Research Article

Some Recurrence Relations and Hilbert Series of Right-Angled Affine Artin Monoid $M(\tilde{D}_n^\infty)$

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We find the Hilbert series of the right-angled affine Artin monoid $M(\tilde{D}_n^\infty)$. We also discuss its recurrence relation and the growth rate.

1. Introduction

Coxeter groups were introduced by Coxeter in 1934 as abstract form of reflection groups. These groups were classified into two categories in 1935 also by Coxeter: spherical and affine. In the list of spherical Coxeter groups, A_n is the first. The Artin group associated with A_n is the braid group. Cardinality is an invariant of graded algebraic structures. Hilbert series deals with the cardinality of elements in the graded algebraic structures. In [1] Iqbal gave a linear system for the reducible and irreducible words of the braid monoid MB_n , which leads to compute the Hilbert series of MB_n . In [2] Iqbal and Yousaf computed the Hilbert series of the braid monoid MB_4 in band generators. In [3] Berceanu and Iqbal proved that the growth rate of all the spherical Artin monoids is less than 4. In [4] Iqbal et al. studied the braid monoid $M(\tilde{A}_n^\infty)$ of the affine type $\tilde{A}_n$ of the Coxeter systems. Authors also found the recurrence relations, the growth series of $M(\tilde{A}_n^\infty)$, and proved that the growth rate of $M(\tilde{A}_n^\infty)$ is unbounded (see Figure 1).

In the present paper we study the affine-type Coxeter group $\tilde{D}_n$ and find the Hilbert series (or spherical growth series) of the associated right-angled affine Artin monoid $M(\tilde{D}_n^\infty)$. We also discuss its recurrence relations and the growth rate. For detailed explanation of related concepts and basic ideas about Coxeter groups and its types, readers are referred to [4] and references therein.

The affine (or infinite) Coxeter groups form another important series of Coxeter groups. These well-known affine Coxeter groups are $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{G}_2$, and $\tilde{I}_1$ (for details, see [5]). In [3] authors proved that the universal upper bound for all the spherical Artin monoids is less than 4.

In this work we discuss right-angled affine Artin monoids; specifically, we study the affine monoid $M(\tilde{D}_n^\infty)$ and compute its Hilbert series. We show that the growth of $M(\tilde{D}_n^\infty)$ series is bounded above by 4. Along with Hilbert series we also compute the recurrence relations related to $M(\tilde{D}_n^\infty)$. We give a conjecture about the growth rate of

FIGURE 1

FIGURE 2

$M(\tilde{D}_n^\infty)$, and the growth rates (maximal roots of the characteristic polynomial $D_n(\lambda)$) are computed using the softwares Drive6 and Mathematica.

The monoid $M(\tilde{D}_n^\infty)$ is represented by its Coxeter graph as shown in Figure 2.

Here $x_1, x_2, \dots, x_n$ are vertices of the graph and all the labels are ∞ . If all the labels in a Coxeter diagram are replaced by ∞ , then there is no relation between the adjacent edges. Hence we have the associated right-angled Artin groups and the associated right-angled Artin monoids denoted by $G(\tilde{D}_n^\infty)$ and $M(\tilde{D}_n^\infty)$, respectively.

In a monoid the relation $\alpha = \beta$ will be written as $\alpha > \beta$ in the length-lexicographic order. Let $\alpha_1 = uw$ and $\alpha_2 = wv$; then the word of the form uwv is said to be an *ambiguity*. If $\alpha_1 v = u\alpha_2$ is in the length-lexicographic order, then we say that the ambiguity uwv is solvable. Such a presentation is *complete* if and only if all the ambiguities are solvable. Corresponding to the relation $\alpha = \beta$, the changes $\gamma\alpha\delta \rightarrow \gamma\beta\delta$ give a rewriting system. A complete presentation is equivalent to a confluent rewriting system. In a complete presentation of a monoid, word containing α will be called *reducible word* and a word that does not contain α will be called an *irreducible word* or *canonical word*. In a presentation of a monoid we fix a total order $x_1 < x_2 < \dots < x_n$ on the generators. Hence clearly we have the following.

