Research Article

Normal Families and Growth of Meromorphic Functions with Their K th Derivatives

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Relying on the normal family theory, we mainly study uniqueness problems of meromorphic functions and their k th derivatives and estimate sharply the growth order of their meromorphic functions. Our theorems improve some previous results.

1. Introduction and Main Results

In this paper, a meromorphic function means a function that is meromorphic in the whole complex plane $\mathbb{C}$. We will use the fundamental results and the standard notation of the Nevanlinna theory of meromorphic functions such as $T(r, f)$, $m(r, f)$, and $N(r, f)$ as explained in [1].

Let $\mathfrak{R}$ be a rational function (or a polynomial) that belongs asymptotically to kr^α as $r \rightarrow \infty$, where $k \neq 0$, α are constants. The degree of $\mathfrak{R}$ at infinity is defined as $\deg \mathfrak{R} := \deg_{\infty} \mathfrak{R} := \max\{0, \alpha\}$.

Suppose that $f(z)$ and $g(z)$ are two nonconstant meromorphic functions. We define $a(z)$ as a meromorphic function or a finite complex number. In this paper, we define $g(z) - a(z) = 0$ whenever $f(z) - a(z) = 0$; then we write $f(z) = a(z) \Rightarrow g(z) = a(z)$. If $f(z) = a(z) \Rightarrow g(z) = a(z)$ and $g(z) = a(z) \Rightarrow f(z) = a(z)$, then we write $f(z) = a(z) \Leftrightarrow g(z) = a(z)$ and say that $f(z)$ and $g(z)$ share $a(z)$ IM (ignoring multiplicity). If $f - a$ and $g - a$ have the same zeros with the same multiplicities, we write $f(z) = a(z) \equiv g(z) = a(z)$ and say that f and g share a CM (counting multiplicity).

The subject on sharing values between entire functions and their derivatives was first studied by Rubel and Yang (cf. [2]). In 1977, they proved a result that if a nonconstant entire function f and f' share two distinct finite numbers a, b CM, then $f \equiv f'$. Since then, sharing value problems have been studied by many authors and a number of profound results

have been obtained (cf. [1, 3, 4]). To state our main results, we also need the following concepts.

Definition 1. The order $\rho(f)$ and the hyperorder $\rho_2(f)$ of a meromorphic function f are both defined as follows:

$$\begin{aligned}\rho(f) &= \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}, \\ \rho_2(f) &= \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.\end{aligned}\tag{1}$$

On estimating the order or hyperorder of meromorphic functions, there has been a lot of work on the growth order of meromorphic function to certain types of complex differential equations and complex difference equations (or complex functional equations) (cf. [3, 5–8]).

In 2011, Lü and Yi ([9]) investigated the estimating order of meromorphic functions with their derivatives about sharing values and obtained the following results.

Theorem A. Let f be a nonconstant meromorphic function with finitely many poles, and let $\alpha = Q_1 e^Q$ and $\beta = Q_2 e^Q$, where $Q_1, Q_2 (\neq Q_1)$ and Q are three polynomials. Let k be a

positive integer. If all the zeros of $f - \alpha$ have multiplicity at least k and

$$\begin{aligned} f(z) &= \alpha(z) \implies \\ f^{(k)}(z) &= \alpha(z), \\ f(z) &= \beta(z) \iff \\ f^{(k)}(z) &= \beta(z), \end{aligned} \quad (2)$$

then $\rho(f) < +\infty$.

Theorem B. Let $Q_1 (\not\equiv 0)$ and Q_2 be two distinct polynomials, let k be a positive integer, and let f be a transcendental entire function with all the zeros of $f - Q_2$ having multiplicity at least k . If $f(z) = Q_1(z) \implies f^{(k)}(z) = Q_1(z)$ and $f(z) = Q_2(z) \implies f^{(k)}(z) = Q_2(z)$, then one of the following cases must occur:

- (1) $f = f^{(k)}$.
- (2) $f(z) = Q_2(z) + Ae^{\lambda z}$ and $(\mu - 1)Q_1 = \mu Q_2 - Q_2^{(k)}$, where A and $\lambda^k = \mu \neq 1$ are two nonzero constants.

In this paper, we continue to investigate the growth order and hyperorder of meromorphic functions with their derivatives about sharing functions and improve Theorems A and B. Now, we state our results as follows.

Theorem 2. Let f be a nonconstant meromorphic function with finitely many poles, and let $\alpha = R_1 e^\gamma$ and $\beta = R_2 e^\gamma$, where $R_1, R_2 (\not\equiv R_1)$ are two rational functions and γ is an entire function. Let k be a positive integer. If all the zeros of $f - \alpha$ have multiplicity at least k and

$$\begin{aligned} f(z) &= \alpha(z) \implies \\ f^{(k)}(z) &= \alpha(z), \\ f(z) &= \beta(z) \iff \\ f^{(k)}(z) &= \beta(z), \end{aligned} \quad (3)$$

then $\rho_2(f) \leq \rho_2(\alpha) = \rho(\gamma)$.

Remark 3. The following examples show that the conclusion $\rho_2(f) \leq \rho(\gamma)$ is sharp.

