le C \sum_{k=k_0+2}^{\infty} |B_k|^{\alpha p/n} |B_{k0}|^{-\alpha p/n} |B_k|^{-p/2} |B_{k_0}|^{p/2} \frac{2^{k_0 p}}{2^{kp}} \\ &\le C \sum_{k=k_0+2}^{\infty} 2^{(k-k_0)(\alpha - n/2 - 1)p} \le C.\end{aligned} \tag{40}$$

If $1 < p < \infty$, then

$$\begin{aligned}&\|S_L(f)\|^p_{\dot K_2^{\alpha,p}(\mathbb{R}^n)} \\ &\le \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=-\infty}^{\infty} |\lambda_l| \|S_L(a_l)\chi_k\|_{L^2(\mathbb{R}^n)} \right)^p \\ &= C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=k-1}^{\infty} |\lambda_l| \|S_L(a_l)\chi_k\|_{L^2(\mathbb{R}^n)} \right)^p \\ &\quad + C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \|S_L(a_l)\chi_k\|_{L^2(\mathbb{R}^n)} \right)^p \\ &:= II_1 + II_2.\end{aligned} \tag{41}$$

For II_1, L^2 boundedness of S_L and the Hölder inequality tell us

$$\begin{aligned}II_1 &\le C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=k-1}^{\infty} |\lambda_l| \|a_l\|_{L^2(\mathbb{R}^n)} \right)^p \\ &\le C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{l=k-1}^{\infty} |\lambda_l| |B_l|^{-\alpha/n} \right)^p \\ &\le C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \sum_{l=k-1}^{\infty} |\lambda_l|^p |B_l|^{-\alpha p/2n} \\ &\cdot \left(\sum_{l=k-1}^{\infty} |B_l|^{-\alpha p'/2n} \right)^{p/p'} \le C \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/2n}\end{aligned}$$

$$\cdot \sum_{l=k-1}^{\infty} |\lambda_l|^p |B_l|^{-\alpha p/2n} = C \sum_{l=-\infty}^{\infty} |\lambda_l|^p |B_l|^{-\alpha p/2n}$$
$$\cdot \sum_{k=-\infty}^{l+1} |B_k|^{\alpha p/2n} = C \sum_{l=-\infty}^{\infty} |\lambda_l|^p . \tag{42}$$

For II_2, similar to the estimate of $S_L(a)(x)$, we can obtain the estimates of $S_L(a_l)$ as (38) and (39). Thus, using the Hölder inequality, we have that

$$\begin{aligned}
II_2 &\le \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \\
&\cdot \left\{ \sum_{l=-\infty}^{k-2} |\lambda_l| |B_l|^{-\alpha/n} |B_l|^{1/2} |B_k|^{-1/2} 2^l 2^{-k} \right\}^p \\
&\le C \sum_{k=-\infty}^{\infty} \left\{ \sum_{l=-\infty}^{k-2} |\lambda_l| 2^{(k-l)(\alpha-n/2-1)} \right\}^p \\
&\le C \sum_{k=-\infty}^{\infty} \left\{ \sum_{l=-\infty}^{k-2} |\lambda_l|^p 2^{(k-l)(\alpha-n/2-1)p/2} \right\} \\
&\cdot \left\{ \sum_{l=-\infty}^{k-2} 2^{(k-l)(\alpha-n/2-1)p'/2} \right\}^{p/p'} \le C \sum_{l=-\infty}^{\infty} |\lambda_l|^p \\
&\cdot \sum_{k=l+2}^{+\infty} 2^{(k-l)(\alpha-n/2-1)p/2} = C \sum_{l=-\infty}^{\infty} |\lambda_l|^p .
\end{aligned} \tag{43}$$

The sufficiency is proved. Then the proof of Theorem 10 for $q = 2$ is finished.

