Research Article

Sarason’s Conjecture of Toeplitz Operators on Fock-Sobolev Type Spaces

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In this note, we will solve Sarason’s conjecture on the Fock-Sobolev type spaces and give a well solution that if Toeplitz product \( T_u T_v \), with the symbols \( u \) and \( v \), is bounded if and only if \( u = e^{q} \), \( v = Ce^{-q} \), where \( q \) is a linear complex polynomial and \( C \) is a nonzero constant.

1. Introduction

Let \( \mathbb{C}^n \) denote the complex \( n \)-space and \( d\nu \) be the ordinary volume measure on \( \mathbb{C}^n \) that is normalized so that
\[
\int_{\mathbb{C}^n} e^{-|z|^2} d\nu(z) = 1.
\]
If given any two points \( z = (z_1, z_2, \ldots, z_n) \) and \( w = (w_1, w_2, \ldots, w_n) \) in \( \mathbb{C}^n \), we denote
\[
|z| = \sqrt{z \cdot \bar{z}}.
\]
For every \( 0 < p < \infty \), \( \alpha \in \mathbb{R} \), we denote by \( L^p_\alpha(\mathbb{C}^n) \) the space of measurable functions \( f \) such that
\[
\|f\|_{L^p_\alpha} = \left( \int_{\mathbb{C}^n} |f(z)|^p e^{-|z|^{2\alpha}} \frac{d\nu(z)}{(1+|z|)^\alpha} \right)^{1/p} < \infty.
\]
Let \( H(\mathbb{C}^n) \) be the set of entire functions on \( \mathbb{C}^n \). Then for a given \( 0 < p < \infty \), the Fock-Sobolev type space \( F^p_\alpha \) with the norm \( \| f \|_{F^p_\alpha} = \| f \|_{L^p_\alpha} \) is defined as
\[
F^p_\alpha = \{ f \in H(\mathbb{C}^n) \mid \| f \|_{L^p_\alpha} < \infty \}.
\]
Obviously, the Fock-Sobolev type space \( F^2_\alpha \) equipped with the natural inner product defined by
\[
\langle f, g \rangle_{L^2_\alpha} = \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-|z|^2} \frac{d\nu(z)}{(1+|z|)^\alpha},
\]
is a reproducing kernel Hilbert space for every real \( \alpha \). As stated in [1], with respect to the above inner product, it is difficult to compute the reproducing kernel of \( F^2_\alpha \) explicitly. So we use the equivalent norm with respect to a new measure \( |z|^{-\alpha} d\nu(z) \). In more detail, for \( \alpha \leq 0 \) we will let
\[
\langle f, g \rangle_\alpha = \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-|z|^2} \frac{d\nu(z)}{|z|^{\alpha}},
\]
and for \( \alpha > 0 \) we let
\[
\langle f, g \rangle_\alpha = \int_{\mathbb{C}^n} f_{\alpha/2}^-(z) \overline{g_{\alpha/2}^-(z)} e^{-|z|^2} d\nu(z)
+ \int_{\mathbb{C}^n} f_{\alpha/2}^+(z) \overline{g_{\alpha/2}^+(z)} e^{-|z|^2} d\nu(z),
\]
where \( f_{\alpha/2}^- \) is the Taylor expansion of \( f \) up to order \( \alpha/2 \) and \( f_{\alpha/2}^+ = f - f_{\alpha/2}^- \). Now we can bravely make sure that the inner product \( \langle \cdot, \cdot \rangle_\alpha \) generates a new Hilbert space norm on \( F^2_\alpha \) that is equivalent to the \( F^2_\alpha \) norm \( \| \cdot \|_{F^2_\alpha} \). In particular, if we define the norm \( \| f \|_{\tilde{F}^2_\alpha} \) on \( F^2_\alpha \) by, when \( \alpha \leq 0 \),
\[
\| f \|_{\tilde{F}^2_\alpha} = \left( \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^{2\alpha}} \frac{d\nu(z)}{|z|^{2\alpha}} \right)^{1/2},
\]
and when $\alpha > 0$,
\[
\|f\|_{F_{\alpha}}^2 = \left( \int_{\mathbb{C}} |(f)_{\alpha/2}(z)|^2 e^{-|z|^2} dv(z) \right)^{1/2} + \left( \int_{\mathbb{C}} |(f)_{\alpha/2}(z)|^2 e^{-|z|^2} dv(z) \right)^{1/2},
\]
and then we have that both $\| \cdot \|_{F_{\alpha}}$ and $\| \cdot \|_{E_{\alpha}}$ are equivalent norms.

