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Research Article

A Hybrid Proximal Algorithm for the Sum of Monotone Operators with Multivalued Mappings

N. Kaewyong,[1] L. Kittiratanawasin,[2] C. Pukdeboon,[3] and K. Sitthithakerngkiet[3]

[1] *Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok (KMUTNB), 1518, Pracharat 1 Road, Wongsawang, Bangsue, Bangkok 10800, Thailand*
[2] *Department of Mathematics, Faculty of Science, Kasetsart University, 50 Ngam Wong Wan Road, Ladyaow, Chatuchak, Bangkok 10900, Thailand*
[3] *Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok (KMUTNB), 1518 Pracharat 1 Road, Wongsawang, Bangsue, Bangkok 10800, Thailand*

Correspondence should be addressed to K. Sitthithakerngkiet; kanokwan.s@sci.kmutnb.ac.th

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We modify a hybrid method and a proximal point algorithm to iteratively find a zero point of the sum of two monotone operators and fixed point of nonspreading multivalued mappings in a Hilbert space by using the technique of forward-backward splitting method. The strong convergence theorem is established and the illustrative numerical example is presented on this work. The results of this paper extend and improve some well-known results in the literature.

1. Introduction

In a Hilbert space, many authors have intensively studied the convergence of finding a zero point for monotone operators, that is, to find a point $x \in H$ such that

$$0 \in Ax, \tag{1}$$

where A is a monotone operator and the set of zero point of A is denoted by $A^{-1}(0)$. The first method for finding a zero point is introduced by Martinet [1] in 1970, it is well known as the *proximal point algorithm* (PPA) which generates a sequence

$$x_{n+1} = J_{r_n}^A(x_n), \quad \forall n \in \mathbb{N}, \tag{2}$$

where $J_{r_n}^A = (I + r_n A)^{-1}$ is the resolvent operator of maximal monotone operator A, I is the identity mapping and $\{r_n\} \subset (0, \infty)$ is a regularization sequence. It can be related to many kinds of important problems, such as convex minimization problems, equilibrium problems, and variational inequality problems. An iterative (2) is equivalent to

$$x_n \in x_{n+1} + r_n A x_{n+1} \quad \forall n \in \mathbb{N}. \tag{3}$$

It is known that $J_{r_n}^A$ can be reduced to

$$x_{n+1} = \operatorname{argmin}\left\{\phi(y) + \frac{1}{2r_n}\|x_n - y\|^2,\ y \in H\right\} \quad \forall n \in \mathbb{N}, \tag{4}$$

if we let $\phi(x) : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex and lower semicontinuous function.

Later, Rockafellar [2] presented an inexact variant of the following method:

$$x_{n+1} = J_{r_n}^A x_n + e_n, \quad \forall n \in \mathbb{N}, \tag{5}$$

where $\{e_n\}$ is an error sequence. Rockafellar [2] proved that if $e_n \to 0$ quickly enough such that $\sum_{n=1}^{\infty}\|e_n\| < \infty$, $\liminf_{n\to\infty} r_n > 0$, and $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a solution of a zero point of A.

In 1979, Lions and Mercier [3] presented the splitting algorithms to iteratively find zero point of the sum of two nonlinear operators. This algorithm is extended to solve

the nonlinear equations seeking a solution of the following inclusion problem:

$$0 \in A + B, \tag{6}$$

where A and B are two monotone operators. The inclusion problem can be formulated to many important problems, such as a stationary solution of the initial value problem of the evolution equation [3], the minimization problem [4], which is widely used in image recovery, signal processing, and machine learning, equilibrium problems, and variational inequality problems; see [5]. A splitting method for solving the inclusion problem (6) intends an iterative method for which each iteration involves only with the individual operators A and B but not $A+B$. Lions and Mercier [3] introduced the nonlinear splitting iterative algorithms to solve the inclusion problem (6), generated by

$$x_{n+1} = \left(2J_r^A - I\right)\left(2J_r^B - I\right)x_n, \quad \forall n \in \mathbb{N}, \tag{7}$$

$$x_{n+1} = \left[J_r^A\left(2J_r^B - I\right) + \left(I - J_r^B\right)\right]x_n, \quad \forall n \in \mathbb{N}, \tag{8}$$

where J_r^A and J_r^B are the resolvent operators of monotone operators A and B, respectively, with $r > 0$. The algorithm (7) is called the nonlinear Peaceman-Rachford splitting iterative algorithm. Since $(2J_r^A - I)(2J_r^B - I)$ is merely nonexpansive operator then it fails, in general, to converge but the mean averages of x_n can be weakly convergent; for more details see [6]. However, the algorithm, known as the nonlinear Douglas-Rachford splitting iterative algorithm (8), always converges in the weak topology to a point $J_r^B x$ because the operator $J_r^A(2J_r^B - I) + (I - J_r^B)$ is firmly nonexpansive.

The extended PPA is introduced by Manaka and Takahashi [7] to the case of sum of two monotone operators A and B by using the technique of forward-backward splitting method which generates a sequence $\{x_n\}$ defined by

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n) T J_{r_n}^A (I - r_n B) x_n, \quad \forall n \geq 1, \end{aligned} \tag{9}$$

where T is a nonexpansive mapping on a nonempty closed convex subset C of H, $J_{r_n}^A$ is the resolvent of a maximal monotone operator A with $\{r_n\}$ being a positive sequence, B is an inverse strongly monotone mapping, and $\{\alpha_n\}$ is a sequence in $(0,1)$. This algorithm shows that a sequence $\{x_n\}$ converges weakly to some point $z \in \mathrm{Fix}(T) \cap (A+B)^{-1}(0)$ provided that the control sequence satisfies some conditions.