If $q \neq 2$, the proof that is exactly similar to the situation of $q = 2$, we only need to slightly modify some formulas above. We should set

$$\lambda_k^l = \left|Q_k^l\right|^{\alpha/n+1/q-1/2} \left(\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dy\,dt}{t} \right)^{1/2} . \tag{44}$$

Inequalities (29) and (30) are replaced with

$$\begin{aligned}
&\sum_{k,l} \left|Q_k^l\right|^{\alpha p/n} \left|Q_k^l\right|^{(1/q-1/2)p} \\
&\cdot \left(\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dy\,dt}{t} \right)^{p/2} \\
&\le C \sum_{k,l} \left|Q_k^l\right|^{\alpha p/n} \left\| S_L(f)(x) \cdot \chi_{l,k} \right\|_{L^q(\mathbb{R}^n)}^p
\end{aligned} \tag{45}$$

and

$$\begin{aligned}
&\left|Q_k^l\right|^{1/q-1/2} \left(\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dy\,dt}{t} \right)^{1/2} \\
&\le C \left\| S_L(f) \cdot \chi_{l,k} \right\|_{L^q(\mathbb{R}^n)} ,
\end{aligned} \tag{46}$$

respectively. To obtain inequality (46), there is

$$\begin{aligned}
&\left\| S_L(f) \cdot \chi_{l,k} \right\|_{L^q(\mathbb{R}^n)} \\
&= \left(\int_{E_k^l} \left(\iint_{|x-y|<t} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right)^{q/2} dx \right)^{1/q} \\
&\ge C \left(\int_{E_k^l} \left(\iint_{|x-y|<t} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dy\,dt}{t} \right)^{q/2} \right. \\
&\left. \cdot \frac{1}{|Q_k^l|^{q/2}} dx \right)^{1/q} \ge C \left|Q_k^l\right|^{-1/2} \\
&\cdot \left(\int_{E_k^l} \left(\iint_{|x-y|<t} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dy\,dt}{t} dx \right)^{q/2} \right)^{1/q} \\
&\ge C \left|Q_k^l\right|^{1/q-1/2} \left(\iint_{\widehat{Q}_k^l} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dy\,dt}{t} \right)^{1/2} .
\end{aligned} \tag{47}$$

Other details are omitted. □

For the nonhomogeneous Herz-type Hardy space $HK_{q,L}^{\alpha,p}(\mathbb{R}^n)$ associated with the operator L, there is the same result as follows.

Theorem 11. *Let $0 < p < \infty$, $1 < q < \infty$, $0 < \alpha < n(1-1/q)+1$. Suppose L satisfies Assumptions (A1) and (A2). Then, $f \in HK_{q,L}^{\alpha,p}(\mathbb{R}^n)$ if and only if there exist a family of the restrictive (α, q, M, L)-atoms $\{a_k\}_{k=0}^{+\infty}$ and a sequence of numbers $\{\lambda_k\}_{k=0}^{+\infty}$ such that f can be represented in the following form:*

$$f(x) = \sum_{k=0}^{+\infty} \lambda_k a_k(x) , \tag{48}$$

and the sum converges in the sense of L^2-norm, $(\sum_{k=0}^{+\infty} |\lambda_k|^p)^{1/p} < \infty$. Moreover,

$$\|f\|_{HK_{q,L}^{\alpha,p}(\mathbb{R}^n)} \sim \inf \left(\sum_{k=0}^{+\infty} |\lambda_k|^p \right)^{1/p} , \tag{49}$$

where the infimum is taken over all of the decompositions of f.

3.2. Molecular Decomposition of the Herz-Type Hardy Space. In this subsection, we first introduce $(\alpha, q, M, L, \epsilon)$-molecule in the following; then we give the molecular decomposition of the Herz-type Hardy space associated with operator.

Definition 12. Let $1 < q < \infty$, $0 < \alpha < \infty$, $M \ge 1$. Set $D(L) = \{u \in L^2(\mathbb{R}^n) : Lu \in L^2(\mathbb{R}^n)\}$, where L satisfies Assumptions (A1) and (A2).