As is well known, $F_{\alpha}$ is indeed a reproducing kernel Hilbert space (see Lemma 2.1 of [1] for more details). Therefore its reproducing kernel is
\[
K_{\alpha}^z (w) = \sum_{\beta} \phi_{\beta} (w) \overline{\phi}_{\beta} (z),
\]
where $\{\phi_\beta\}$ is any orthonormal basis for $F_{\alpha}^2$ with respect to $\langle \cdot , \cdot \rangle_\alpha$. Note that polynomials form a dense subset of $F_{\alpha}^2$ (see Proposition 2.3 in [2]). Also note that monomials are mutually orthogonal, which means that $\{z^\beta / \sqrt{\beta!} \}_{\beta \in \mathbb{N}}$ is an orthonormal basis for $F_{\alpha}^2$. The arguments that are identical to the ones in the proof of Theorem 4.5 in [2] then give us that
\[
K_{\alpha}^z (w) = \begin{cases} \mathcal{I}^{-\alpha/2} K_{\alpha} (w), & \text{if } \alpha \leq 0; \\ \mathcal{I}^{-\alpha/2} K_{\alpha} (w) + (K_{\alpha/2})_\alpha (w), & \text{if } \alpha > 0. \end{cases}
\]
Here $\mathcal{I}^s$ is the fractional integration operator defined as
\[
\mathcal{I}^s f(z) = \left\{ \sum_{k=s}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+s+k)} f_k (z), \right. & \text{if } s \geq 0; \\ \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+s+k)} f_k (z), & \text{if } s < 0,
\]
where each $f_k$ is a polynomial of degree $k$. Moreover, for $s > 0$, $f_s^+$ is the tail part of the Taylor expansion of $f$ of degree higher than $|s|$ given by
\[
 f_s^+ (z) = \sum_{k>|s|} f_k (z)
\]
and we let $f_s^- = f - f_s^+$ (see [2] for more information on fractional differentiation and integration).

Now it is easy to see that if $\alpha \leq 0$, $(F_{\alpha}^2 , \| \cdot \|_{F_{\alpha}})$ is a closed subspace of $L_{\alpha}^2$ with respect to $\langle \cdot , \cdot \rangle_\alpha$. In this case, let $P_{\alpha}$ denote the orthogonal projection, so that
\[
P_{\alpha} f (z) = \langle f , K_{\alpha}^z \rangle_\alpha
\]
for any $f \in L_{\alpha}^2$. Unfortunately, the inner product $\langle \cdot , \cdot \rangle_\alpha$ does not make sense on $L_{\alpha}^2$ when $\alpha > 0$. That means we cannot define the Toeplitz operator on $F_{\alpha}^2$ in the usual way in terms of this inner product. However according to the ideas of [1], it makes sense to define the Toeplitz operator with the symbols in $F_{\alpha}^2$ by the following formula:
\[
T_{\phi}^\alpha f (z) = \int_{\mathbb{C}} \phi (w) \overline{K}_{\alpha}^w (w) e^{-|w|^2} dv(w) / |w|^\alpha.
\]
functions in the Fock(-Sobolev) space, then the Toeplitz product $T_f T_g$ is bounded if and only if $f = e^q$ and $g = C e^{-q}$, where $C$ is a nonzero constant and $q$ is a linear polynomial. More properties about Toeplitz operators on Fock-Sobolev spaces are referred to in [13]. Sequentially, Bommier-Hato et al. in [14] continued to research Cho’s results on the general Fock-type space with the weight functions $\exp \{ - | \cdot |^m \}$. They took full advantage of the exact form of the reproducing kernel of the general Fock-type space and concluded that if $u$ and $v$ are two non-zero functions, then the Toeplitz product $T_u T_v$ is bounded if and only if $u = e^q$ and $v = C e^{-q}$, where $C$ is a nonzero constant and $q$ is a polynomial of degree at most $m$. The similar techniques are founded in [15, 16]. However, the translations appearing to the classical Fock spaces are not suitable to the generalized Fock space. To tackle the main theorem, we have to use the main ideas of [14], that is, making good use of the explicit properties of the reproducing kernel $K^\alpha z$ in Fock-Sobolev type spaces $F^\alpha_z$ instead of the Weyl operators defined by translations on the complex plane.

