We establish some new Hermite-Hadamard type integral inequalities for functions whose second-order mixed derivatives are coordinated \((s, m)\)-\(P\)-convex. An expression form of Hermite-Hadamard type integral inequalities via the beta function and the hypergeometric function is also presented. Our results provide a significant complement to the work of Wu et al. involving the Hermite-Hadamard type inequalities for coordinated \((s, m)\)-\(P\)-convex functions in an earlier article.

1. Introduction

Let \(f : I \to \mathbb{R}\) be a convex mapping. Then for any \(a, b \in I\) with \(a < b\), we have the following double inequality:

\[
\frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \tag{1}
\]

This celebrated inequality is known in the literature as the Hermite-Hadamard inequality. As we all know, some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping \(f\). Indeed, Hermite-Hadamard’s inequality (1) has already found many applications in mathematical analysis and optimization (see, for example, [1–8]).

In recent years, the applications of various properties of extended convex functions in establishing and improving Hermite-Hadamard type inequalities have attracted the attention of many researchers (see [10–15] and references cited therein).

In [16], Wu et al. established some Hermite-Hadamard type inequalities under the assumption that the function \(f\) is a coordinated \((s, m)\)-\(P\)-convex function. Motivated by the ideas of work [16], in this paper we study Hermite-Hadamard type inequalities related to the convexity of second-order mixed derivatives of \(f\). More precisely, we focus on establishing some new Hermite-Hadamard type inequalities for functions whose second-order mixed derivatives are coordinated \((s, m)\)-\(P\)-convex. For convenience of our discussions in subsequent sections, we begin with recalling some relevant definitions.

Definition 1. A function \(f : I \subseteq \mathbb{R} \to \mathbb{R}\) is said to be convex function if

\[
f (tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{2}
\]

holds for all \(x, y \in I\) and \(t \in [0, 1]\).

Definition 2 (see [5]). We say that a map \(f : I \subseteq \mathbb{R} \to \mathbb{R}\) belongs to the class \(P(I)\) if it is nonnegative and for all \(x, y \in I\) and \(t \in [0, 1]\) satisfies the following inequality:

\[
f (tx + (1-t)y) \leq f(x) + f(y). \tag{3}
\]

In [17], the concept of \(m\)-convex functions was introduced as follows.
**Definition 3** (see [17]). For \( f : [0, b] \subseteq \mathbb{R}_0 = [0, +\infty) \rightarrow \mathbb{R} \) and \( m \in (0, 1) \), if
\[
f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)
\] is valid for all \( x, y \in [0, b] \) and \( t \in [0, 1] \), then we say that \( f \) is a \( m \)-convex function on \([0, b]\).

In [18], the concept of \( s \)-convex functions was presented as follows.

**Definition 4** (see [18]). Let \( s \in (0, 1] \). A function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is said to be \( s \)-convex (in the second sense) if
\[
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\] holds for all \( x, y \in I \) and \( t \in [0, 1] \).

**Definition 5** (see [19]). For \((s, m) \in (0, 1] \times (0, 1] \), a function \( f : [0, b] \rightarrow \mathbb{R} \) is said to be \((s, m)\)-convex if
\[
f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y)
\] holds for all \( x, y \in [0, b] \) and \( t \in [0, 1] \).

**Definition 6** (see [20]). For some \( s \in [-1, 1] \), a function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is said to be extended \( s \)-convex if
\[
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\] is valid for all \( x, y \in I \) and \( t \in (0, 1) \).

Dragomir [21] and Dragomir and Pearce [22] considered the convexity of a function on the coordinates and put forward the following definition.

**Definition 7** (see [21, 22]). A function \( f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) is said to be convex on the coordinates on \( \Delta \) with \( a < b \) and \( c < d \) if the partial functions
\[
f_y : [a, b] \rightarrow \mathbb{R},

f_y(u) = f(u, y),

f_x : [c, d] \rightarrow \mathbb{R},

f_x(v) = f(x, v),
\]
are convex for all \( x \in (a, b) \) and \( y \in (c, d) \).