Lemma 1. *The monoid $M(\tilde{D}_n^\infty)$ has the following presentation:*

$$M(\tilde{D}_n^\infty) = \left\langle x_1, x_2, \dots, x_n \mid \begin{array}{l} x_2 x_1 = x_1 x_2, \\ x_j x_i = x_i x_j, \quad 3 \leq i+2 \leq j \leq n-1, \quad j \neq 3 \\ x_n x_k = x_k x_n, \quad k \neq n-2 \end{array} \right\rangle. \quad (1)$$

2. Recurrence Relations of the Monoid $M(\tilde{D}_n^\infty)$

In this section we discuss few interesting results relating to the recurrence relations of $M(\tilde{D}_n^\infty)$. First we talk about the solution of the system of linear recurrences.

Consider a system [6] of linear recurrences

$$\begin{aligned} u_1(t+1) &= a_{11}(t)u_1(t) + \dots + a_{1n}(t)u_n(t) + f_1(t) \\ u_2(t+1) &= a_{21}(t)u_1(t) + \dots + a_{2n}(t)u_n(t) + f_2(t) \\ &\vdots \end{aligned} \quad (2)$$

$$u_n(t+1) = a_{n1}(t)u_1(t) + \dots + a_{nn}(t)u_n(t) + f_n(t).$$

This system can be written as $u(t+1) = A(t)u(t) + f(t)$, where

$$\begin{aligned} u(t) &= \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}, \\ A(t) &= \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix}, \\ f(t) &= \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}. \end{aligned} \quad (3)$$

The solution (which we need in our work) of the homogeneous equation $u(t+1) = A(t)u(t)$ is given by $u(t) = c_1 \lambda_1^t u^1 + \dots + c_k \lambda_k^t u^k$, where $\lambda_1, \dots, \lambda_k$ are the eigenvalues of $A(t)$ and u^i is an eigenvector corresponding to λ_i . The largest eigenvalue is the growth rate of the sequence $(u_i(t))$ (by the definition of growth rate).

Let $c_k = \#\{\text{canonical words of length } k\}$ and $c_{k;i} = \#\{\text{canonical words starting with } x_i \text{ of length } k\}$. Then we have the following.

Lemma 2. *The monoid $M(\tilde{D}_n^\infty)$ satisfies the following recurrence relations:*

- (a) $c_0 = 1$, $c_{1;i} = 1$, and $c_k = \sum_{i=1}^n c_{k;i}$, ($k \geq 1$).
- (b) $c_{k;i}$ ($k \geq 1$) are given by the recurrence

$$c_{k;i} = \begin{cases} \sum_{j=i}^n c_{k-1;j}, & i = 1, 2, \\ c_{k;1}, & i = 3, \\ \sum_{j=i-1}^n c_{k-1;j}, & i = 4, 5, \dots, n-1, \\ c_{k-1;n-2} + c_{k-1;n}, & i = n. \end{cases} \quad (4)$$

Let K_n be a Coxeter diagram (K_n is similar to the standard Coxeter diagram D_n but opposite in direction) (see Figure 3).

We use $M(K_n^\infty)$ in the solution of the recurrence of $M(\tilde{D}_n^\infty)$. Therefore we have the following.

Lemma 3. *The monoid $M(K_n^\infty)$ satisfies the recurrence relations*

FIGURE 3

- (a) $c_0 = 1$, $c_{1;i} = 1$, and $c_k = \sum_{i=1}^n c_{k;i}$, ($k \geq 1$).
 (b) $c_{k;i}$ ($k \geq 1$) are given by the recurrence

$$c_{k;i} = \begin{cases} \sum_{j=i}^n c_{k-1;j}, & i = 1, 2, \\ c_{k;1}, & i = 3, \\ \sum_{j=i-1}^n c_{k-1;j}, & i = 4, 5, \dots, n \end{cases} \quad (5)$$

Let $K_n(\lambda)$ and $D_n(\lambda)$ denote the characteristic polynomials of the system of recurrence relations of the monoids $M(K_n^{\infty})$ and $M(\bar{D}_n^{\infty})$, respectively. Then we have the following.

