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Research Article

A New Inequality for Frames in Hilbert Spaces

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We obtain a new inequality for frames in Hilbert spaces associated with a scalar and a bounded linear operator induced by two Bessel sequences. It turns out that the corresponding results due to Balan et al. and Găvruţa can be deduced from our result.

1. Introduction

A frame for a Hilbert space firstly emerged in the work on nonharmonic Fourier series owing to Duffin and Schaeffer [1], which has made great contributions to various fields because of its nice properties; the reader can examine the papers [2–12] for background and details of frames.

Balan et al. in [13] showed us a surprising inequality when they further investigated the Parseval frame identity derived from their study on efficient algorithms for signal reconstruction, which was then extended to general frames and alternate dual frames by Găvruţa [14]. In this paper, we establish a new inequality for frames in Hilbert spaces, where a scalar and a bounded linear operator with respect to two Bessel sequences are involved, and it is shown that our result can lead to the corresponding results of Balan et al. and Găvruţa.

The notations $\mathscr{H}$, $\mathrm{Id}_{\mathscr{H}}$, and $\mathbb{J}$ are reserved, respectively, for a complex Hilbert space, the identity operator on $\mathscr{H}$, and an index set which is finite or countable. The algebra of all bounded linear operators on $\mathscr{H}$ is designated as $B(\mathscr{H})$.

One says that a family $\{f_j\}_{j\in\mathbb{J}}$ of vectors in $\mathscr{H}$ is a frame, if there are two positive constants $C, D > 0$ satisfying

$$C\|f\|^2 \le \sum_{j\in\mathbb{J}} \left|\left\langle f, f_j\right\rangle\right|^2 \le D\|f\|^2, \quad \forall f \in \mathscr{H}. \tag{1}$$

The frame $\{f_j\}_{j\in\mathbb{J}}$ is said to be Parseval if $C = D = 1$. If $\{f_j\}_{j\in\mathbb{J}}$ satisfies the inequality to the right in (1), we call that $\{f_j\}_{j\in\mathbb{J}}$ is a Bessel sequence for $\mathscr{H}$.

For a given frame $\mathscr{F} = \{f_j\}_{j\in\mathbb{J}}$, the frame operator $S_{\mathscr{F}}$, a positive, self-adjoint, and invertible operator on $\mathscr{H}$, is defined by

$$\begin{gathered} S_{\mathscr{F}} : \mathscr{H} \longrightarrow \mathscr{H}, \\ S_{\mathscr{F}} f = \sum_{j\in\mathbb{J}} \left\langle f, f_j\right\rangle f_j, \\ \forall f \in \mathscr{H}, \end{gathered} \tag{2}$$

from which we see that

$$f = \sum_{j\in\mathbb{J}} \left\langle f, f_j\right\rangle S_{\mathscr{F}}^{-1} f_j = \sum_{j\in\mathbb{J}} \left\langle f, S_{\mathscr{F}}^{-1} f_j\right\rangle f_j, \quad \forall f \in \mathscr{H}, \tag{3}$$

where the involved frame $\{\widetilde{f}_j = S_{\mathscr{F}}^{-1} f_j\}_{j\in\mathbb{J}}$ is said to be the canonical dual of $\{f_j\}_{j\in\mathbb{J}}$.

For any $\mathbb{I} \subset \mathbb{J}$, denote $\mathbb{I}^c = \mathbb{J} \setminus \mathbb{I}$. A positive, bounded linear, and self-adjoint operator induced by $\mathbb{I}$ and the frame $\mathscr{F} = \{f_j\}_{j\in\mathbb{J}}$ is given below

$$\begin{gathered} S_{\mathbb{I}}^{\mathscr{F}} : \mathscr{H} \longrightarrow \mathscr{H}, \\ S_{\mathbb{I}}^{\mathscr{F}} f = \sum_{j\in\mathbb{I}} \left\langle f, f_j\right\rangle f_j, \\ \forall f \in \mathscr{H}. \end{gathered} \tag{4}$$