It should be noted that a formal definition for coordinated convex functions is stated as follows.

**Definition 8** (see [21, 22]). A function \( f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) is said to be convex on the coordinates on \( \Delta \) with \( a < b \) and \( c < d \) if the partial function
\[
f(tx + (1-t)z, \lambda y + (1-\lambda)w)
\]
\[
\leq t\lambda f(x, y) + t(1-\lambda) f(x, w) + (1-t)\lambda f(z, y) + (1-t)(1-\lambda) f(z, w)
\] holds for all \( t, \lambda \in [0, 1], (x, y, z, w) \in \Delta \).

**Definition 9** (see [16]). For some \( m \in (0, 1] \) and \( s \in [-1, 1] \), a function \( f : [0, b] \times [c, d] \rightarrow \mathbb{R} \) is said to be coordinated \((s, m)\)-\( P \)-convex on \([0, b] \times [c, d] \) with \( 0 < b \) and \( c < d \), if
\[
f(tx + m(1-t)z, \lambda y + (1-\lambda)w)
\]
\[
\leq t^s [f(x, y) + f(x, w)]
\]
\[
+ m(1-t)^s [f(z, y) + f(z, w)]
\]
holds for all \( t \in (0, 1), \lambda \in (0, 1] \) and \((x, y), (z, w) \in [0, b] \times [c, d] \).

Dragomir [21] and Dragomir and Pearce [22] established the following result.

**Theorem 10** (see [21, 22]). Let \( f : \Delta = [a, b] \times [c, d] \) be convex on the coordinates on \( \Delta = [a, b] \times [c, d] \) with \( a < b \) and \( c < d \).

Then, one has the inequalities:
\[
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right]
\]
\[
+ \frac{1}{d-c} \left[ \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right]
\]
\[
\leq \frac{1}{4} \left[ \frac{1}{b-a} \left( \int_a^b f(x, c) dx + \int_a^b f(x, d) dx \right) + \frac{1}{d-c} \left( \int_c^d f(a, y) dy + \int_c^d f(b, y) dy \right) \right]
\]
\[
\leq \frac{f(a,c) + f(b,c) + f(a,d) + f(b,d)}{4}.
\]
\[-\frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy = (b-a)(d-c)\]

\[
\cdot \int_0^1 \int_0^1 K(t, \lambda) \frac{\partial^2}{\partial x \partial y} f(ta + (1-t)b, \lambda c + (1-\lambda)d) dt \, d\lambda,
\]

(12)

where

\[
K(t, \lambda) = \begin{cases} t\lambda, & (t, \lambda) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right], \\ t(\lambda-1), & (t, \lambda) \in \left[0, \frac{1}{2}\right] \times \left(\frac{1}{2}, 1\right], \\ (t-1)\lambda, & (t, \lambda) \in \left(\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right], \\ (t-1)(\lambda-1), & (t, \lambda) \in \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2}, 1\right].
\]

(13)

3. Main Results

In this section, we establish some Hermite-Hadamard type integral inequalities for functions whose second-order mixed derivatives are coordinated \((s, m)\)-co-\(\beta\)-convex on the plane \(\mathbb{R}_0 \times \mathbb{R}\).

**Theorem 12.** Suppose that the function \(f : \mathbb{R}_0 \times \mathbb{R} \rightarrow \mathbb{R}\) has continuous partial derivatives of the second-order and \(\frac{\partial^2 f}{\partial x \partial y} \in L_1([0, b^* / m] \times [c, d])\) with \(0 \leq a < b \leq b^*, c < d,\) for some \(m \in (0, 1)\) and \(s \in [-1, 1].\) If \(\frac{\partial^2 f}{\partial x \partial y}\) is coordinated \((s, m)\)-\(\beta\)-convex functions on \([0, b^* / m] \times [c, d]\) for \(q \geq 1,\) then

\[
|P(a, b, c, d)| \leq \frac{(b-a)(d-c)}{2^s} \left\{ \frac{1}{s+2} \Delta_1(q) \right. + \left. \frac{m}{s+2} \Delta_2(m, q) \right\}^{1/q}
\]