Lemma 4. The polynomials $(K_n(\lambda))_{n \geq 3}$ satisfy the recurrence relation

$$K_n(\lambda) = \lambda K_{n-1}(\lambda) - \lambda K_{n-2}(\lambda), \quad n \geq 3 \quad (6)$$

with the initial values $K_3(\lambda) = \lambda^3 - 3\lambda^2 + \lambda$ and $K_4(\lambda) = \lambda^4 - 4\lambda^3 + 3\lambda^2 - \lambda$.

Proof. Let P_n be the matrix of order $n \times n$ of the recurrences given in the Lemma 3. Then the characteristic polynomial $K_n(\lambda)$ of P_n is

$$K_n(\lambda) = \begin{vmatrix} \lambda - 1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & \lambda - 1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & \lambda - 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 & \lambda - 1 \end{vmatrix}. \quad (7)$$

We write $K_n(\lambda) = S_n(\lambda) + T_n(\lambda)$, where the determinants $S_n(\lambda)$ and $T_n(\lambda)$ are obtained by splitting $K_n(\lambda)$ such that the last rows of $S_n(\lambda)$ and $T_n(\lambda)$ are $(0, \dots, 0, \lambda)$ and $(0, \dots, 0, -1, -1)$, respectively. Therefore easily we have $S_n(\lambda) = \lambda K_{n-1}(\lambda)$ and

$$T_n(\lambda) = \begin{vmatrix} \lambda - 1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & \lambda - 1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & \lambda - 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 & -1 \end{vmatrix}. \quad (8)$$

Subtracting last column from 2nd last column of $T_n(\lambda)$ and after few easy computations we have the recurrence relation $K_n(\lambda) = \lambda K_{n-1}(\lambda) - \lambda K_{n-2}(\lambda)$, $n \geq 3$. $\square$

Here we have an explicit formula to compute $K_n(\lambda)$.

Lemma 5. Let $q = \sqrt{\lambda^2 - 4\lambda}$; then we have the following:

- (1) $K_{2m}(\lambda) = (1/2^{2m}) \sum_{k=0}^{2m} \binom{2m}{k} t^{2m-k} q^{2k} (a+b)$,
 (2) $K_{2m+1}(\lambda) = (1/2^{2m+1}) \sum_{k=0}^{2m+1} \binom{2m+1}{k} t^{2m-k+1} q^{2k+1} (a-b)$, where

$$a = \frac{4t(q(t^3 - 3t + 1) + (t^3 - 5t^2 + 5t - 2))}{q(t+q)^3}, \quad (9)$$

$$b = \frac{4t(q(t^3 - 3t + 1) - (t^3 - 5t^2 + 5t - 2))}{q(t-q)^3}.$$

Proof. Let $K_n(\lambda) = \alpha^n$. Then $K_n(\lambda) = \lambda K_{n-1}(\lambda) - \lambda K_{n-2}(\lambda)$ can be written as $\alpha^n = \lambda \alpha^{n-1} - \lambda \alpha^{n-2}$; i.e., $\alpha^2 - \lambda \alpha + \lambda = 0$. This characteristic equation has the roots $\alpha = (\lambda \pm \sqrt{\lambda^2 - 4\lambda})/2$. Let $\alpha_1 = (\lambda + \sqrt{\lambda^2 - 4\lambda})/2$ and $\alpha_2 = (\lambda - \sqrt{\lambda^2 - 4\lambda})/2$. Then we have the solution of the recurrence given by $K_n(\lambda) = a\alpha_1^n + b\alpha_2^n$, where a and b are constants to determine. Since $K_3(\lambda) = \lambda^3 - 3\lambda^2 + \lambda$ and $K_4(\lambda) = \lambda^4 - 4\lambda^3 + 3\lambda^2 - \lambda$, therefore