Suppose that $\mathscr{F} = \{f_j\}_{j\in\mathbb{J}}$ and $\mathscr{G} = \{g_j\}_{j\in\mathbb{J}}$ are two Bessel sequences for $\mathscr{H}$. An application of the Cauchy-Schwartzinequality can show that the operator

$$S_{\mathcal{F}\mathcal{G}} : \mathcal{H} \longrightarrow \mathcal{H},$$
$$S_{\mathcal{F}\mathcal{G}} f = \sum_{j\in\mathbb{J}} \left\langle f, g_j \right\rangle f_j \tag{5}$$

is well-defined and further $S_{\mathcal{F}\mathcal{G}} \in B(\mathcal{H})$. Particularly, if $S_{\mathcal{F}\mathcal{G}} = \mathrm{Id}_{\mathcal{H}}$, then both $\{f_j\}_{j\in\mathbb{J}}$ and $\{g_j\}_{j\in\mathbb{J}}$ are frames for $\mathcal{H}$. In this case we say that $\{g_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of $\{f_j\}_{j\in\mathbb{J}}$, and the pair $(\{f_j\}_{j\in\mathbb{J}}, \{g_j\}_{j\in\mathbb{J}})$ is called an alternate dual frame pair.

2. The Main Results

We need the following simple result on operators to present our main result.

Lemma 1. *Suppose that $U, V, L \in B(\mathcal{H})$ and that $U + V = L$. Then for each $\lambda \in [0,1]$ we have*

$$\begin{aligned} &U^*U + \lambda\left(V^*L + L^*V\right) \\ &\quad = V^*V + (1-\lambda)\left(U^*L + L^*U\right) + (2\lambda - 1) L^*L \\ &\quad \geq \left(2\lambda - \lambda^2\right) L^*L. \end{aligned} \tag{6}$$

Proof. A direct calculation gives

$$\begin{aligned} &U^*U + \lambda\left(V^*L + L^*V\right) \\ &\quad = U^*U + \lambda\left(\left(L^* - U^*\right)L + L^*\left(L - U\right)\right) \\ &\quad = U^*U + \lambda\left(L^*L - U^*L + L^*L - L^*U\right) \\ &\quad = U^*U - \lambda\left(U^*L + L^*U\right) + 2\lambda L^*L. \end{aligned} \tag{7}$$

From this fact and taking into account that

$$\begin{aligned} &V^*V + (1-\lambda)\left(U^*L + L^*U\right) + (2\lambda - 1) L^*L \\ &\quad = \left(L^* - U^*\right)\left(L - U\right) + (1-\lambda)\left(U^*L + L^*U\right) \\ &\qquad + (2\lambda - 1) L^*L \\ &\quad = L^*L - \left(L^*U + U^*L\right) + U^*U \\ &\qquad + (1-\lambda)\left(U^*L + L^*U\right) + (2\lambda - 1) L^*L \\ &\quad = U^*U - \lambda\left(U^*L + L^*U\right) + 2\lambda L^*L \\ &\quad = (U - \lambda L)^*(U - \lambda L) + \left(2\lambda - \lambda^2\right) L^*L \\ &\quad \geq \left(2\lambda - \lambda^2\right) L^*L, \end{aligned} \tag{8}$$

we arrive at the relation stated in the lemma. □

We can immediately get the following result obtained by Poria in [15], when putting $L = \mathrm{Id}_{\mathcal{H}}$ in Lemma 1.

Corollary 2. *Suppose that $U, V \in B(\mathcal{H})$ and that $U + V = \mathrm{Id}_{\mathcal{H}}$. Then for every $\lambda \in [0,1]$ we have*

$$\begin{aligned} U^*U + \lambda\left(V^* + V\right) &= V^*V + (1-\lambda)\left(U^* + U\right) \\ &\quad + (2\lambda - 1)\,\mathrm{Id}_{\mathcal{H}} \\ &\geq \left(2\lambda - \lambda^2\right) \mathrm{Id}_{\mathcal{H}}. \end{aligned} \tag{9}$$