Example 4. Let $f(z) = Ae^z$, where A is a nonzero constant. Let $\alpha(z) = e^{-z+z}$ and $\beta(z) = ze^{-z+z}$. Noting that $f = f^{(k)}$, one has

$$\begin{aligned} f(z) &= \alpha(z) \implies \\ f^{(k)}(z) &= \alpha(z), \\ f(z) &= \beta(z) \iff \\ f^{(k)}(z) &= \beta(z). \end{aligned} \quad (4)$$

Thus, it satisfies the assumptions of Theorem 2 and $\rho_2(f) = 0 < \rho_2(\alpha) = 1$.

Example 5. Let $f(z) = ze^z + Ae^{3z}$, $\alpha(z) = ze^z$, and $\beta(z) = (z - 1/2)e^z$, where A is a nonzero constant. Differentiating f yields $f'(z) = (z + 1)e^z + 3Ae^{3z}$, and then $f(z) = \alpha(z) \implies f'(z) = \alpha(z)$ and $f(z) = \beta(z) \iff f'(z) = \beta(z)$. Thus, $\rho_2(f) \leq \rho_2(\alpha)$.

Theorem 6. Let f be a nonconstant meromorphic function with finitely many poles, and let $\alpha = Q_1 e^Q$ and $\beta = Q_2 e^Q$, where $Q_1, Q_2 (\not\equiv Q_1)$, and Q are three polynomials with $\deg Q_1 = q_1$, $\deg Q_2 = q_2$, and $\deg Q = q$, where q_1, q_2, q are three nonnegative integers. Let k be a positive integer. If all the zeros of $f - \alpha$ have multiplicity at least k and

$$\begin{aligned} f(z) &= \alpha(z) \implies \\ f^{(k)}(z) &= \alpha(z), \\ f(z) &= \beta(z) \iff \\ f^{(k)}(z) &= \beta(z), \end{aligned} \quad (5)$$

then $\rho(f) \leq \max\{q, q - (q_2 - q_1)/k\}$.

Remark 7. The next two examples can illustrate that the conclusion of Theorem 6 is meaningful.

Example 8. Let $f(z) = z^2 e^{z+1} + Ae^{2z}$, $\alpha(z) = z^2 e^{z+1}$, and $\beta(z) = (z^2 - (4/3)z - 2/3)e^{z+1}$, where A is a nonzero constant. Differentiating f twice yields $f'(z) = (z^2 + 2z)e^{z+1} + 2Ae^{2z}$ and $f''(z) = (z^2 + 4z + 2)e^{z+1} + 4Ae^{2z}$, and then $f(z) = \alpha(z) \implies f''(z) = \alpha(z)$ and $f(z) = \beta(z) \iff f''(z) = \beta(z)$. Thus, it satisfies the assumption of Theorem 6. Noting that $k = 2$, $\lambda = 2$, and $\mu = 4$, we obtain that $\rho(f) = 1 \leq \max\{q = 1, q - (q_2 - q_1)/k = 1 - (2 - 2)/2 = 1\}$ holds.

Example 9. Let $f(z) = z^2 e^{z^2+1} + Ae^{\sqrt{2}z}$, $\alpha(z) = z^2 e^{z^2+1}$, and $\beta(z) = (-4z^4 - 8z^2 - 2)e^{z^2+1}$, where A is a nonzero constant. Differentiating f twice yields $f'(z) = (2z^3 + 2z)e^{z^2+1} + \sqrt{2}Ae^{\sqrt{2}z}$ and $f''(z) = (4z^4 + 10z^2 + 2)e^{z^2+1} + 2Ae^{\sqrt{2}z}$, and then $f(z) = \alpha(z) \implies f''(z) = \alpha(z)$ and $f(z) = \beta(z) \iff f''(z) = \beta(z)$. Thus, it satisfies the assumption of Theorem 6. We have $k = 2$, $\lambda = \sqrt{2}$, $\mu = 2$. We obtain that $\rho(f) = 2 \leq \max\{q = 2, q - (q_2 - q_1)/k = 2 - (4 - 2)/2 = 1\}$ holds.

Theorem 10. Let f be a transcendental meromorphic function with finitely many poles, and let $\alpha = Q_1 e^Q$ and $\beta = Q_2 e^Q$, where $Q_1, Q_2 (\not\equiv Q_1)$, and Q are three polynomials. Let k be a positive integer. If all the zeros of $f - \alpha$ have multiplicity at least k and $\rho(\beta) < \rho(f)$ and

$$\begin{aligned} f(z) &= \alpha(z) \implies \\ f^{(k)}(z) &= \alpha(z), \\ f(z) &= \beta(z) \iff \\ f^{(k)}(z) &= \beta(z), \end{aligned} \quad (6)$$

then the conclusions of Theorem B still hold, and $Q(z)$ must be a constant.

The next example will show if f is rational; then Theorem 10 fails.

Example 11. Let $f(z) = z^3$, $Q_2(z) = (3/2)z^3 - (3/2)z^2$, and $Q_1 = Q = 0$. Note that

$$\begin{aligned} \frac{f' - Q_2}{f - Q_2} &= 3, \\ f &= 0 \implies \\ f' &= 0. \end{aligned} \quad (7)$$

However, it does not satisfy any case of Theorem B.