$$\begin{aligned} \lambda^3 - 3\lambda^2 + \lambda &= \alpha_1^3 a + \alpha_2^3 b, \\ \lambda^4 - 4\lambda^3 + 3\lambda^2 - \lambda &= \alpha_1^4 a + \alpha_2^4 b. \end{aligned} \quad (10)$$

Solving these equations we get $a = (4t/q(t+q)^3)(q(t^3 - 3t + 1) + (t^3 - 5t^2 + 5t - 2))$ and $b = (4t/q(t-q)^3)(q(t^3 - 3t + 1) - (t^3 - 5t^2 + 5t - 2))$. Hence

$$\begin{aligned} K_n(\lambda) &= \frac{a(t+q)^n + b(t-q)^n}{2^n} \\ &= \frac{a}{2^n} \sum_{k=0}^n \binom{n}{k} t^{n-k} q^k \\ &\quad + \frac{b}{2^n} \sum_{k=0}^n \binom{n}{k} t^{n-k} (-q)^k. \end{aligned} \quad (11)$$

For even and odd values of n we have

$$\begin{aligned} K_{2m}(\lambda) &= \frac{1}{2^{2m}} \sum_{k=0}^{2m} \binom{2m}{k} t^{2m-k} q^{2k} (a+b), \\ K_{2m+1}(\lambda) &= \frac{1}{2^{2m+1}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} t^{2m-k+1} q^{2k+1} (a-b). \end{aligned} \quad (12)$$

$\square$

Theorem 6. The polynomials $(D_n(\lambda))_{n \geq 5}$ satisfy the recurrence relation

$$D_n(\lambda) = (\lambda - 1) K_{n-1}(\lambda) - \lambda^2 K_{n-4}(\lambda), \quad n \geq 5 \quad (13)$$

with $D_0(\lambda) = 1$, $D_1(\lambda) = \lambda - 1$, $D_2(\lambda) = (\lambda - 1)^2$, $D_3(\lambda) = \lambda(\lambda^2 - 3\lambda + 1)$, and $D_4(\lambda) = \lambda^2(\lambda^2 - 4\lambda + 3)$ being the initial values.

Proof. Let M_n be the matrix of order $n \times n$ of the recurrences given in Lemma 2. Then the characteristic polynomial of M_n is

$$D_n(\lambda) = \begin{vmatrix} \lambda - 1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & \lambda - 1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda - 1 & -1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & \lambda - 1 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \lambda - 1 \end{vmatrix}. \quad (14)$$

Write $D_n(\lambda) = U_n(\lambda) + V_n(\lambda)$, where the determinants $U_n(\lambda)$ and $V_n(\lambda)$ are obtained by splitting $D_n(\lambda)$ such that the last row of $U_n(\lambda)$ is $(0, \dots, 0, \lambda - 1)$ and the last row of $V_n(\lambda)$ is $(0, \dots, 0, -1, 0, 0)$. Hence easily we have $U_n(\lambda) = (\lambda - 1)K_{n-1}(\lambda)$. Subtracting last column from 3rd last and 2nd last columns of $V_n(\lambda)$, respectively, we have $V_n(\lambda) = -\lambda^2 K_{n-4}(\lambda)$. Hence

$$D_n(\lambda) = (\lambda - 1)K_{n-1}(\lambda) - \lambda^2 K_{n-4}(\lambda), \quad n \geq 5. \quad (15)$$