(1) A function $a(x) \in L^2(\mathbb{R}^n)$ is said to be an $(\alpha, q, M, L, \epsilon)$-molecule, if there exists $b \in D(L^M)$, such that

(i) $a = L^M b$;

(ii) $\|(r^2 L)^j b\|_{L^q(S_k(B))} \le 2^{-k\epsilon} r^{2M} |2^k B|^{-\alpha/n}$, $j = 0, 1, \cdots, M$ and $k = 0, 1, \cdots$, where $B = B(0,r) = \{x \in \mathbb{R}^n : |x| \le r\}$, $r > 0$; and

$$S_0(B) = B,$$
$$S_k(B) = 2^k B \backslash 2^{k-1} B \quad \text{for } k \in \mathbb{N}. \tag{50}$$

(2) Function $a(x) \in L^2(\mathbb{R}^n)$ is said to be a restrictive $(\alpha, q, M, L, \epsilon)$-molecule, if there exists $b \in D(L^M)$, satisfying (i), (ii), and $B(0,r) = \{x \in \mathbb{R}^n : |x| \le r\}$, $r \ge 1$.

It is not difficult to check that an (α, q, M, L)-atom associated with ball B is also an $(\alpha, q, M, L, \epsilon)$-molecule associated with the same ball B.

The following molecular decomposition of the Herz-type Hardy spaces associated with the operator L is the main result in this subsection.

Theorem 13. *Let $0 < p < \infty$, $1 < q < \infty$, $0 < \alpha < n(1-1/q)+1$. Suppose L satisfies Assumptions (A1) and (A2). Then, $f \in H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)$ if and only if there exist a family of $(\alpha, q, M, L, \epsilon)$-molecules $\{a_k\}$ and a sequence of numbers $\{\lambda_k\}$ such that f can be represented in the following form:*

$$f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x), \tag{51}$$

and the sum converges in the sense of L^2-norm, $(\sum_{k=-\infty}^{\infty} |\lambda_k|^p)^{1/p} < \infty$. Moreover,

$$\|f\|_{H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)} \sim \inf\left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p\right)^{1/p}, \tag{52}$$

where the infimum is taken over all of the decompositions of f.

Proof. (i) The proof of necessity is a direct consequence of the necessity in Theorem 10, since an (α, q, M, L)-atom is also an $(\alpha, q, M, L, \epsilon)$-molecule for all $\epsilon > 0$.

(ii) The proof of sufficiency is similar to that of the sufficiency in Theorem 10. The main difference is that the support of $(\alpha, q, M, L, \epsilon)$-molecule is not the ball B. However, we can overcome this difficulty by decomposing $\mathbb{R}^n$ into annuli associated with the ball B, then using the same argument as in Theorem 10 to get sufficiency. We omit the details here. □

Similarly, for the nonhomogeneous Herz-type Hardy space associated with operator $HK^{\alpha,p}_{q,L}(\mathbb{R}^n)$, there is the same result as follows.

Theorem 14. *Let $0 < p < \infty$, $1 < q < \infty$, $0 < \alpha < n(1-1/q)+1$. Suppose L satisfies Assumptions (A1) and (A2). Then, $f \in HK^{\alpha,p}_{q,L}(\mathbb{R}^n)$ if and only if there exist a family of the restrictive $(\alpha, q, M, L, \epsilon)$-molecules $\{a_k\}_{k=0}^{+\infty}$ and a sequence of numbers $\{\lambda_k\}_{k=0}^{+\infty}$ such that f can be represented in the following form:*

$$f(x) = \sum_{k=0}^{+\infty} \lambda_k a_k(x), \tag{53}$$

and the sum converges in the sense of L^2-norm, $(\sum_{k=0}^{+\infty} |\lambda_k|^p)^{1/p} < \infty$. Moreover,

$$\|f\|_{HK^{\alpha,p}_{q,L}(\mathbb{R}^n)} \sim \inf\left(\sum_{k=0}^{+\infty} |\lambda_k|^p\right)^{1/p}, \tag{54}$$

where the infimum is taken over all of the decompositions of f.