Remark 12. The condition $\rho(\beta) < \rho(f)$ plays an important pole in the proof of Theorem 10. Unfortunately, we do not know whether the condition could be weakened.

2. Some Lemmas

In order to prove our theorems, we need the following lemmas.

A family $\mathcal{F}$ of meromorphic functions defined on a domain $\Omega \subset \widehat{\mathbb{C}}$ (here $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) is said to be normal in Ω if every sequence of elements of $\mathcal{F}$ contains a subsequence that converges locally uniformly in Ω with respect to the spherical metric to a meromorphic function or ∞ (cf. [10]).

Lemma 13 (see [3, 11]). *Let $\{f_n\}$ be a family of meromorphic (analytic) functions on the unit disc Δ . If $a_n \rightarrow a$, $|a| < 1$, and $f_n^\#(a_n) \rightarrow \infty$, then there exist*

- (a) *a subsequence of f_n (which we still write as $\{f_n\}$);*
- (b) *points $z_n \rightarrow z_0$, $|z_0| < 1$;*
- (c) *positive numbers $\rho_n \rightarrow 0$,*

such that $f_n(z_n + \rho_n \xi) = g_n(\xi) \rightarrow g(\xi)$ locally uniformly, where g is a nonconstant meromorphic (entire) function on $\mathbb{C}$, such that

$$\rho_n \leq \frac{M}{f_n^\#(a_n)}, \quad (8)$$

where M is a constant that is independent of n .

Here, as usual, $f^\#(\xi) = |f'(\xi)|/(1 + |f(\xi)|^2)$ is the spherical derivative.

The next lemma is an extending result obtained by Lü and Qi in [12].

Lemma 14. *Let f be a meromorphic function of hyperorder $\rho_2(f) > 0$. Then, for any $\epsilon > 0$, there exists a sequence $z_n \rightarrow \infty$ such that*

$$f^\#(z_n) > e^{|z_n|^{d-\epsilon}} \quad (9)$$

for large enough n , where $d = \rho_2(f)$ if $\rho_2(f) < \infty$ or d is an arbitrary positive number.

Lemma 15 (cf. [5]). *Let $f(z)$ be an entire function; $\rho(f) > 1$. Then, for each $0 < \mu < \rho(f) - 1$, there exist points $a_n \rightarrow \infty$ ($n \rightarrow \infty$), such that*

$$\lim_{n \rightarrow \infty} \frac{f^\#(a_n)}{|a_n|^\mu} = +\infty. \quad (10)$$

Lemma 16 (cf. [1]). *Suppose that $f(z)$ and $g(z)$ are two nonconstant meromorphic functions in the complex plane with $\rho(f)$ and $\rho(g)$ as their orders, respectively. Then*

$$\begin{aligned} \rho(fg) &\leq \max\{\rho(f), \rho(g)\}, \\ \rho(f+g) &\leq \max\{\rho(f), \rho(g)\}. \end{aligned} \quad (11)$$

Lemma 17 (cf. [13]). *Let $f(z)$ and $a(z)$ be meromorphic functions of finite order such that both of $f(z)$ and $a(z)$ have only finite many poles, $f(z)$ and $a(z)$ have no common poles, and the order of $a(z)$ is less than the order of $f(z)$. If $f(z)$ and $f^{(k)}(z)$ share $a(z)$ CM, then $f^{(k)}(z) - a(z) = c(f(z) - a(z))$ for some nonzero constant c .*

3. The Proof of Theorem 2

By the method in [3, 12, 14–16], we will present our proof as follows.

Note that $\alpha = R_1 e^\gamma$; then $\rho_2(\alpha) = \rho(\gamma)$. It suffices to show $\rho_2(f) \leq \rho(\gamma)$.

By reduction to absurdity, we assume that $\rho_2(f) > \rho(\gamma)$. Set $\rho_2(f) = d > c := \rho(\gamma)$, and set $H := f - \alpha$; then

- (i) $H = 0 \implies H^{(k)} = \alpha - \alpha^{(k)}$,
- (ii) $H = \beta - \alpha \iff H^{(k)} = \beta - \alpha^{(k)}$,

where $\alpha^{(k)} = \phi e^\gamma$ and ϕ is a differential polynomial of R_1 and γ . Let

$$\begin{aligned} Y &= \beta - \alpha = (R_2 - R_1) e^\gamma =: Re^\gamma, \\ \theta &= \beta - \alpha^{(k)} = (R_2 - \phi) e^\gamma =: \Psi e^\gamma. \end{aligned} \quad (12)$$

We may set $F = H/Y$ and get $\rho_2(F) = \rho_2(f) = d$. By Lemma 14, for $0 < \epsilon < (d - c)/2$, there exists a sequence $w_n \rightarrow \infty$ as $n \rightarrow \infty$ such that

$$F^\#(w_n) > e^{|w_n|^{\rho_2(F)-\epsilon}} = e^{|w_n|^{d-\epsilon}}. \quad (13)$$

Note that $Y = \beta - \alpha = (R_2 - R_1)e^\gamma$ has at most finitely many zeros and poles and $f - \alpha$ has at most finitely many poles. Then there exists a positive number r such that F has no poles in $D = \{z : |z| > r\}$.