□

3. The Hilbert Series of the Monoid $M(\widetilde{D}_n^\infty)$

Now we compute the Hilbert series of $M(\widetilde{D}_n^\infty)$. For this we need to fix some notations first. Let $\mathcal{H}_M^{(n)}(t) = \sum_{k \geq 0} c_k t^k$ denote the Hilbert series of $M(\widetilde{D}_n^\infty)$, where $c_k = \#\{\text{canonical words of length } k\}$, and $\mathcal{H}_{M;i}^{(n)}(t) = \sum_{k \geq 0} c_{k;i}^{(n)} t^k$ denote the Hilbert series of $M(\widetilde{D}_n^\infty)$ of words starting with x_i , where $c_{k;i} = \#\{\text{canonical words starting with } x_i \text{ of length } k\}$.

Theorem 7. *The Hilbert series of $M(\widetilde{D}_n^\infty)$ is given by the following:*

- (1) $\mathcal{H}_M^{(n)}(t) = 1 + \sum_{i=1}^n \mathcal{H}_{M;i}^{(n)}(t)$,
- (2) $\mathcal{H}_{M;i}^{(n)}(t) = t + t \sum_{j=i}^n \mathcal{H}_{M;j}^{(n)}(t)$, $i = 1, 2$,
- (3) $\mathcal{H}_{M;1}^{(n)}(t) = \mathcal{H}_{M;3}^{(n)}(t)$,
- (4) $\mathcal{H}_{M;i}^{(n)}(t) = t + t \sum_{j=i-1}^n \mathcal{H}_{M;j}^{(n)}(t)$, $i = 4, 5, \dots, n-1$,
- (5) $\mathcal{H}_{M;n}^{(n)}(t) = t + t \mathcal{H}_{M;n-2}^{(n)}(t) + t \mathcal{H}_{M;n}^{(n)}(t)$.

Proof. (1) From Lemma 2 we have $c_k = \sum_{i=1}^n c_{k;i}$, ($k \geq 1$). Therefore

$$\begin{aligned} \mathcal{H}_M^{(n)}(t) &= \sum_{k \geq 0} c_k t^k = c_0 + \sum_{k \geq 1} c_k t^k = 1 + \sum_{k \geq 1} \sum_{i=1}^n c_{k;i} t^k \\ &= 1 + \sum_{i=1}^n \sum_{k \geq 1} c_{k;i} t^k = 1 + \sum_{i=1}^n \mathcal{H}_{M;i}^{(n)}(t). \end{aligned} \quad (16)$$

(2) From Lemma 2 we have $c_{k;i} = \sum_{j=i}^n c_{k-1;j}$ ($i = 1, 2$). Hence

$$\begin{aligned} \mathcal{H}_{M;i}^{(n)}(t) &= \sum_{k \geq 1} c_{k;i} t^k = c_{1;i} t + \sum_{k \geq 2} c_{k;i} t^k \\ &= t + \sum_{k \geq 2} \sum_{j=i}^n c_{k-1;j} t^k \\ &= t + t \sum_{j=i}^n \sum_{k \geq 2} c_{k-1;j} t^{k-1} \\ &= t + t \sum_{j=i}^n \mathcal{H}_{M;j}^{(n)}(t). \end{aligned} \quad (17)$$

Similarly we can easily prove (3), (4), and (5). □

The system of equations in Theorem 7 can be written in matrix form as $W_n X = B$, where

$$\begin{aligned} W_n &= \begin{bmatrix} 1-t & -t & \cdots & -t & -t & -t & -t \\ 0 & 1-t & \cdots & -t & -t & -t & -t \\ -t & -t & \cdots & -t & -t & -t & -t \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & 1-t & -t & -t \\ 0 & 0 & \cdots & 0 & -t & 1-t & -t \\ 0 & 0 & \cdots & 0 & -t & 0 & 1-t \end{bmatrix}, \\ X &= \begin{bmatrix} \mathcal{H}_{M;1}^{(n)}(t) \\ \mathcal{H}_{M;2}^{(n)}(t) \\ \mathcal{H}_{M;3}^{(n)}(t) \\ \vdots \\ \mathcal{H}_{M;n}^{(n)}(t) \end{bmatrix}, \\ B &= \begin{bmatrix} t \\ t \\ t \\ \vdots \\ t \end{bmatrix}. \end{aligned} \quad (18)$$