(14)

\[
\times \left\{ \left[ \int_0^{1/2} \int_0^{1/2} t\lambda \left| \frac{\partial^2 f}{\partial x \partial y} f(ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^q dt \, d\lambda \right]^{1/q}
\]

(17)

where

\[
\Delta_1(q) = \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^q + \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^q,
\]

\[
\Delta_2(m, q) = \left| \frac{\partial^2 f(b/m, c)}{\partial x \partial y} \right|^q + \left| \frac{\partial^2 f(b/m, d)}{\partial x \partial y} \right|^q.
\]

**Proof.** By Lemma 11 and Hölder’s integral inequality, we have

\[
|P(a, b, c, d)| \leq (b-a)(d-c) \left( \int_0^{1/2} \int_0^{1/2} K(t, \lambda) \, dt \, d\lambda \right)^{1-1/q}
\]

(11) if \(s \in (-1, 1],\) we have

\[
|P(a, b, c, d)| \leq \frac{(b-a)(d-c)}{2^s} \left\{ \frac{1}{s+2} \Delta_1(q) \right. + \left. \frac{m}{s+2} \Delta_2(m, q) \right\}^{1/q}
\]

(14)

\[
\times \left\{ \left[ \int_0^{1/2} \int_0^{1/2} t\lambda \left| \frac{\partial^2 f(ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^q dt \, d\lambda \right]^{1/q}
\]

(17)

(2) if \(s = -1,\) we have

\[
|P(a, b, c, d)| \leq \frac{(b-a)(d-c)}{8} \left\{ \left[ \Delta_1(q) + m(2 \ln 2 - 1) \Delta_2(m, q) \right]^{1/q}
\]

\[
+ \left[ (2 \ln 2 - 1) \Delta_1(q) + m \Delta_2(m, q) \right]^{1/q} \right\},
\]

(15)

where

\[
\Delta_1(q) = \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^q + \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^q,
\]

\[
\Delta_2(m, q) = \left| \frac{\partial^2 f(b/m, c)}{\partial x \partial y} \right|^q + \left| \frac{\partial^2 f(b/m, d)}{\partial x \partial y} \right|^q.
\]

Proof. By Lemma 11 and Hölder’s integral inequality, we have

\[
|P(a, b, c, d)| \leq (b-a)(d-c) \left( \int_0^{1/2} \int_0^{1/2} K(t, \lambda) \, dt \, d\lambda \right)^{1-1/q}
\]

(11) if \(s \in (-1, 1],\) we have

\[
|P(a, b, c, d)| \leq \frac{(b-a)(d-c)}{2^s} \left\{ \frac{1}{s+2} \Delta_1(q) \right. + \left. \frac{m}{s+2} \Delta_2(m, q) \right\}^{1/q}
\]

(14)

\[
\times \left\{ \left[ \int_0^{1/2} \int_0^{1/2} t\lambda \left| \frac{\partial^2 f(ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^q dt \, d\lambda \right]^{1/q}
\]

(17)

(2) if \(s = -1,\) we have

\[
|P(a, b, c, d)| \leq \frac{(b-a)(d-c)}{8} \left\{ \left[ \Delta_1(q) + m(2 \ln 2 - 1) \Delta_2(m, q) \right]^{1/q}
\]

\[
+ \left[ (2 \ln 2 - 1) \Delta_1(q) + m \Delta_2(m, q) \right]^{1/q} \right\},
\]

(15)

where

\[
\Delta_1(q) = \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^q + \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^q,
\]