In view of $w_n \rightarrow \infty$ as $n \rightarrow \infty$, without loss of generality, we assume that $|w_n| \geq r + 1$ for all n . Define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) := F(w_n + z) = \frac{H(w_n + z)}{Y(w_n + z)}, \quad (14)$$

and then every F_n is analytic in D_1 . Also $F_n^\#(0) = F^\#(w_n) \rightarrow \infty$ as $n \rightarrow \infty$. It follows from Marty's criterion that F_n is not normal at $z = 0$.

By Lemma 13 and choosing an appropriate subsequence of F_n if necessary, we may assume that there exist sequences z_n and ρ_n with $|z_n| < r < 1$ and $\rho_n \rightarrow 0$ such that the sequence g_n is defined by

$$g_n(\zeta) = F_n(z_n + \rho_n \zeta) = \frac{H(w_n + z_n + \rho_n \zeta)}{Y(w_n + z_n + \rho_n \zeta)} \rightarrow g(\zeta) \quad (15)$$

locally uniformly in $\mathbb{C}$, where g is a nonconstant entire function of order at most 1, whose zeros have multiplicity at least k and

$$\rho_n \leq \frac{M}{F_n^\#(0)} = \frac{M}{F^\#(w_n)} \leq M e^{-|w_n|^{d-\epsilon}} \quad (16)$$

for a positive number M .

We assert that

$$\rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n \zeta)}{Y(w_n + z_n + \rho_n \zeta)} \rightarrow g^{(k)}(\zeta). \quad (17)$$

We prove the assertion by induction. By (15), the assertion holds for $k = 0$. We assume that the assertion is right for $k = l$; that is,

$$\rho_n^l \frac{H^{(l)}(w_n + z_n + \rho_n \zeta)}{Y(w_n + z_n + \rho_n \zeta)} \rightarrow g^{(l)}(\zeta). \quad (18)$$

Now we will prove that the assertion is also right for $k = l + 1$. Let

$$G_n(\zeta) = \rho_n^l \frac{H^{(l)}(w_n + z_n + \rho_n \zeta)}{Y(w_n + z_n + \rho_n \zeta)}, \quad (19)$$

and then

$$\begin{aligned} G_n'(\zeta) &= \rho_n^{l+1} \frac{H^{(l+1)}(w_n + z_n + \rho_n \zeta)}{Y(w_n + z_n + \rho_n \zeta)} \\ &\quad - \rho_n^{l+1} \frac{H^{(l)}(w_n + z_n + \rho_n \zeta) Y'(w_n + z_n + \rho_n \zeta)}{Y^2(w_n + z_n + \rho_n \zeta)} \\ &= \rho_n^{l+1} \frac{H^{(l+1)}(w_n + z_n + \rho_n \zeta)}{Y(w_n + z_n + \rho_n \zeta)} \\ &\quad - \rho_n \frac{G_n(\zeta) Y'(w_n + z_n + \rho_n \zeta)}{Y(w_n + z_n + \rho_n \zeta)} \rightarrow g^{(l+1)}(\zeta). \end{aligned} \quad (20)$$

Recall the above definition of degree of rational function; we have

$$\begin{aligned} \left| \frac{Y'}{Y} \right|_{z=w_n+z_n+\rho_n \zeta} &= \left| \frac{R' + RY'}{R} \right|_{z=w_n+z_n+\rho_n \zeta} \\ &\leq |w_n|^q M (|w_n + z_n + \rho_n \zeta|, \gamma') \\ &\leq |w_n|^q M (2|w_n|, \gamma') \\ &\leq |w_n|^q e^{A|w_n|^{c+\epsilon}}, \end{aligned} \quad (21)$$

and here A is a positive constant and q is an integer. Noting that $0 < \epsilon < (d-c)/2$, we have $d-\epsilon > c+\epsilon$. Then, combining (16) and (21), we conclude that

$$\left| \rho_n G_n(\zeta) \frac{Y'(w_n + z_n + \rho_n \zeta)}{Y(w_n + z_n + \rho_n \zeta)} \right| \leq M |G_n(\zeta)| |w_n|^q e^{A|w_n|^{c+\epsilon} - |w_n|^{d-\epsilon}} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (22)$$

Then

$$\rho_n^{l+1} \frac{H^{(l+1)}(w_n + z_n + \rho_n \zeta)}{Y(w_n + z_n + \rho_n \zeta)} \rightarrow g^{(l+1)}(\zeta). \quad (23)$$

Hence, we complete the proof of the assertion.

Obviously, $g^{(k)} \neq 0$. Otherwise g would be a polynomial of degree less than k and could not have zeros of multiplicity at least k . In the following, we will prove that

$$(a) \quad g = 0 \Rightarrow g^{(k)} = 0,$$

$$(b) \quad g = 1 \Leftrightarrow g^{(k)} = 0.$$

Firstly, we prove (a). Suppose that $g(\zeta_0) = 0$. By Hurwitz's theorem, there exists a sequence $(\zeta_n)_n, \zeta_n \rightarrow \zeta_0$, such that (for n being sufficiently large)

$$g_n(\zeta_n) = \frac{H(w_n + z_n + \rho_n \zeta_n)}{Y(w_n + z_n + \rho_n \zeta_n)} = 0. \quad (24)$$

Thus, $H(w_n + z_n + \rho_n \zeta_n) = 0$ and $H^{(k)}(w_n + z_n + \rho_n \zeta_n) = (\alpha - \alpha^{(k)})(w_n + z_n + \rho_n \zeta_n) = (R_1 - \phi)e^\gamma(w_n + z_n + \rho_n \zeta_n)$.