Theorem 3. *Suppose that $\{f_j\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{H}$, that $\{g_j\}_{j\in\mathbb{J}}$ and $\{h_j\}_{j\in\mathbb{J}}$ are two Bessel sequences for $\mathcal{H}$, and that the operator $S_{\mathcal{F}\mathcal{G}}$ is defined by (5). Then for each $\lambda \in [0,1]$ and each $f \in \mathcal{H}$, we have*

$$\begin{aligned} &\left\|\sum_{j\in\mathbb{J}} \left\langle f, g_j - h_j \right\rangle f_j\right\|^2 + \operatorname{Re}\sum_{j\in\mathbb{J}} \left\langle f, h_j \right\rangle \left\langle f_j, S_{\mathcal{F}\mathcal{G}} f \right\rangle \\ &\quad = \left\|\sum_{j\in\mathbb{J}} \left\langle f, h_j \right\rangle f_j\right\|^2 \\ &\qquad + \operatorname{Re}\sum_{j\in\mathbb{J}} \left\langle f, g_j - h_j \right\rangle \left\langle f_j, S_{\mathcal{F}\mathcal{G}} f \right\rangle \\ &\quad \geq \left(2\lambda - \lambda^2\right) \operatorname{Re}\sum_{j\in\mathbb{J}} \left\langle f, g_j - h_j \right\rangle \left\langle f_j, S_{\mathcal{F}\mathcal{G}} f \right\rangle \\ &\qquad + \left(1 - \lambda^2\right) \operatorname{Re}\sum_{j\in\mathbb{J}} \left\langle f, h_j \right\rangle \left\langle f_j, S_{\mathcal{F}\mathcal{G}} f \right\rangle. \end{aligned} \tag{10}$$

Moreover, if $S_{\mathcal{F}\mathcal{G}}$ is self-adjoint, then for any $\lambda \in [0,1]$ and any $f \in \mathcal{H}$,

$$\begin{aligned} &\left\|\sum_{j\in\mathbb{J}} \left\langle f, f_j \right\rangle \left(g_j - h_j\right)\right\|^2 + \operatorname{Re}\sum_{j\in\mathbb{J}} \left\langle f, S_{\mathcal{F}\mathcal{G}} h_j \right\rangle \left\langle f_j, f \right\rangle \\ &\quad = \left\|\sum_{j\in\mathbb{J}} \left\langle f, f_j \right\rangle h_j\right\|^2 \\ &\qquad + \operatorname{Re}\sum_{j\in\mathbb{J}} \left\langle f, S_{\mathcal{F}\mathcal{G}}\left(g_j - h_j\right) \right\rangle \left\langle f_j, f \right\rangle \\ &\quad \geq \left(2\lambda - \lambda^2\right) \operatorname{Re}\sum_{j\in\mathbb{J}} \left\langle f, S_{\mathcal{F}\mathcal{G}}\left(g_j - h_j\right) \right\rangle \left\langle f_j, f \right\rangle \\ &\qquad + \left(1 - \lambda^2\right) \operatorname{Re}\sum_{j\in\mathbb{J}} \left\langle f, S_{\mathcal{F}\mathcal{G}} h_j \right\rangle \left\langle f_j, f \right\rangle. \end{aligned} \tag{11}$$

Proof. We take $Uf = \sum_{j\in\mathbb{J}} \langle f, g_j - h_j \rangle f_j$ and $Vf = \sum_{j\in\mathbb{J}} \langle f, h_j \rangle f_j$ for any $f \in \mathcal{H}$. Then $U, V \in B(\mathcal{H})$ and further

$$\begin{aligned} Uf + Vf &= \sum_{j\in\mathbb{J}} \left\langle f, g_j - h_j \right\rangle f_j + \sum_{j\in\mathbb{J}} \left\langle f, h_j \right\rangle f_j \\ &= \sum_{j\in\mathbb{J}} \left\langle f, g_j \right\rangle f_j = S_{\mathcal{F}\mathcal{G}} f. \end{aligned} \tag{12}$$