Lemma 8. *In the monoid $M(\widetilde{D}_n^\infty)$*

$$\det(W_n) = t^n D_n \left(\frac{1}{t} \right). \quad (19)$$

Proof. The result follows immediately by factoring out t from each row of $\det(W_n)$. □

Lemma 9. In $M(\widetilde{D}_n^\infty)$

$$\mathcal{H}_{M;m}^{(n)}(t) = \begin{cases} \frac{t^{m-1}K_{m-2}(1/t)}{t^n D_n(1/t)}, & 5 \leq m \leq n-1 \\ \frac{(1-t)t^{m-2}K_{m-3}(1/t)}{t^n D_n(1/t)}, & m = n. \end{cases} \quad (20)$$

Proof. The system given in Theorem 7 of n equations in n variables $\mathcal{H}_{M;i}^{(n)}(t)$, $1 \leq i \leq n$ is already written in the form $W_n X = B$, where $X = [\mathcal{H}_{M;1}^{(n)}(t), \dots, \mathcal{H}_{M;n}^{(n)}(t)]^t$, $\det(W_n) = t^n D_n(1/t)$, and $B = [t, \dots, t]^t$. Here we have only two cases.

Case I ($5 \leq m \leq n-1$). By using Cramer's rule we have

$$\mathcal{H}_{M;m}^{(n)}(t) = \frac{T_m}{\det(W_n)}, \quad (21)$$

where T_m is a determinant obtained by replacing m th column of W_n by column of B ; i.e.,

$$\mathcal{H}_{M;m}^{(n)}(t) = \frac{1}{t^n D_n(1/t)} \begin{vmatrix} 1-t & -t & \cdots & -t & t & \cdots & -t & -t \\ 0 & 1-t & \cdots & -t & t & \cdots & -t & -t \\ -t & -t & \cdots & -t & t & \cdots & -t & -t \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & t & \cdots & 0 & 1-t \end{vmatrix}. \quad (22)$$

Let C_i denote the i th column of T_m . Adding C_m in $C_{m+1}, C_{m+2}, \dots, C_n$ of T_m and simplifying it we have a determinant of order m . Hence

$$\mathcal{H}_{M;m}^{(n)}(t) = \frac{1}{t^n D_n(1/t)} \begin{vmatrix} 1-t & -t & \cdots & -t & t \\ 0 & 1-t & \cdots & -t & t \\ -t & -t & \cdots & -t & t \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -t & t \end{vmatrix}. \quad (23)$$

Now by adding $(m-1)$ th column of the last determinant in its m th column and simplifying it we finally have

$$\mathcal{H}_{M;m}^{(n)}(t) = \frac{t^{m-1}K_{m-2}(1/t)}{t^n D_n(1/t)}. \quad (24)$$

Case II ($m = n$). Using again Cramer's rule, we have

$$\mathcal{H}_{M;n}^{(n)}(t) = \frac{1}{t^n D_n(1/t)} \begin{vmatrix} 1-t & -t & \cdots & -t & -t & -t & t \\ 0 & 1-t & \cdots & -t & -t & -t & t \\ -t & -t & \cdots & -t & -t & -t & t \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -t & 1-t & -t & t \\ 0 & 0 & \cdots & 0 & -t & 1-t & t \\ 0 & 0 & \cdots & 0 & -t & 0 & t \end{vmatrix}. \quad (25)$$

Adding C_n in C_{n-2} and simplifying we have the result

$$\mathcal{H}_{M;n}^{(n)}(t) = \frac{(1-t)t^{n-2}K_{n-3}(1/t)}{t^n D_n(1/t)}. \quad (26)$$

□

Now we have our main result.