By (16), (17), (21), and (22), we derive that

$$\begin{aligned} g^{(k)}(\zeta_0) &= \lim_{n \rightarrow \infty} \rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n \zeta_n)}{Y(w_n + z_n + \rho_n \zeta_n)} \\ &= \lim_{n \rightarrow \infty} \rho_n^k \frac{(R_1 - \phi)(w_n + z_n + \rho_n \zeta_n)}{(R_2 - R_1)(w_n + z_n + \rho_n \zeta_n)} = 0. \end{aligned} \quad (25)$$

Thus, $g(\zeta) = 0 \Rightarrow g^{(k)}(\zeta) = 0$. So (a) holds.

Similarly, we can prove $g = 1 \Rightarrow g^{(k)} = 0$. Now we prove

$$\begin{aligned} g^{(k)} &= 0 \Rightarrow \\ g &= 1. \end{aligned} \quad (26)$$

From (16) and (17), we have

$$\rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n \zeta) - \theta(w_n + z_n + \rho_n \zeta)}{\Upsilon(w_n + z_n + \rho_n \zeta)} \longrightarrow g^{(k)}(\zeta). \quad (27)$$

Suppose that $g^{(k)}(\eta_0) = 0$. Then $g^{(k)} \not\equiv 0$. Hence, by (27) and Hurwitz's theorem, there exists a sequence $(\eta_n)_n$, $\eta_n \rightarrow \eta_0$, such that (for n being sufficiently large)

$$H^{(k)}(w_n + z_n + \rho_n \eta_n) = \theta(w_n + z_n + \rho_n \eta_n). \quad (28)$$

Then, it follows from the assumption that we get $H(w_n + z_n + \rho_n \eta_n) = \Upsilon(w_n + z_n + \rho_n \eta_n)$. It implies that

$$g(\eta_0) = \lim_{r \rightarrow \infty} \frac{H(w_n + z_n + \rho_n \eta_n)}{\Upsilon(w_n + z_n + \rho_n \eta_n)} = 1. \quad (29)$$

Thus, we prove that $g^{(k)} = 0 \Rightarrow g = 1$ and obtain (b).

From (a) and (b), it is easy to deduce that $g \neq 0$. Thus, $g = Ae^{\lambda z}$; it contradicts with $g = 1 \Rightarrow g^{(k)} = 0$. So $\rho_2(f) \leq \rho(\gamma)$. Therefore, we complete the proof of Theorem 2.

4. The Proof of Theorem 6

Using some ideas of [3, 14], we give the following proof of Theorem 6.

Let $H = f - \alpha$. Then all the zeros of H have multiplicities at least k and

$$\begin{aligned} H = 0 &\implies \\ H^{(k)} &= \alpha - \alpha^{(k)}, \\ H &= \beta - \alpha \iff \\ H^{(k)} &= \beta - \alpha^{(k)}, \end{aligned} \quad (30)$$

where $\alpha^{(k)} = P_1 e^Q$ and P_1 is a polynomial. A careful calculation leads us to obtain $\deg P_1 = k \deg Q' + \deg Q_1 = k(q-1) + q_1$. Let

$$\begin{aligned} \gamma &= \beta - \alpha = (Q_2 - Q_1) e^Q := P e^Q, \\ \theta &= \beta - \alpha^{(k)} = (Q_2 - P_1) e^Q := P_2 e^Q, \end{aligned} \quad (31)$$

where P_2 is a polynomial. Set $F = H/\gamma$.

If $\rho(F) > \max\{q, q - (q_2 - q_1)/k\}$, by Lemma 15, for each $0 < \mu < \rho(F) - 1$, there exist $w_n \rightarrow \infty$ such that, for $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{F^\#(w_n)}{|w_n|^\mu} = +\infty. \quad (32)$$

In view of $w_n \rightarrow \infty$ as $n \rightarrow \infty$, without loss of generality, we may assume that $|w_n| \geq r+1$ for all n . Define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) := F(w_n + z) = \frac{H(w_n + z)}{\gamma(w_n + z)}. \quad (33)$$

Note that $\gamma = \beta - \alpha = (Q_2 - Q_1)e^Q$ has at most finitely many zeros and $f - \alpha$ has at most finitely poles; then there exists a positive number r such that F has no poles in $D = \{z : |z| > r\}$, so $F_n(z) = F(w_n + z)$ is analytic in D_1 . Therefore, all $F_n(z)$ are analytic in D_1 . Thus, we have structured a family $(F_n)_n$ of holomorphic functions.