By Lemma 1 we have

$$\begin{aligned} &\|Uf\|^2 + 2\lambda \operatorname{Re}\left\langle S_{\mathcal{F}\mathcal{G}}^* V f, f \right\rangle \\ &\quad = \|Vf\|^2 + 2(1-\lambda)\operatorname{Re}\left\langle S_{\mathcal{F}\mathcal{G}}^* U f, f \right\rangle \\ &\qquad + (2\lambda - 1)\operatorname{Re}\left\langle S_{\mathcal{F}\mathcal{G}} f, S_{\mathcal{F}\mathcal{G}} f \right\rangle. \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned}\|Uf\|^2 &= \|Vf\|^2 + 2(1-\lambda)\operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Uf, f\rangle \\ &\quad + (2\lambda-1)\operatorname{Re}\langle S_{\mathcal{F}\mathcal{G}}f, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &\quad - 2\lambda\operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Vf, f\rangle \\ &= \|Vf\|^2 + 2\operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Uf, f\rangle \\ &\quad - 2\lambda\operatorname{Re}\langle (S^*_{\mathcal{F}\mathcal{G}}U + S^*_{\mathcal{F}\mathcal{G}}V) f, f\rangle \\ &\quad + (2\lambda-1)\operatorname{Re}\langle S_{\mathcal{F}\mathcal{G}}f, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &= \|Vf\|^2 + 2\operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Uf, f\rangle \\ &\quad - \operatorname{Re}\langle S_{\mathcal{F}\mathcal{G}}f, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &= \|Vf\|^2 + 2\operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Uf, f\rangle \\ &\quad - \operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}(U+V) f, f\rangle \\ &= \|Vf\|^2 + \operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Uf, f\rangle - \operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Vf, f\rangle .\end{aligned} \tag{14}$$

It follows that

$$\begin{aligned}&\left\|\sum_{j\in\mathbb{J}}\langle f, g_j - h_j\rangle f_j\right\|^2 + \operatorname{Re}\sum_{j\in\mathbb{J}}\langle f, h_j\rangle\langle f_j, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &= \|Uf\|^2 + \operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Vf, f\rangle \\ &= \|Vf\|^2 + \operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Uf, f\rangle \\ &= \left\|\sum_{j\in\mathbb{J}}\langle f, h_j\rangle f_j\right\|^2 \\ &\quad + \operatorname{Re}\sum_{j\in\mathbb{J}}\langle f, g_j - h_j\rangle\langle f_j, S_{\mathcal{F}\mathcal{G}}f\rangle .\end{aligned} \tag{15}$$

We now prove the inequality in (10). Again by Lemma 1,

$$\begin{aligned}&\|Uf\|^2 + 2\lambda\operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Vf, f\rangle \\ &\quad \geq (2\lambda-\lambda^2)\langle S^*_{\mathcal{F}\mathcal{G}}S_{\mathcal{F}\mathcal{G}}f, f\rangle\end{aligned} \tag{16}$$

for every $f \in \mathcal{H}$. Hence,

$$\begin{aligned}\|Uf\|^2 &\geq (2\lambda-\lambda^2)\langle S^*_{\mathcal{F}\mathcal{G}}S_{\mathcal{F}\mathcal{G}}f, f\rangle \\ &\quad - 2\lambda\operatorname{Re}\langle S^*_{\mathcal{F}\mathcal{G}}Vf, f\rangle \\ &= (2\lambda-\lambda^2)\langle S_{\mathcal{F}\mathcal{G}}f, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &\quad - 2\lambda\operatorname{Re}\langle Vf, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &= (2\lambda-\lambda^2)\operatorname{Re}\langle (U+V) f, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &\quad - 2\lambda\operatorname{Re}\langle Vf, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &= (2\lambda-\lambda^2)\operatorname{Re}\langle Uf, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &\quad - \lambda^2\operatorname{Re}\langle Vf, S_{\mathcal{F}\mathcal{G}}f\rangle ,\end{aligned} \tag{17}$$