\[
\Delta_2(m, q) = \left| \frac{\partial^2 f(b/m, c)}{\partial x \partial y} \right|^q + \left| \frac{\partial^2 f(b/m, d)}{\partial x \partial y} \right|^q.
\]
A straightforward computation gives
\[
\int_0^{1/2} \lambda \, d\lambda = \int_0^{1} (1 - \lambda) \, d\lambda = \frac{1}{8},
\]
\[
\int_0^{1/2} t^{1/2} \, dt = \int_0^{1/2} (1 - t)^{1/2} \, dt = \frac{2^{s+2} - s - 3}{2^{s+5} (s + 2)} \Delta_2 (m, q) \quad \text{for } s \in (-1, 1),
\]
\[
\int_0^{1/2} t (1 - t)^{1/2} \, dt = \int_0^{1/2} (1 - t) t^{1/2} \, dt = \frac{2^{s+2} - s - 3}{2^{s+5} (s + 1) (s + 2)} \Delta_2 (m, q),
\]
(18)
\[
\int_0^{1/2} t^{-1} \, dt = \int_0^{1/2} (1 - t)^{-1} \, dt = \frac{1}{2},
\]
\[
\int_0^{1/2} t (1 - t)^{-1} \, dt = \int_0^{1/2} (1 - t) t^{-1} \, dt = \ln 2 - \frac{1}{2},
\]
\[
\int_0^{1/2} \int_0^{1/2} |K(t, \lambda)| \, dt \, d\lambda = \frac{1}{16}.
\]

Now, by using the coordinated \((s, m)\)-P-convexity of \(|\partial^2 f/\partial x \partial y|^q\), it follows that if \(-1 < s \leq 1\), we have
\[
\int_0^{1/2} \int_0^{1/2} t \lambda \left| \frac{\partial^2}{\partial x \partial y} f (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right|^q \, dt \, d\lambda \leq \frac{1}{16} [\Delta_1 (q) + m (2 \ln 2 - 1) \Delta_2 (m, q)],
\]
(19)
and if \(s = -1\), we have
\[
\int_0^{1/2} \int_0^{1/2} t \lambda \left| \frac{\partial^2}{\partial x \partial y} f (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right|^q \, dt \, d\lambda \leq \frac{1}{16} [\Delta_1 (q) + m (2 \ln 2 - 1) \Delta_2 (m, q)].
\]
(20)

By a similar argument, we obtain
\[
\int_{1/2}^{1} \int_0^{1/2} t (1 - t) \, dt \lambda \left| \frac{\partial^2}{\partial x \partial y} f (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right|^q \, dt \, d\lambda \leq \frac{1}{2^{s+3}} \Delta_1 (q) + m \frac{2^{s+2} - s - 3}{(s + 1) (s + 2)} \Delta_2 (m, q), \quad -1 < s \leq 1,
\]
\[
\int_{1/2}^{1} \int_0^{1/2} t (1 - t) \, dt \lambda \left| \frac{\partial^2}{\partial x \partial y} f (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right|^q \, dt \, d\lambda \leq \frac{1}{2^{s+3}} \Delta_1 (q) + m \frac{2^{s+2} - s - 3}{(s + 1) (s + 2)} \Delta_2 (m, q), \quad s = -1,
\]
(21)

Applying (18) and inequalities (19)–(21) into inequality (17), we get (14) and (15). This completes the proof of Theorem 12. \(\square\)

**Corollary 13.** Under the assumptions of Theorem 12, if \(q = 1\), then
\[
(1) \text{if } s \in (-1, 1], \quad |P(a, b, c, d)| \leq \frac{(b - a) (d - c) (2^{s+1} - 1)}{2^{s+3} (s + 1) (s + 2)} \left[ \Delta_1 (1) + m \Delta_2 (m, 1) \right],
\]
(22)
\[
(2) \text{if } s = -1, \quad |P(a, b, c, d)| \leq \frac{(b - a) (d - c) \ln 2}{4} \left[ \Delta_1 (1) + m \Delta_2 (m, 1) \right].
\]
(23)

**Corollary 14.** Under the assumptions of Theorem 12, if \(q = m = 1\), then
\[
(1) \text{if } s \in (-1, 1], \quad |P(a, b, c, d)| \leq \frac{(b - a) (d - c) (2^{s+1} - 1)}{2^{s+3} (s + 1) (s + 2)} \times \left[ \left| \frac{\partial^2 f (a, c)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f (a, d)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f (b, c)}{\partial x \partial y} \right| \right] \left[ \left| \frac{\partial^2 f (a, c)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f (a, d)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f (b, c)}{\partial x \partial y} \right| \right],
\]
(24)
\[
(2) \text{if } s = -1, \quad |P(a, b, c, d)| \leq \frac{(b - a) (d - c) \ln 2}{4} \times \left[ \left| \frac{\partial^2 f (a, c)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f (a, d)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f (b, c)}{\partial x \partial y} \right| \right].
\]
(25)
Furthermore, if \( q = m = 1, s = 0 \), then
\[
\begin{align*}
|P(a, b, c, d)| & \leq \frac{(b - a) (d - c)}{16} \times \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right| .
\end{align*}
\]
(26)