Also $F_n^\#(0) = F^\#(w_n) \rightarrow \infty$ as $n \rightarrow \infty$. It follows from Marty's criterion that F_n is not normal at $z = 0$. Therefore, by applying Lemma 13 and choosing an appropriate subsequence of F_n if necessary, we may assume that there exist subsequences z_n and ρ_n with $|z_n| < r < 1$ and $\rho_n \rightarrow 0$ such that the sequence g_n is defined by

$$g_n(\zeta) := F_n(z_n + \rho_n \zeta) = \frac{H(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \longrightarrow g(\zeta) \quad (34)$$

locally uniformly in $\mathbb{C}$, where g is a nonconstant entire function of order at most 1, whose zeros have multiplicity at least k and

$$\rho_n \leq \frac{M}{F_n^\#(0)} = \frac{M}{F^\#(w_n)} \leq M |w_n|^{-\mu} \quad (35)$$

for a positive number M .

We assert that

$$\rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \longrightarrow g^{(k)}(\zeta). \quad (36)$$

We will use the mathematical induction to prove the assertion. By (34), the assertion is right for $k = 0$. We assume that the assertion is right for $k = l$; that is,

$$\rho_n^l \frac{H^{(l)}(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \longrightarrow g^{(l)}(\zeta). \quad (37)$$

Now we prove that the assertion is also right when $k = l + 1$.
Let

$$G_n(\zeta) = \rho_n^l \frac{H^{(l)}(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)}, \quad (38)$$

and then

$$\begin{aligned} G_n'(\zeta) &= \rho_n^{l+1} \frac{H^{(l+1)}(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \\ &\quad - \rho_n^{l+1} \frac{H^{(l)}(w_n + z_n + \rho_n \zeta) \gamma'(w_n + z_n + \rho_n \zeta)}{\gamma^2(w_n + z_n + \rho_n \zeta)} \quad (39) \\ &= \rho_n^{l+1} \frac{H^{(l+1)}(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \\ &\quad - \rho_n \frac{G_n(\zeta) \gamma'(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \rightarrow g^{(l+1)}(\zeta). \end{aligned}$$

Since $\gamma = Pe^Q$, we have

$$\begin{aligned} &\rho_n \frac{G_n(\zeta) \gamma'(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \\ &= \rho_n \frac{G_n(\zeta) (P' + PQ')(w_n + z_n + \rho_n \zeta)}{P(w_n + z_n + \rho_n \zeta)}. \end{aligned} \quad (40)$$

In view of the degree of a polynomial function, one has

$$\begin{aligned} \left| \frac{\gamma'}{\gamma} \right|_{z=w_n+z_n+\rho_n\zeta} &= \left| \frac{P' + PQ'}{P} \right|_{z=w_n+z_n+\rho_n\zeta} \\ &= \left| \left(\frac{P'}{P} + Q' \right) \right|_{z=w_n+z_n+\rho_n\zeta} \quad (41) \\ &= O(|w_n|^{l_1}) \quad (\text{as } n \rightarrow \infty), \end{aligned}$$

where $l_1 := \deg Q'$ is a nonnegative integer. Then, combining (35) and (41) yields

$$\left| \rho_n G_n(\zeta) \frac{\gamma'(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \right| \leq M |G_n(\zeta)| |w_n|^{l_1 - \mu} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (42)$$

Furthermore,

$$\rho_n^{l+1} \frac{H^{(l+1)}(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \rightarrow g^{(l+1)}(\zeta). \quad (43)$$

So we complete the proof of the assertion.

Obviously, $g^{(k)} \neq 0$. Otherwise, g would be a polynomial of degree less than k and so could not have zeros of multiplicity at least k . In the following, we will prove

$$(c) \ g = 0 \Rightarrow g^{(k)} = 0,$$

$$(d) \ g = 1 \Leftrightarrow g^{(k)} = 0.$$

Firstly, we prove (c). Suppose that $g(\zeta_0) = 0$. Then, by Hurwitz's theorem, we get that there exists a sequence $(\zeta_n)_n$, $\zeta_n \rightarrow \zeta_0$, such that (for n being sufficiently large)

$$g_n(\zeta_n) = \frac{H(w_n + z_n + \rho_n \zeta_n)}{\gamma(w_n + z_n + \rho_n \zeta_n)} = 0. \quad (44)$$

Thus, $H(w_n + z_n + \rho_n \zeta_n) = 0$ and $H^{(k)}(w_n + z_n + \rho_n \zeta_n) = (Q_1 - P_1)e^Q(w_n + z_n + \rho_n \zeta_n)$. By (36) and (39), we derive that

$$\begin{aligned} g^{(k)}(\zeta_0) &= \lim_{n \rightarrow \infty} \rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n \zeta_n)}{\gamma(w_n + z_n + \rho_n \zeta_n)} \\ &= \lim_{n \rightarrow \infty} \rho_n^k \frac{(Q_1 - P_1)(w_n + z_n + \rho_n \zeta_n)}{P(w_n + z_n + \rho_n \zeta_n)}. \end{aligned} \quad (45)$$

Also,

$$\begin{aligned} \left| \frac{Q_1 - P_1}{P} \right|_{z=w_n+z_n+\rho_n\zeta} &= \left| \frac{Q_1 - P_1}{Q_2 - Q_1} \right|_{z=w_n+z_n+\rho_n\zeta} \\ &= \left| \left(\frac{1 - P_1/Q_1}{Q_2/Q_1 - 1} \right) \right|_{z=w_n+z_n+\rho_n\zeta} \quad (46) \\ &= O(|w_n|^{l_2}), \quad (\text{as } n \rightarrow \infty) \end{aligned}$$

where $l_2 := \max\{k(q-1) - (q_2 - q_1), 0\}$ is a nonnegative integer. Then, combining (35) and (46) yields

$$\left| \rho_n^k \frac{(Q_1 - P_1)(w_n + z_n + \rho_n \zeta_n)}{P(w_n + z_n + \rho_n \zeta_n)} \right| \leq M^k |w_n|^{l_2 - k\mu} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (47)$$