from which we conclude that

$$\begin{aligned}&\left\|\sum_{j\in\mathbb{J}}\langle f, g_j - h_j\rangle f_j\right\|^2 + \operatorname{Re}\sum_{j\in\mathbb{J}}\langle f, h_j\rangle\langle f_j, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &= \|Uf\|^2 + \operatorname{Re}\langle Vf, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &\geq (2\lambda-\lambda^2)\operatorname{Re}\langle Uf, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &\quad + (1-\lambda^2)\operatorname{Re}\langle Vf, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &= (2\lambda-\lambda^2)\operatorname{Re}\sum_{j\in\mathbb{J}}\langle f, g_j - h_j\rangle\langle f_j, S_{\mathcal{F}\mathcal{G}}f\rangle \\ &\quad + (1-\lambda^2)\operatorname{Re}\sum_{j\in\mathbb{J}}\langle f, h_j\rangle\langle f_j, S_{\mathcal{F}\mathcal{G}}f\rangle .\end{aligned} \tag{18}$$

The proof of (11) is similar to the proof of (10); we leave the details to the reader. □

Corollary 4. *Suppose that $\{f_j\}_{j\in\mathbb{J}}$ is a frame for $\mathcal{H}$ with frame operator $S_{\mathcal{F}}$ and that $\widetilde{f}_j = S_{\mathcal{F}}^{-1}f_j$ for any $j \in \mathbb{J}$. Then for all $\lambda \in [0,1]$, for any $\mathbb{I} \subset \mathbb{J}$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned}&\sum_{j\in\mathbb{I}^c}\left|\langle f, f_j\rangle\right|^2 + \sum_{j\in\mathbb{J}}\left|\langle S^{\mathcal{F}}_{\mathbb{I}}f, \widetilde{f}_j\rangle\right|^2 \\ &\quad = \sum_{j\in\mathbb{I}}\left|\langle f, f_j\rangle\right|^2 + \sum_{j\in\mathbb{J}}\left|\langle S^{\mathcal{F}}_{\mathbb{I}^c}f, \widetilde{f}_j\rangle\right|^2 \\ &\quad \geq (2\lambda-\lambda^2)\sum_{j\in\mathbb{I}}\left|\langle f, f_j\rangle\right|^2 \\ &\quad\quad + (1-\lambda^2)\sum_{j\in\mathbb{I}^c}\left|\langle f, f_j\rangle\right|^2 .\end{aligned} \tag{19}$$

Proof. Setting $g_j = S_{\mathcal{F}}^{-1/2}f_j$ for each $j \in \mathbb{J}$, then $S_{\mathcal{F}\mathcal{G}} = S_{\mathcal{F}}^{1/2}$. Taking

$$h_j = \begin{cases} 0, & j \in \mathbb{I}, \\ g_j, & j \in \mathbb{I}^c, \end{cases} \tag{20}$$

then $\{g_j\}_{j\in\mathbb{J}}$ and $\{h_j\}_{j\in\mathbb{J}}$ are both Bessel sequences for $\mathcal{H}$. For any $f \in \mathcal{H}$ we have

$$\begin{aligned}&\left\|\sum_{j\in\mathbb{J}}\langle f, f_j\rangle(g_j - h_j)\right\|^2 = \left\|\sum_{j\in\mathbb{I}}\langle f, f_j\rangle S_{\mathcal{F}}^{-1/2}f_j\right\|^2 \\ &\quad = \left\|S_{\mathcal{F}}^{-1/2}S^{\mathcal{F}}_{\mathbb{I}}f\right\|^2 = \langle S_{\mathcal{F}}^{-1/2}S^{\mathcal{F}}_{\mathbb{I}}f, S_{\mathcal{F}}^{-1/2}S^{\mathcal{F}}_{\mathbb{I}}f\rangle \\ &\quad = \langle S^{\mathcal{F}}_{\mathbb{I}}f, S_{\mathcal{F}}^{-1}S^{\mathcal{F}}_{\mathbb{I}}f\rangle = \langle S_{\mathcal{F}}S_{\mathcal{F}}^{-1}S^{\mathcal{F}}_{\mathbb{I}}f, S_{\mathcal{F}}^{-1}S^{\mathcal{F}}_{\mathbb{I}}f\rangle \\ &\quad = \sum_{j\in\mathbb{J}}\langle S_{\mathcal{F}}^{-1}S^{\mathcal{F}}_{\mathbb{I}}f, f_j\rangle\langle f_j, S_{\mathcal{F}}^{-1}S^{\mathcal{F}}_{\mathbb{I}}f\rangle \\ &\quad = \sum_{j\in\mathbb{J}}\langle S^{\mathcal{F}}_{\mathbb{I}}f, S_{\mathcal{F}}^{-1}f_j\rangle\langle S_{\mathcal{F}}^{-1}f_j, S^{\mathcal{F}}_{\mathbb{I}}f\rangle \\ &\quad = \sum_{j\in\mathbb{J}}\left|\langle S^{\mathcal{F}}_{\mathbb{I}}f, \widetilde{f}_j\rangle\right|^2 .\end{aligned} \tag{21}$$