**Theorem 15.** Suppose that the function \( f: \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R} \) has continuous partial derivatives of the second-order and \( \partial^2 f/\partial x \partial y \in L_1([0, b^*/m] \times [c, d]) \) with \( 0 \leq a < b \leq b^*, c < d, \) and \( 0 \leq r \leq q, -1 < \ell \leq q. \) If \( |\partial^2 f/\partial x \partial y|^q \) is coordinated \((s, m)-P\)-convex functions on \([0, b^*/m] \times [c, d]\) for some \( m \in (0, 1], s \in (-1, 1], \) and \( q > 1, \) then
\[
|P(a, b, c, d)| \leq (b - a) (d - c)
\]
\[
\times \left\{ \frac{\Delta_1(q)}{2^{r+1} (r + s + 1)} + 2^{-r} m \Delta_2(m, q) \frac{r}{r + 1} \left( \begin{array}{c} 1 \\ + \end{array} \right) \right\}^{1/q},
\]
(27)
where \( \Delta_1(q) \) and \( \Delta_2(m, q) \) are defined as in (16), and \( B(\alpha, \beta) \) is the beta function defined by
\[
B(\alpha, \beta) = \int_0^1 t^{\alpha - 1} (1 - t)^{\beta - 1} \, dt, \quad \alpha, \beta > 0,
\]
(28)
and \( 2F_1(c, d, e, z) \) is the hypergeometric function defined by
\[
2F_1(c, d, e, z) = \frac{\Gamma(e)}{\Gamma(d) \Gamma(e - d)} \int_0^1 t^{e - d - 1} (1 - zt)^{-c} \, dt,
\]
(29)
for \( e > d > 0, |z| < 1, c \in \mathbb{R}, u > 0. \)