At last, it is remarked that, as stated in [1], the Fock-Sobolev type spaces $F^\alpha_z$ are in fact very natural generalization of the Fock-Sobolev spaces and the Fock-Sobolev spaces of fractional order. For example, when $\alpha = 0$, $F^0_z$ is the classical Fock space $F^2$. Thus in this paper, we always omit discussing the case of $\alpha = 0$ and the similar result of this case is obtained in [11, 14].

Throughout this paper we write $X \subseteq Y$ or $Y \supseteq X$ for nonnegative quantities $X$ and $Y$ whenever there is a constant $C > 0$ independent of $X$ and $Y$ such that $X \leq CY$. Similarly we write $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$.

2. Proof of the Main Result

We begin with some properties of the Fock-Sobolev spaces $F^\alpha_z$. See [1] for more information.

Lemma 1. Suppose that $f$ belongs to the Fock-Sobolev type space $F^\alpha_z$ for any real $\alpha$. Then for any $z, w \in \mathbb{C}^n$, we have

$$|f(z)|^p e^{-|z|^2/4} \leq \|f\|_{F^\alpha_z}^p,$$

and when $\alpha < 0$,

$$|K^\alpha_z (w)| \leq (1 + |z||w|)^{\alpha/2} \exp \left( \frac{1}{2} |z|^2 + \frac{1}{2} |w|^2 - \frac{1}{8} |z - w|^2 \right),$$

when $\alpha > 0$,

$$|K^\alpha_z (w)| \leq (1 + |z||w|)^{\alpha/2} \exp \left( \frac{1}{2} |z|^2 + \frac{1}{2} |w|^2 - \frac{1}{8} |z - w|^2 \right).$$

More specifically,

$$|K^\alpha_z (z)| = (1 + |z|^2)^{\alpha/2} e^{(|z|^2)^2/2}$$

for any $z \in \mathbb{C}^n$ and there is a $r > 0$ such that

$$|K^\alpha_z (w)| \geq (1 + |z|^2)^{\alpha} \exp \left( \frac{1}{2} |z|^2 + \frac{1}{2} |w|^2 \right),$$

for any $z \in B(w, r)$.

A consequence of the first estimate in Lemma 1 is that, for any function $u \in F^2_z$, the Toeplitz operators $T_u$ and $T_v$ are both densely defined on $F^\alpha_z$.

Lemma 2. If the function $u$ belongs to the Fock-Sobolev type space $F^2_z$, then we have $(T^* u)^\alpha = T^\alpha u$.

Proof. In view of the Lemma 3.4 in [1], we can calculate that, for any polynomial $f$ and $g \in F^\alpha_z$ (see [1] for the definition of $F^\alpha_z$),

$$\langle (T^* u)^\alpha \rangle_{\alpha} = \int_{\mathbb{C}^n} \overline{u}(z) f(z) \frac{e^{-|z|^2}}{|z|^\alpha} dv(z)$$

$$= \langle T^\alpha u, f \rangle_{\alpha},$$

if $\alpha \leq 0$, and if $\alpha > 0$,

$$\langle (T^* u)^\alpha \rangle_{\alpha} = \int_{\mathbb{C}^n} \overline{u}(z) f_{\alpha/2} (z) \frac{e^{-|z|^2}}{|z|^\alpha} dv(z)$$

$$+ \int_{\mathbb{C}^n} \overline{u}(z) f_{\alpha/2} (z) \frac{e^{-|z|^2}}{|z|^\alpha} dv(z).$$

Lastly the fact that the set of all holomorphic polynomials is dense in $F^\alpha_z$ completes the proof.

Lemma 3. For given $u, v \in F^2_z$, if $T^\alpha u T^\alpha v$ is bounded on $F^\alpha_z$, then $T^\alpha u T^\alpha v (w) = \overline{u}(w) \overline{v}(w) K^\alpha_z (w)$ for any $z, w \in \mathbb{C}^n$.