A similar discussion yields

$$\left\|\sum_{j\in\mathbb{J}}\left\langle f, f_j\right\rangle h_j\right\|^2=\sum_{j\in\mathbb{J}}\left|\left\langle S_{\mathbb{I}^c}^{\mathcal{F}} f, \widetilde{f}_j\right\rangle\right|^2. \tag{22}$$

We also have

$$\begin{gathered}\operatorname{Re}\sum_{j\in\mathbb{J}}\left\langle f, S_{\mathcal{F}\mathcal{G}} h_j\right\rangle\left\langle f_j, f\right\rangle=\sum_{j\in\mathbb{I}^c}\left|\left\langle f, f_j\right\rangle\right|^2, \\ \operatorname{Re}\sum_{j\in\mathbb{J}}\left\langle f, S_{\mathcal{F}\mathcal{G}}\left(g_j-h_j\right)\right\rangle\left\langle f_j, f\right\rangle=\sum_{j\in\mathbb{I}}\left|\left\langle f, f_j\right\rangle\right|^2.\end{gathered} \tag{23}$$

Thus the result follows from Theorem 3. □

Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval frame for $\mathcal{H}$; then $S_{\mathcal{F}}=\mathrm{Id}_{\mathcal{H}}$. Thus for any $\mathbb{I}\subset\mathbb{J}$,

$$\begin{aligned}\sum_{j\in\mathbb{J}}\left|\left\langle S_{\mathbb{I}}^{\mathcal{F}} f, \widetilde{f}_j\right\rangle\right|^2&=\sum_{j\in\mathbb{J}}\left|\left\langle S_{\mathbb{I}}^{\mathcal{F}} f, f_j\right\rangle\right|^2=\left\|S_{\mathbb{I}}^{\mathcal{F}} f\right\|^2 \\ &=\left\|\sum_{j\in\mathbb{I}}\left\langle f, f_j\right\rangle f_j\right\|^2, \quad \forall f\in\mathcal{H}.\end{aligned} \tag{24}$$

Similarly we have

$$\sum_{j\in\mathbb{J}}\left|\left\langle S_{\mathbb{I}^c}^{\mathcal{F}} f, \widetilde{f}_j\right\rangle\right|^2=\left\|\sum_{j\in\mathbb{I}^c}\left\langle f, f_j\right\rangle f_j\right\|^2. \tag{25}$$

This together with Corollary 4 leads to a result as follows.

Corollary 5. *Suppose that $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval frame for $\mathcal{H}$. Then for each $\lambda\in[0,1]$, for any $\mathbb{I}\subset\mathbb{J}$ and any $f\in\mathcal{H}$, we have*

$$\begin{aligned}&\sum_{j\in\mathbb{I}^c}\left|\left\langle f, f_j\right\rangle\right|^2+\left\|\sum_{j\in\mathbb{I}}\left\langle f, f_j\right\rangle f_j\right\|^2 \\ &\quad=\sum_{j\in\mathbb{I}}\left|\left\langle f, f_j\right\rangle\right|^2+\left\|\sum_{j\in\mathbb{I}^c}\left\langle f, f_j\right\rangle f_j\right\|^2 \\ &\quad\geq\left(2\lambda-\lambda^2\right)\sum_{j\in\mathbb{I}}\left|\left\langle f, f_j\right\rangle\right|^2 \\ &\qquad+\left(1-\lambda^2\right)\sum_{j\in\mathbb{I}^c}\left|\left\langle f, f_j\right\rangle\right|^2.\end{aligned} \tag{26}$$