**Proof.** Using Lemma 11 and Hölder’s integral inequality, we obtain
\[
|P(a, b, c, d)| \leq (b - a) (d - c)
\]
\[
\times \left\{ \left( \frac{\Delta_1(q)}{2^{r+1} (r + s + 1)} + 2^{-r} m \Delta_2(m, q) \frac{r}{r + 1} \right)^{1/q}
\right\},
\]
\[
\times \left\{ \left( \frac{\Delta_1(q)}{2^{r+1} (r + s + 1)} + 2^{-r} m \Delta_2(m, q) \frac{r}{r + 1} \right)^{1/q}
\right\},
\]
\[
\times \left\{ \left( \frac{\Delta_1(q)}{2^{r+1} (r + s + 1)} + 2^{-r} m \Delta_2(m, q) \frac{r}{r + 1} \right)^{1/q}
\right\},
\]
\[
\times \left\{ \left( \frac{\Delta_1(q)}{2^{r+1} (r + s + 1)} + 2^{-r} m \Delta_2(m, q) \frac{r}{r + 1} \right)^{1/q}
\right\},
\]
(30)
After some calculations, it follows that
\[
\int_0^{1/2} \int_0^{1/2} t^{(q-r)/(q-1)} (1-\lambda)^{q-1} \, dt \, d\lambda \\
= \int_0^{1/2} \int_0^{1/2} t^{(q-r)/(q-1)} (1-\lambda)^{q-1} \, dt \, d\lambda \\
= \int_0^{1/2} \int_0^{1/2} (1-t)^{(q-r)/(q-1)} (1-\lambda)^{q-1} \, dt \, d\lambda \\
= \int_0^{1/2} \int_0^{1/2} (1-t)^{(q-r)/(q-1)} (1-\lambda)^{q-1} \, dt \, d\lambda \\
= \frac{(q-1)^2}{(2q-r-1)(2q-\ell-1)} \times 2^{-2(4q-r-\ell-2)/(q-1)}.
\]
From the coordinated \((s, m)\)-\(P\)-convexity of \(|\frac{\partial^2 f}{\partial x \partial y}|^q\), we deduce that
\[
\int_0^{1/2} \int_0^{1/2} t^{(q-1)/2} \lambda^q \left| \frac{\partial^2}{\partial x \partial y} \right|^q \\
\cdot f (ta + (1-t) b, \lambda c + (1-\lambda) d) \, dt \, d\lambda \\
\leq 2^{-2(\ell+1)} \int_0^{1/2} t^{(q-1)/2} \left[ t^{(q-1)/2} \Delta_1(q) + m(1-t)^{\ell} \right] \\
\cdot \Delta_2 (m, q) \, dt \\
= 2^{-2(\ell+1)} \left[ \frac{\Delta_1(q)}{2^{\ell+1}(r+s+1)} + \frac{\Delta_1(q)+m\Delta_2(m,q)}{r+1} \right] \\
\int_0^{1/2} (1-t)^{\ell} (1-\lambda)^{q-1} \left| \frac{\partial^2}{\partial x \partial y} \right|^q \\
\cdot f (ta + (1-t) b, \lambda c + (1-\lambda) d) \, dt \, d\lambda \\
\leq 2^{-2(\ell+1)} \left[ B(r+1,s+1) - 2^{-2} 2F_1 \left( -s, r+1, s+2, 2^{-1} \right) \right] \\
\cdot \left| \frac{\partial^2}{\partial x \partial y} \right|^q \\
\cdot f (ta + (1-t) b, \lambda c + (1-\lambda) d) \, dt \, d\lambda \\
\leq 2^{-2(\ell+1)} \left[ B(r+1,s+1) - 2^{-2} 2F_1 \left( -s, r+1, s+2, 2^{-1} \right) \right] \times \left[ \frac{\Delta_1(q)+m\Delta_2(m,q)}{r+1} \right].
\]

Applying (31) and inequalities (32) into inequality (30), we get inequality (27). The proof of Theorem 15 is complete.

Corollary 16. Under the assumptions of Theorem 15, if \(r = 0\), then
\[
|P (a, b, c, d)| \leq \frac{(b-a)(d-c)}{[2^{\ell+1}(s+1)(\ell+1)]^{1/q}} \left( \frac{q-1}{2q-\ell-1} \right)^{1-1/q} \\
\times \left[ \left( \frac{1}{2} \right)^{(q-1)/(q-1)} \left( \frac{1}{2q-\ell-1} \right)^{(4q-\ell-2)/(q-1)} \right]^{1-1/q} \\
\times \left[ \left( \frac{1}{2} \right)^{(q-1)/(q-1)} \left( \frac{1}{2q-\ell-1} \right)^{(4q-\ell-2)/(q-1)} \right]^{1-1/q} \\
\times \left[ \frac{\Delta_1(q)+m\Delta_2(m,q)}{r+1} \right]^{1-1/q}.
\]

In particular, if \(r = \ell = 0\), then
\[
|P (a, b, c, d)| \\
\leq \frac{(b-a)(d-c)}{[2^{\ell+1}(s+1)(\ell+1)]^{1/q}} \left( \frac{q-1}{2q-\ell-1} \right)^{1-1/q} \\
\times \left[ \left( \frac{1}{2} \right)^{(q-1)/(q-1)} \left( \frac{1}{2q-\ell-1} \right)^{(4q-\ell-2)/(q-1)} \right]^{1-1/q} \\
\times \left[ \left( \frac{1}{2} \right)^{(q-1)/(q-1)} \left( \frac{1}{2q-\ell-1} \right)^{(4q-\ell-2)/(q-1)} \right]^{1-1/q} \\
\times \left[ \frac{\Delta_1(q)+m\Delta_2(m,q)}{r+1} \right]^{1-1/q}.
\]

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Authors’ Contributions
All authors read and approved the final manuscript.

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