Theorem 10. The Hilbert series of the monoid $M(\widetilde{D}_n^\infty)$ is

$$\mathcal{H}_M^{(n)}(t) = \frac{1}{t^n D_n(1/t)}. \quad (27)$$

Proof. From Theorem 7 we have

$$\begin{aligned} \mathcal{H}_M^{(n)}(t) &= 1 + \mathcal{H}_{M;1}^{(n)}(t) + \mathcal{H}_{M;2}^{(n)}(t) + \cdots \\ &+ \mathcal{H}_{M;n-1}^{(n)}(t) + \mathcal{H}_{M;n}^{(n)}(t) = \frac{1}{t^n D_n(1/t)} \left(t^n D_n \left(\frac{1}{t} \right) \right. \\ &+ t^3 - 3t^2 + 4t + t^4 K_3 \left(\frac{1}{t} \right) + t^5 K_4 \left(\frac{1}{t} \right) + \cdots \\ &+ t^{n-1} K_{n-2} \left(\frac{1}{t} \right) + (1-t) t^{n-1} K_{n-3} \left(\frac{1}{t} \right) \Big) \\ &= \frac{1}{t^n D_n(1/t)} \left((1-t) t^{n-1} K_{n-1} \left(\frac{1}{t} \right) \right. \\ &- t^{n-2} K_{n-2} \left(\frac{1}{t} \right) + t^3 - 3t^2 + 4t + t^4 B_3 \left(\frac{1}{t} \right) + \cdots \\ &+ t^{n-1} K_{n-2} \left(\frac{1}{t} \right) + (1-t) t^{n-1} K_{n-3} \left(\frac{1}{t} \right) \Big) \\ &= \frac{1}{t^n D_n(1/t)} \left(t^4 K_4 \left(\frac{1}{t} \right) + t^3 - 3t^2 + 4t \right) \\ &= \frac{1}{t^n D_n(1/t)} \left(t^4 \left(\frac{1}{t^4} - \frac{4}{t^3} + \frac{3}{t^2} - \frac{1}{t} \right) + t^3 - 3t^2 \right. \\ &\left. + 4t \right) = \frac{1}{t^n D_n(1/t)}. \end{aligned} \quad (28)$$

□

FIGURE 4

4. Conjecture on the Upper Bound of Growth Rate of $M(\tilde{D}_n^\infty)$

In this section we compute the growth rates of $M(\tilde{D}_n^\infty)$ and show them on the graph. We see that the growth rate is bounded above by 4. Let r_n be the growth rate (or the maximal root of the polynomial $D_n(\lambda)$). We compute few initial growth rates (using Mathematica and derive 6) for $M(\tilde{D}_n^\infty)$. Let r_n denote the growth rate of $M(\tilde{D}_n^\infty)$; then we have the following few initial values of r_n : $r_5 = 3.629$, $r_6 = 3.732$, $r_7 = 3.791$, $r_8 = 3.831$, $r_9 = 3.858$, $r_{10} = 3.879$, $r_{11} = 3.895$, $r_{12} = 3.907$, $r_{13} = 3.917$, $r_{14} = 3.926$, $r_{15} = 3.933$, $r_{16} = 3.939$, $r_{17} = 3.944$, $r_{18} = 3.948$, $r_{19} = 3.952$, $r_{20} = 3.956$. We also compute $r_{40} = 3.984$, $r_{60} = 3.992$, $r_{80} = 3.995$, $r_{100} = 3.996$, and $r_{120} = 3.997$ (using Mathematica). We have Figure 4 representing the growth rate of $M(\tilde{D}_n^\infty)$.

We observe that the growth rate for $M(\tilde{D}_n^\infty)$ approaching 4 as n approaches ∞ . Hence at the end we have the following.

Conjecture. The growth rate of $M(\tilde{D}_n^\infty)$ is bounded above by 4.

Data Availability

No such data has been used to prove these results.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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