4. Boundedness of Singular Integral Operators

This section is based on the decompositions of $f \in H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)$ in the previous section; as applications, we give some boundedness of sublinear operators satisfying certain conditions on Herz-type Hardy spaces associated with operator.

Theorem 15. *Let $0 < p < \infty$, $1 < q < \infty$, $0 < \alpha < n(1-1/q)$. If a sublinear operator T satisfies that*

(i) *T is bounded on $L^q(\mathbb{R}^n)$;*

(ii) *Tg satisfies the size condition*

$$|Tg(x)| \le C|x|^{-n} \|g\|_1, \tag{55}$$

for suitable function g with $dist(x, \operatorname{supp} g) \ge |x|/2$.

Then T is bounded from $H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$, that is,

$$\|Tf\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} \le C\|f\|_{H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)}. \tag{56}$$

Proof. Suppose $f \in H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)$. By Theorem 10, we have

$$f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x), \tag{57}$$

where each a_j is a (α, q, M, L)-atom with $a_j = L^M b_j$, and $\operatorname{supp} L^m b_j \subset B_j$, $m = 0, 1, \ldots, M$, $\|f\|_{H\dot{K}^{\alpha,p}_{q,L}(\mathbb{R}^n)} \sim \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p}$.

Thus,

$$\begin{aligned}
\|Tf\|^p_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \|(Tf)\cdot\chi_k\|^p_{L^q(\mathbb{R}^n)} \\
&\le C\sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \, \|(Ta_j)\cdot\chi_k\|_{L^q(\mathbb{R}^n)}\right)^p \\
&\quad + C\sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \left\|(Ta_j)\cdot\chi_k\right\|_{L^q(\mathbb{R}^n)}\right)^p \\
&\triangleq J_1 + J_2
\end{aligned} \tag{58}$$

First, we estimate J_2. By the boundedness of T in $L^q(\mathbb{R}^n)$, we can infer that

$$\begin{aligned}
J_2 &\le C\sum_{k=-\infty}^{\infty} |B_k|^{\alpha p/n} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \left\|(Ta_j)\cdot\chi_k\right\|_{L^q(\mathbb{R}^n)}\right)^p \\
&\le C\sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha}\right)^p.
\end{aligned} \tag{59}$$

Therefore, if $0 < p \le 1$, then, by the Jensen inequality, we have

$$\begin{aligned} J_2 &\le C \sum_{k=-\infty}^{\infty} \sum_{j=k-1}^{\infty} \left|\lambda_j\right|^p 2^{(k-j)\alpha p} \\ &\le C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p \sum_{k\le j+1} 2^{(k-j)\alpha p} \le C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p. \end{aligned} \tag{60}$$

If $1 < p < \infty$, let $1/p + 1/p' = 1$. Then we can obtain

$$\begin{aligned} &J_2 \\ &\le C \sum_{k=-\infty}^{\infty} \sum_{j=k-1}^{\infty} \left|\lambda_j\right|^p 2^{(k-j)\alpha p/2} \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p'/2} \right)^{p/p'} \\ &\le C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p \sum_{k\le j+1} 2^{(k-j)\alpha p/2} \le C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p. \end{aligned} \tag{61}$$

Hence, $J_2 \le C\|f\|_{H\dot{K}_{q,L}^{\alpha,p}(\mathbb{R}^n)}$.

Second, we estimate J_1. By the Hölder inequality and (55), we obtain

$$\begin{aligned} &\left\|Ta_j \cdot \chi_k\right\|_{L^q(\mathbb{R}^n)} \\ &\le C \left\{ \int_{C_k} |x|^{-nq} \left(\int_{B_j} \left|a_j\right| dy \right)^q dx \right\}^{1/q} \\ &\le C 2^{-nk} \left|B_k\right|^{1/q} \left\{ \left(\int_{B_j} \left|a_j\right|^q dy \right) \left(\int_{B_j} dy \right)^{q/q'} \right\}^{1/q} \\ &\le C 2^{-nk} \left|B_k\right|^{1/q} \left|B_j\right|^{1/q'} \left\|a_j\right\|_{L^q(\mathbb{R}^n)} \\ &\le C 2^{(j-k)(n/q')} \left|B_j\right|^{-\alpha/n}. \end{aligned} \tag{62}$$