Corollary 6. *Suppose that $(\{f_j\}_{j\in\mathbb{J}},\{g_j\}_{j\in\mathbb{J}})$ is an alternate dual frame pair for $\mathcal{H}$. Then for each $\lambda\in[0,1]$, for any $\mathbb{I}\subset\mathbb{J}$ and any $f\in\mathcal{H}$, we have*

$$\begin{aligned}&\left\|\sum_{j\in\mathbb{I}}\left\langle f, g_j\right\rangle f_j\right\|^2+\operatorname{Re}\sum_{j\in\mathbb{I}^c}\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle \\ &\quad=\left\|\sum_{j\in\mathbb{I}^c}\left\langle f, g_j\right\rangle f_j\right\|^2+\operatorname{Re}\sum_{j\in\mathbb{I}}\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle \\ &\quad\geq\left(2\lambda-\lambda^2\right)\operatorname{Re}\sum_{j\in\mathbb{I}}\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle \\ &\qquad+\left(1-\lambda^2\right)\operatorname{Re}\sum_{j\in\mathbb{I}^c}\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle.\end{aligned} \tag{27}$$

Proof. Since $\{g_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of $\{f_j\}_{j\in\mathbb{J}}$, $S_{\mathcal{F}\mathcal{G}}=\mathrm{Id}_{\mathcal{H}}$. For any $j\in\mathbb{J}$, let

$$h_j=\begin{cases}0, & j\in\mathbb{I}, \\ g_j, & j\in\mathbb{I}^c.\end{cases} \tag{28}$$

On the one hand we have

$$\begin{gathered}\left\|\sum_{j\in\mathbb{J}}\left\langle f, g_j-h_j\right\rangle f_j\right\|^2=\left\|\sum_{j\in\mathbb{I}}\left\langle f, g_j\right\rangle f_j\right\|^2, \\ \left\|\sum_{j\in\mathbb{J}}\left\langle f, h_j\right\rangle f_j\right\|^2=\left\|\sum_{j\in\mathbb{I}^c}\left\langle f, g_j\right\rangle f_j\right\|^2.\end{gathered} \tag{29}$$

On the other hand we have

$$\begin{aligned}&\operatorname{Re}\sum_{j\in\mathbb{J}}\left\langle f, h_j\right\rangle\left\langle f_j, S_{\mathcal{F}\mathcal{G}} f\right\rangle=\operatorname{Re}\sum_{j\in\mathbb{I}^c}\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle, \\ &\operatorname{Re}\sum_{j\in\mathbb{J}}\left\langle f, g_j-h_j\right\rangle\left\langle f_j, S_{\mathcal{F}\mathcal{G}} f\right\rangle \\ &\quad=\operatorname{Re}\sum_{j\in\mathbb{I}}\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle.\end{aligned} \tag{30}$$

By Theorem 3 the conclusion follows. □

Remark 7. Theorems 2.2 and 3.2 in [14] and Proposition 4.1 in [13] can be obtained when taking $\lambda=1/2$, respectively, in Corollaries 4, 6, and 5.

As a matter of fact, we can establish a more general inequality for alternate dual frames than that shown in Corollary 6.