So,

$$\begin{aligned} g^{(k)}(\zeta_0) &= \lim_{n \rightarrow \infty} \rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n \zeta_n)}{\gamma(w_n + z_n + \rho_n \zeta_n)} \\ &= \lim_{n \rightarrow \infty} \rho_n^k \frac{(Q_1 - P_1)(w_n + z_n + \rho_n \zeta_n)}{P(w_n + z_n + \rho_n \zeta_n)} = 0. \end{aligned} \quad (48)$$

Thus, $g(\zeta) = 0 \Rightarrow g^{(k)}(\zeta) = 0$ and (c) holds.

Next, we will prove that $g = 1 \Rightarrow g^{(k)} = 0$.

Similar to what was stated above, suppose that $g(\zeta_0) = 1$. Then, by Hurwitz's theorem, there exists a sequence $(\zeta_n)_n$, $\zeta_n \rightarrow \zeta_0$, such that (for n being sufficiently large)

$$g_n(\zeta_n) = \frac{H(w_n + z_n + \rho_n \zeta_n)}{\gamma(w_n + z_n + \rho_n \zeta_n)} = 1. \quad (49)$$

Thus, $H(w_n + z_n + \rho_n \zeta_n) = \gamma(w_n + z_n + \rho_n \zeta_n)$ and $H^{(k)}(w_n + z_n + \rho_n \zeta_n) = (Q_2 - P_1)e^Q(w_n + z_n + \rho_n \zeta_n)$. By (35) and (36), we deduce that

$$\begin{aligned} g^{(k)}(\zeta_0) &= \lim_{n \rightarrow \infty} \rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n \zeta_n)}{\gamma(w_n + z_n + \rho_n \zeta_n)} \\ &= \lim_{n \rightarrow \infty} \rho_n^k \frac{(Q_2 - P_1)(w_n + z_n + \rho_n \zeta_n)}{(Q_2 - Q_1)(w_n + z_n + \rho_n \zeta_n)}. \end{aligned} \quad (50)$$

By calculation,

$$\begin{aligned} \left| \frac{Q_2 - P_1}{P} \right|_{z=w_n+z_n+\rho_n\zeta_n} &= \left| \frac{Q_2 - P_1}{Q_2 - Q_1} \right|_{z=w_n+z_n+\rho_n\zeta_n} \\ &= \left| \left(\frac{Q_2/Q_1 - P_1/Q_1}{Q_2/Q_1 - 1} \right) \right|_{z=w_n+z_n+\rho_n\zeta_n} = O(|w_n|^{l_3}), \end{aligned} \quad (51)$$

(as $n \rightarrow \infty$)

where $l_3 := \max\{k(q-1) - (q_2 - q_1), 0\} = l_2$ is a nonnegative integer. Then, combining (35) and (51) shows that

$$\left| \rho_n^k \frac{(Q_2 - P_1)(w_n + z_n + \rho_n \zeta_n)}{P(w_n + z_n + \rho_n \zeta_n)} \right| \leq M^k |w_n|^{l_3 - k\mu} \rightarrow 0, \quad (52)$$

as $n \rightarrow \infty$.

So,

$$\begin{aligned} g^{(k)}(\zeta_0) &= \lim_{n \rightarrow \infty} \rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n \zeta_n)}{\gamma(w_n + z_n + \rho_n \zeta_n)} \\ &= \lim_{n \rightarrow \infty} \rho_n^k \frac{(Q_2 - P_1)(w_n + z_n + \rho_n \zeta_n)}{P(w_n + z_n + \rho_n \zeta_n)} = 0. \end{aligned} \quad (53)$$

It implies that $g(\zeta) = 1 \Rightarrow g^{(k)}(\zeta) = 0$. Now, we prove

$$\begin{aligned} g^{(k)} &= 0 \implies \\ g &= 1. \end{aligned} \quad (54)$$

From (35), (36), (51), and (52), we have

$$\rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n \zeta) - \theta(w_n + z_n + \rho_n \zeta)}{\gamma(w_n + z_n + \rho_n \zeta)} \rightarrow g^{(k)}(\zeta). \quad (55)$$

Suppose that $g^{(k)}(\eta_0) = 0$. Since $g^{(k)} \neq 0$, by (55) and Hurwitz's theorem, there exists a sequence η_n , $\eta_n \rightarrow \eta_0$, such that (for n being sufficiently large)

$$H^{(k)}(w_n + z_n + \rho_n \eta_n) = \theta(w_n + z_n + \rho_n \eta_n). \quad (56)$$

By the assumption, we get $H(w_n + z_n + \rho_n \eta_n) = \gamma(w_n + z_n + \rho_n \eta_n)$ and

$$g(\eta_0) = \lim_{r \rightarrow \infty} \frac{H(w_n + z_n + \rho_n \eta_n)}{\gamma(w_n + z_n + \rho_n \eta_n)} = 1. \quad (57)$$

Thus, we prove $g^{(k)} = 0 \Rightarrow g = 1$ and obtain (d).