Thus,

$$J_1 \le C \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{k-2} \left|\lambda_j\right| 2^{(j-k)(n/q'-\alpha)} \right\}^p. \tag{63}$$

Therefore, if $0 < p \le 1$, by the Jensen inequality, we have

$$\begin{aligned} J_1 &\le C \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} \left|\lambda_j\right|^p 2^{(j-k)(n/q'-\alpha)p} \\ &\le C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p \sum_{k\ge j} 2^{(j-k)(n/q'-\alpha)p} \le C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p. \end{aligned} \tag{64}$$

If $1 < p < \infty$, by the Hölder inequality, we obtain

$$\begin{aligned} J_1 &\le C \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} \left|\lambda_j\right|^p 2^{(j-k)(n/q'-\alpha)p/2} \\ &\quad \cdot \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n/q'-\alpha)p'/2} \right)^{1/p'} \\ &\le C \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} \left|\lambda_j\right|^p 2^{(j-k)(n/q'-\alpha)p/2} \\ &\le C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p \sum_{k\ge j} 2^{(j-k)(n/q'-\alpha)p/2} \le C \sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p. \end{aligned} \tag{65}$$

Hence,

$$\|Tf\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} \le C \|f\|_{H\dot{K}_{q,L}^{\alpha,p}(\mathbb{R}^n)}. \tag{66}$$

□

Theorem 16. *Let $0 < p < \infty$, $1 < q < \infty$, $0 < \alpha < n(1 - 1/q)$. Suppose that T is a sublinear operator as Theorem 15 and T and L are commutative. Then T is bounded on $H\dot{K}_{q,L}^{\alpha,p}(\mathbb{R}^n)$.*

Proof. Suppose a is an (α, q, M, L)-atom. According to Definition 9, there exists $b \in D(L^M)$, such that

(i) $a = L^M b$;

(ii) supp $L^j b \subset B(0, 2^l)$, $j = 0, 1, \cdots, M$;

(iii) $\|(r^2 L)^j b\|_{L^q(\mathbb{R}^n)} \le r^{2M} |B|^{-\alpha/n}$, $j = 0, 1, \cdots, M$.

Ta being an $(\alpha, q, M, L, \epsilon)$-molecule only needs to be prove, such that

(1) $Ta = L^M Tb$;

(2) $\|(r^2 L)^j Tb\|_{L^q(S_k(B))} \le 2^{-k\epsilon} r^{2M} |2^k B|^{-\alpha/n}$, $j = 0, 1, \cdots, M$ and $k = 0, 1, \cdots$.

In fact, it is enough to check (2). For any $j = 0, 1, \cdots, M$ and $k = 0, 1, \cdots$, we have

$$\begin{aligned} &\left\|\left(r^2 L\right)^j Tb\right\|_{L^q(S_k(B))} = \left(\int_{2^k B \setminus 2^{k-1} B} \left| T \left(r^2 L\right)^j b \right|^q dx \right)^{1/q} \\ &\le \left\{ \int_{2^k B \setminus 2^{k-1} B} |x|^{-nq} \left| \int_B \left(r^2 L\right)^j b \, dy \right|^q dx \right\}^{1/q} \\ &\le C 2^{-(k+l-1)n} \left\|2^k B\right\|^{1/q} \left\|\left(r^2 L\right)^j\right\|_{L^q(\mathbb{R}^n)} |B|^{1/q'} \\ &\le C 2^{-kn} 2^{-ln} 2^{n(k+l)/q} |B|^{-\alpha/n} r^{2M} 2^{ln/q'} \\ &\le C 2^{-kn(1-1/q-\alpha/n)} \left|2^k B\right|^{-\alpha/n} r^{2M}. \end{aligned} \tag{67}$$

Thus, we complete the proof of Theorem 16. □

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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