Proof. When $\alpha \leq 0$, in view of reproducing properties of $K^\alpha_z (w)$, Lemma 3.4, the claim $(d) \Rightarrow (a)$ of Lemma 3.10 in [1], and Lemma 2, we see that

$$T^\alpha u T^\alpha v (w) = \langle T^\alpha u, T^\alpha v, K^\alpha_z \rangle_{\alpha}$$

$$= \langle \overline{v}(z) \overline{u}(w), K^\alpha_z (w) \rangle_{\alpha}$$

$$= u(w) \overline{v}(w) K^\alpha_z (w).$$

On the other side, when $\alpha > 0$, we have to use the Lemma 3.4, the claim $(d) \Rightarrow (a)$ of Lemma 3.10 in [1] to achieve that, if $u, v \in F^2_z$, $T^\alpha u T^\alpha v$ is bounded,

$$\langle T^\alpha u T^\alpha v, K^\alpha_z \rangle_{\alpha}$$

$$= \int_{\mathbb{C}^n} u(\lambda) \overline{v}(\lambda) \overline{u}(\lambda) \frac{e^{-|\lambda|^2}}{|\lambda|^\alpha} dv(\lambda)$$

$$= \int_{\mathbb{C}^n} u(\lambda) \overline{v}(\lambda) \overline{u}(\lambda) \frac{e^{-|\lambda|^2}}{|\lambda|^\alpha} dv(\lambda).$$
Together with (3.5) in [1], Fubini’s theorem, and the reproducing property, we can see that

$$\int_{C^w} u(\lambda) \left( T^\alpha w K^\alpha_z \right)^+ \frac{e^{-|\lambda|^2}}{|\lambda|^n} d\nu(\lambda)$$

$$= \int_{C^w} \left( \frac{e^{-|\lambda|^2}}{|\lambda|^n} \right) d\nu(\lambda) \frac{e^{-|\lambda|^2}}{|\lambda|^n} d\nu(\lambda)$$

$$= \int_{C^w} u(\lambda) \left( \frac{e^{-|\lambda|^2}}{|\lambda|^n} \right) d\nu(\lambda)$$

$$\left( \frac{e^{-|\lambda|^2}}{|\lambda|^n} \right) d\nu(\lambda)$$

$$= \int_{C^w} \left( \frac{e^{-|\lambda|^2}}{|\lambda|^n} \right) d\nu(\lambda)$$

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On the other side, we give the representation of quadratic polynomial in the case of real inner product as follows: $q(z) = \langle A z, z \rangle$, where $q = q_1 + q_2$, $q_1$ is linear, $q_2$ is a homogeneous polynomial of degree 2, and $A = A_{\alpha\beta}$ is a complex matrix symmetric in the real sense. After we choose $w = r\xi$ and $z = r\xi + (e_\alpha/2)\eta$, where $r$ is any real positive number, we achieve that

$$\text{det}(e^{z-\overline{\eta}w}) = M \exp(r_0 \langle A\xi, \eta \rangle)$$

(32) is not bounded as $r \to \infty$. This contradiction finishes the proof.

Theorem 5 ((2) ⇒ (1)). If $u = e^q$ and $\nu = e^{-q}$ where $q$ is a complex linear polynomial on $C^n$, then $T^\alpha_u T^\alpha_w$ is bounded on the Fock-Sobolev type space $F^\alpha_z$.

Proof. To prove the boundedness of $T^\alpha_u T^\alpha_w$, we will sufficiently obtain that $\|T^\alpha_u T^\alpha_w f\|_{F^\alpha_z}$ is bounded by means of the idea of [1]. In fact we only discuss the case of $\alpha > 0$ because the other case is the same as the proof of Theorem 2.5 in [12]. Using the similar ways, our goal is to obtain that

$$\|T^\alpha_u T^\alpha_w f\|_{F^\alpha_z} = \left( \int_{C^w} \left| \left( T^\alpha_u T^\alpha_w f \right) \right|^2 (z) \right)^{1/2}$$

$$\left( \int_{C^w} \left| \left( T^\alpha_u T^\alpha_w f \right) \right|^2 (z) \right)^{1/2}$$

is bounded for any $f \in F^\alpha_z$ in view of the definition of the norm $\| \cdot \|_{F^\alpha_z}$. To the end, we focus our attention on the integrands in it. By formulae (3.4) (3.5) in [1] and the definition of Toeplitz operator, the integrands in the norm are

$$\|T^\alpha_u T^\alpha_w f\|_{F^\alpha_z}$$

$$= \left( \int_{C^w} \left| \left( T^\alpha_u T^\alpha_w f \right) \right|^2 (z) \right)^{1/2}$$

$$\left( \int_{C^w} \left| \left( T^\alpha_u T^\alpha_w f \right) \right|^2 (z) \right)^{1/2}$$

is bounded for any $f \in F^\alpha_z$.
and then, similarly,
\[
\left| (T_uT_f)_{\alpha/2} (z) \right| \leq \int_{C^n} |u(z)| |\bar{v}(\eta)| \left| (f)_{\alpha/2} (\eta) \right| \\
\cdot \left| (K_{2}^{\alpha})_{\alpha/2} (\eta) \right| e^{-|\eta|^2} d\eta.
\]
(36)