Theorem 8. *Suppose that* $(\{f_j\}_{j\in\mathbb{J}}, \{g_j\}_{j\in\mathbb{J}})$ *is an alternate dual frame pair for* $\mathscr{H}$*. Then for every bounded sequence* $\{\omega_j\}_{j\in\mathbb{J}}$*, for all* $\lambda \in [0,1]$ *and all* $f \in \mathscr{H}$*, we have*

$$\begin{aligned} &\operatorname{Re}\sum_{j\in\mathbb{J}}\omega_j\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle + \left\|\sum_{j\in\mathbb{J}}\left(1-\omega_j\right)\left\langle f, g_j\right\rangle f_j\right\|^2 \\ &\geq \left(2\lambda-\lambda^2\right)\operatorname{Re}\sum_{j\in\mathbb{J}}\left(1-\omega_j\right)\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle \\ &\quad + \left(1-\lambda^2\right)\operatorname{Re}\sum_{j\in\mathbb{J}}\omega_j\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle. \end{aligned} \tag{31}$$

Proof. We define the operators F_ω and $F_{1-\omega}$ by

$$\begin{aligned} F_\omega f &= \sum_{j\in\mathbb{J}}\omega_j\left\langle f, g_j\right\rangle f_j, \\ F_{1-\omega} f &= \sum_{j\in\mathbb{J}}\left(1-\omega_j\right)\left\langle f, g_j\right\rangle f_j. \end{aligned} \tag{32}$$

Then both series converge unconditionally and $F_\omega, F_{1-\omega} \in B(\mathscr{H})$. Since $F_\omega + F_{1-\omega} = \mathrm{Id}_{\mathscr{H}}$, by Corollary 2 we obtain

$$\begin{aligned} &\left\langle F_{1-\omega}^* F_{1-\omega} f, f\right\rangle + \lambda\overline{\left\langle F_\omega f, f\right\rangle} + \lambda\left\langle F_\omega f, f\right\rangle \\ &\geq \left(2\lambda-\lambda^2\right)\|f\|^2 \end{aligned} \tag{33}$$

for each $f \in \mathscr{H}$. Hence

$$\left\|F_{1-\omega} f\right\|^2 + 2\lambda\operatorname{Re}\left\langle F_\omega f, f\right\rangle \geq \left(2\lambda-\lambda^2\right)\langle f, f\rangle. \tag{34}$$

Therefore,

$$\begin{aligned} \left\|F_{1-\omega} f\right\|^2 &\geq \left(2\lambda-\lambda^2\right)\langle f, f\rangle - 2\lambda\operatorname{Re}\left\langle F_\omega f, f\right\rangle \\ &= \left(2\lambda-\lambda^2\right)\operatorname{Re}\left\langle\left(F_\omega + F_{1-\omega}\right) f, f\right\rangle \\ &\quad - 2\lambda\operatorname{Re}\left\langle F_\omega f, f\right\rangle \\ &= \left(2\lambda-\lambda^2\right)\operatorname{Re}\left\langle F_{1-\omega} f, f\right\rangle \\ &\quad - \lambda^2\operatorname{Re}\left\langle F_\omega f, f\right\rangle. \end{aligned} \tag{35}$$

It follows that

$$\begin{aligned} &\operatorname{Re}\sum_{j\in\mathbb{J}}\omega_j\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle + \left\|\sum_{j\in\mathbb{J}}\left(1-\omega_j\right)\left\langle f, g_j\right\rangle f_j\right\|^2 \\ &= \operatorname{Re}\left\langle F_\omega f, f\right\rangle + \left\|F_{1-\omega} f\right\|^2 \\ &\geq \left(2\lambda-\lambda^2\right)\operatorname{Re}\left\langle F_{1-\omega} f, f\right\rangle \\ &\quad + \left(1-\lambda^2\right)\operatorname{Re}\left\langle F_\omega f, f\right\rangle \\ &= \left(2\lambda-\lambda^2\right)\operatorname{Re}\sum_{j\in\mathbb{J}}\left(1-\omega_j\right)\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle \\ &\quad + \left(1-\lambda^2\right)\operatorname{Re}\sum_{j\in\mathbb{J}}\omega_j\left\langle f, g_j\right\rangle\left\langle f_j, f\right\rangle. \end{aligned} \tag{36}$$

This completes the proof. □

Remark 9. If we take $\lambda = 1/2$ in Theorem 8, then we can obtain Theorem 3.3 in [14].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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