From (c) and (d), it is easy to deduce that $g \neq 0$. Thus, $g = Ae^{\lambda z}$. But it contradicts with $g = 1 \Rightarrow g^{(k)} = 0$. So $\rho(F) \leq \max\{q, q - (q_2 - q_1)/k\}$.

Next, we will prove $\rho(f) \leq \rho(F) \leq \max\{q, q - (q_2 - q_1)/k\}$.

If $\rho(\alpha) < \rho(f)$, noting the fact that $\rho(\beta - \alpha) \leq \rho(\alpha)$ and Lemma 16, then $\rho(F(\beta - \alpha)) \leq \max\{\rho(F), \rho(\alpha)\}$. Rewrite the function $F = (f - \alpha)/(\beta - \alpha)$ as $f = \alpha + F(\beta - \alpha)$. Thus, by Lemma 16, we conclude that $\rho(f) \leq \max\{\rho(\alpha), \rho(F(\beta - \alpha))\} \leq \max\{\rho(\alpha), \rho(F)\}$. Furthermore, it follows from $\rho(\alpha) < \rho(f)$ that $\rho(f) \leq \rho(F) \leq \max\{q, q - (q_2 - q_1)/k\}$.

If $\rho(\alpha) = \rho(f)$, noting that $\rho(\alpha) = \deg Q$, then $\rho(f) = \rho(\alpha) = \deg Q = q \leq \max\{q, q - (q_2 - q_1)/k\}$.

If $\rho(\alpha) > \rho(f)$, noting that $F = (f - \alpha)/(\beta - \alpha) = (f - Q_1 e^Q)/(Q_2 - Q_1)e^Q = f/Pe^Q - Q_1/P$ and Q_1/P is a rational function, then $\rho(Q_1/P) = 0$ and $\rho(\alpha) = \deg Q = \rho(Pe^Q) = q > \rho(f)$. So, $\rho(f) < \deg Q = \rho(F) \leq \max\{q, q - (q_2 - q_1)/k\}$.

Therefore, $\rho(f) \leq \rho(F) \leq \max\{q, q - (q_2 - q_1)/k\}$ holds. This completes the proof of Theorem 6.

5. The Proof of Theorem 10

Based on some ideas in [9], we will prove the theorem.

By Theorem 6, Lemma 17, and the assumption of Theorem 10, we have

$$\frac{f^{(k)} - \beta}{f - \beta} = \mu, \quad (58)$$

where μ is a nonzero constant. It follows that

$$f^{(k)} = \mu f + (1 - \mu) Q_2 e^Q. \quad (59)$$

If $\mu = 1$, then $f = f^{(k)}$; this is result (1) in Theorem B. Assuming that $\mu \neq 1$ and z_0 is a zero of $f - Q_1 e^Q$, we have $f^{(k)}(z_0) - Q_1(z_0)e^{Q(z_0)} = 0$. Substituting z_0 into (59), we have

$$Q_2(z_0) - Q_1(z_0) = 0. \quad (60)$$

Noting that $Q_2 - Q_1 \neq 0$, we know that all the zeros of $f - Q_1 e^Q$ are the zeros of $Q_2 - Q_1$, so $f - Q_1 e^Q$ has only finitely many zeros. Set $f = Q_1 e^Q + P_1 e^{Q_1}$, where P_1 is a nonzero polynomial

and Q_* is a polynomial. Then $f^{(k)} = P_2 e^Q + (P_1 Q_*^k + P_3) e^{Q_*}$, where $P_2 = (Q_1 e^Q)^{(k)}/e^Q$ is a polynomial with $\deg P_2 = \deg Q_1 Q_*^k$ and P_3 is a polynomial with $\deg P_3 < \deg P_1 Q_*^k$. Substituting them into (59), we have

$$\begin{aligned} & P_2 e^Q + (P_1 Q_*^k + P_3) e^{Q_*} \\ &= \mu Q_1 e^Q + \mu P_1 e^{Q_*} + (1 - \mu) Q_2 e^Q. \end{aligned} \quad (61)$$

Then,

$$P_2 = \mu Q_1 + (1 - \mu) Q_2, \quad (62)$$

$$P_1 (Q'_*)^k + P_3 = \mu P_1. \quad (63)$$

From (63), we get $Q'_* = \mu$ and $P_3 = 0$. Thus Q'_* is a constant, say $Q'_* = \lambda$. So $Q_*(z) = \lambda z + b$ and $\mu = \lambda^k$, where b is a constant. Furthermore,

$$f(z) = Q_1(z) e^{Q(z)} + P_1(z) e^{\lambda z + b}, \quad (64)$$

$$f^{(k)}(z) = P_2 e^Q + (P_1 \lambda^k + P_3) e^{\lambda z + b}. \quad (65)$$

We deduce that $\deg(P_3) = \deg(P'_1)$. Since $P_3 = 0$, we get that P_1 is a constant. Set $A = P_1 e^b$ as a constant. Due to the fact that $\rho(\beta) < \rho(f)$ and (64), $Q(z)$ must be a constant. Hence, we complete the proof of Theorem 10.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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