Therefore, using the Cauchy-Schwarz inequality, we have the estimation of the first term of the norm as follows:
\[
\int_{C^n} \left| (T_uT_f)_{\alpha/2} (z) e^{-(1/2)|z|^2} \right|^2 d\nu(z)
\leq \int_{C^n} \left( \int_{C^n} H^1_u (w,z) d\nu(w) \int_{C^n} H^1_f (w,z) \right)
\cdot \left| (f)_{\alpha/2} (w) \right|^2 e^{-|w|^2} d\nu(w) d\nu(z),
\]
(37)
where $H^1_u (w,z) = e^{-(1/2)|z|^2} |(K_{2}^{\alpha})_{\alpha/2} (w)e^{-(1/2)|w|^2} \exp(\text{Re}(q(z) - \bar{q}(w)))$. Similarly the estimation of the second norm has been achieved that
\[
\int_{C^n} \left| (T_uT_f)_{\alpha/2} (z) e^{-(1/2)|z|^2} |z|^{-\alpha/2} \right|^2 d\nu(z)
\leq \int_{C^n} \left( \int_{C^n} H^2_u (w,z) V d\nu(w) \right) \int_{C^n} H^2_f (w,z)
\cdot \left| (f)_{\alpha/2} (w) \right|^2 e^{-|w|^2} |w|^{-\alpha/2} d\nu(w) d\nu(z),
\]
(38)

This implies that $\|T_uT_f\|_{\alpha/2}$ is bounded and completes the proof.

\[\Box\]

**Theorem 6** $(1) \Rightarrow (3)$. If $u$ and $v$ are two functions in the Fock-Sobolev type space $F^{\alpha}_a$, not identically zero, such that the operator $T_uT_f$ is bounded on $F^{\beta}_a$, then $\|u(z)v(z)\|^{2}$ is a bounded function on the complex space.

**Proof.** We omit the proof here for it is analogous to Theorem 2.6 in [12].

\[\Box\]

**Theorem 7** $(3) \Rightarrow (2)$. Suppose that $u$ and $v$ are two functions in the Fock-Sobolev type space $F^{\alpha}_a$, not identically zero, such that $\|u(z)v(z)\|^{2}$ is bounded on $\mathbb{C}^n$. Then there is a complex linear polynomial $q(z)$ on $\mathbb{C}^n$ satisfying $u = e^q$ and $v = Ce^{-q}$, where $C$ is a nonzero complex constant.

**Proof.** Now we only consider the case of $\alpha > 0$ while the other case would be referred to in Theorem 2.7 in [12].

It is easy to see that, for any $u \in F^{\alpha}_a$, $\bar{u}(z) = (T_uT_f)_{\alpha/2} = u(z)$. When $\alpha > 0$, we use the triangle inequality and Hölder's inequality to calculate
\[
\|u(z)\|^2 \leq \int_{C^n} |u(w)(K_{2}^{\alpha})_{\alpha/2} (w)|^2 e^{-|w|^2} d\nu(w)
\cdot \left( \int_{C^n} \left| (K_{2}^{\alpha})_{\alpha/2} (w) \right|^2 \right)^{1/2} d\nu(w)
\cdot \left( \int_{C^n} \left| (K_{2}^{\alpha})_{\alpha/2} (w) \right|^2 e^{-|w|^2} |w| \right)^{1/2} d\nu(w).
\]
(42)

Because $K_{2}^{\alpha}$ is a unit element, that is,
\[
\|K_{2}^{\alpha}\|_{\alpha/2} = \left( \int_{C^n} \left| (K_{2}^{\alpha})_{\alpha/2} (w) \right|^2 e^{-|w|^2} d\nu(w) \right)^{1/2}
+ \left( \int_{C^n} \left| (K_{2}^{\alpha})_{\alpha/2} (w) \right|^2 e^{-|w|^2} |w| \right)^{1/2}
\]
(43)

we can see that
\[
\int_{C^n} \left| (K_{2}^{\alpha})_{\alpha/2} (w) \right|^2 e^{-|w|^2} d\nu(w) \leq 1,
\]
(44)

\[
\int_{C^n} \left| (K_{2}^{\alpha})_{\alpha/2} (w) \right|^2 e^{-|w|^2} |w| \right)^{1/2} d\nu(w) \leq 1.
\]
From the above inequations, the estimate of $|u(z)|^2$ turns into
\[
|u(z)|^2 \leq \int_{C^n} |u(\omega)|^2 \left( \left| \left( k^\alpha_{z/2} \right)^- (\omega) \right|^2 e^{-|w|^2} d\nu(\omega) + \int_{C^n} |u(\omega)|^2 \left( \left| \left( k^\alpha_{z/2} \right)^+ (\omega) \right|^2 e^{-|w|^2} \left| |w|^{-\alpha} \right|^2 d\nu(\omega) \right) \right)
= |u|^2(z).
\]

If $|u|^2(z)|\bar{u}|^2(z)$ is a bounded function on $C^n$, $|\bar{u}|^2(z)|u|^2(z)$ and $|u|^2(z)|\bar{u}|^2(z)$ are both bounded on $C^n$. By Liouville's theorem, the boundedness of $|u|^2(z)|\bar{u}|^2(z)$ implies that there exists a constant $C$ such that $uv = C$. Since neither $u$ nor $v$ is identically zero, we have $C \neq 0$. That is, both $u$ and $v$ are nonvanishing. By Lemma 1, there exists a complex polynomial $q(z)$ on $C^n$ with $\deg(q) \leq 2$ such that $u = e^q$ and $v = Ce^{-q}$.

On the other side, by the definition of Berezin transformation in this case,
\[
|u|^2(z)|\bar{u}|^2(z) = \int_{C^n} |u(\omega)|^2 \bar{v}(z) \left( \left| \left( k^\alpha_{z/2} \right)^- (\omega) \right|^2 e^{-|w|^2} d\nu(\omega) + \int_{C^n} |u(\omega)|^2 \bar{v}(z) \left( \left| \left( k^\alpha_{z/2} \right)^+ (\omega) \right|^2 e^{-|w|^2} \left| |w|^{-\alpha} \right|^2 d\nu(\omega) \right) \right).
\]

Now giving a sufficiently small $\delta > 0$, we further obtain that
\[
|u|^2(z)|\bar{u}|^2(z) \geq \int_{|w|<\delta} |u(\omega)|^2 \bar{v}(z) \left( \left| \left( k^\alpha_{z/2} \right)^- (\omega) \right|^2 \left| |w|^{-\alpha} \right|^2 d\nu(\omega) \right)
\geq e^{-|w|^2} \left( 1 + [z]^{-\alpha/2} e^{-1/2} |z|^2 K^2_{\alpha} (w) e^{-1/2} |w|^2 \right) \left| \frac{e^{q(w)-q(z)}}{|w|^\alpha/2} \right|^2 d\nu(\omega).
\]

When choosing a constant $\varepsilon > 0$ satisfying Lemma 1, we can see that
\[
|u|^2(z)|\bar{u}|^2(z) \geq \int_{|w|<\delta} \left( 1 + [z]^{-\alpha/2} \left| e^{-1/2} |w|^2 \right|^2 \left| \frac{e^{q(w)-q(z)}}{|w|^\alpha/2} \right|^2 d\nu(\omega) \right)
\geq \int_{|z|<\varepsilon} \left( 1 + [z]^{-\alpha/2} \left| e^{-1/2} |w|^2 \right|^2 \left| \frac{e^{q(w)-q(z)}}{|w|^\alpha/2} \right|^2 d\nu(\omega) \right).
\]

Using the similar method like the case $\alpha < 0$, we can get the desired results and the proof is finished at this moment. 

\section{Conclusions}

In this content, we deal with the Sarason’s problem on the Fock-Sobolev type spaces and have a complete solution that $u = e^q, v = Ce^{-q}$, where $q$ is a linear complex polynomial and $C$ is a nonzero constant. As stated in [1], we know that the Fock-Sobolev type space $F^2_{\alpha}$ clearly does not fall under the class of weighted Fock spaces $F^2_\varphi$. Therefore the Sarason’s problem of weighted Fock spaces $F^2_\varphi$ is still open. We will focus on this open problem in the future study.

\section*{Conflicts of Interest}

The authors declare that they have no conflicts of interest.

\section*{Authors’ Contributions}

All authors contributed equally. All authors read and approved the final manuscript.

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