In 2014, Cho et al. [8] presented the strong convergence theorem for the solution set $\mathrm{Fix}(T) \cap (A+B)^{-1}(0)$ in a Hilbert space by using the following iterative scheme:

$$\begin{aligned} &x_1 \in C, \\ &z_n = \alpha_n f(x_n) + (1-\alpha_n) x_n, \\ &y_n = J_{r_n}^A (z_n - r_n B z_n + e_n), \\ &x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n y_n + (1-\gamma_n) T y_n) \\ &\qquad \forall n \in \mathbb{N}, \end{aligned} \tag{10}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$, $\{r_n\}$ is a positive sequence, T is a strictly pseudo-contractive mapping with $k \in [0,1)$, and f is a contractive mapping.

An iterative algorithm for finding an approximate solution of the sum of two monotone operators and fixed point of several type mappings has received a lot of attention more recently; for more details, see [9–11].

On the other hand, Iemoto and Takahashi [12] study the approximation of common fixed points of a nonexpansive mapping T and a nonspreading mapping S (i.e., $2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2$) in a Hilbert space by using iterative scheme:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n) x_n + \alpha_n \left[\beta_n S x_n + (1-\beta_n) T x_n\right], \end{aligned} \tag{11}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. They proved that if $\mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ is nonempty then the sequence generated by (11) converges weakly to common fixed point of S and T. For the extension of mappings, many authors have studied the convergence theorems of multivalued mappings (see [13–15]).

In 2016, Suantai et al. [16] considered iterative schemes for solving split equilibrium problems and fixed point problems of nonspreading multivalued mappings in Hilbert spaces and proved that the modified Mann iteration converges weakly to a common solution of the considered problems.

Inspired by [8, 16], in this paper, we present the convergence analysis on the set $\mathrm{Fix}(T) \cap (A+B)^{-1}(0)$, where T is a nonspreading multivalued mapping in a Hilbert space. The results of this paper extend and improve some well-known results in the literature. Furthermore, the illustrative numerical example is presented.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. For any $x, y \in H$ and $\lambda \in [0,1]$, we see that

$$\begin{aligned} \|x+y\|^2 &\leq \|x\|^2 + 2\langle y, x+y\rangle, \\ \|\lambda x + (1-\lambda) y\|^2 &= \lambda \|x\|^2 + (1-\lambda)\|y\|^2 \\ &\quad - \lambda(1-\lambda)\|x-y\|^2. \end{aligned} \tag{12}$$

An operator $T : H \to H$ is called a nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \|x-y\|^2, \quad \forall x, y \in H, \tag{13}$$

and is called a firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in H. \tag{14}$$

Clearly, the above inequality is equivalent to

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \|x-y\|^2 - \|(I-T)x - (I-T)y\|^2 \\ &\qquad \forall x, y \in H, \end{aligned} \tag{15}$$

where I is identity mapping. For any point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \forall y \in C. \tag{16}$$

The operator P_C denotes the metric projection from H onto C. It is known that P_C is a firmly nonexpansive mapping; that is,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \tag{17}$$

Furthermore, for any $x \in H$ and $z \in C$, we note that $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \ge 0 \quad \forall y \in C. \tag{18}$$

Any subset C of a Hilbert space H is said to be *proximinal* if, for all $x \in H$, there exists $y \in C$ such that

$$\|x - y\| = d(x, C) = \inf\{\|x - z\| : z \in C\}. \tag{19}$$

In this paper, we denote the sets $CB(C)$, $K(C)$, and $P(C)$ are the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal subsets of C, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$\mathscr{H}(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, \quad \forall A, B \in CB(C), \tag{20}$$

where $d(x, B) = \inf_{b\in B}\|x - b\|$. Let $T : C \to CB(C)$ be a multivalue mapping, an element $p \in C$ is called a *fixed point* of T if $p \in Tp$ and we denote the fixed point set of a multivalue operator T by $\mathrm{Fix}(T)$. A multivalued mapping $T : C \to K(C)$ is said to be *nonexpansive* if $\mathscr{H}(Tx, Ty) \le \|x - y\|$ for all $x, y \in C$ and said to be *quasi-nonexpansive* if $\mathscr{H}(Tx, Tp) \le \|x - p\|$ for all $x \in C$ and $p \in \mathrm{Fix}(T)$. In this paper, we focus on a *k-nonspreading* multivalued mapping T that satisfies, for all $x, y \in C$,

$$\mathscr{H}(Tx, Ty)^2 \le k\left(d(Tx, y)^2 + d(x, Ty)^2\right), \tag{21}$$

for some $k > 0$.

Condition (I). Let H be a Hilbert space and C be a subset of H. A multivalued mapping $T : C \to CB(C)$ is said to satisfy *Condition (I)* if $\|x - p\| = d(x, Tp)$ for all $x \in H$ and $p \in \mathrm{Fix}(T)$.

Remark 1. It is easy to see that T satisfies Condition (I) if and only if $Tp = \{p\}$ for all $p \in \mathrm{Fix}(T)$. We know that if T is nonexpansive, then T is quasi-nonexpansive. Clearly, if T is a 1/2-nonspreading and $\mathrm{Fix}(T) \neq 0$, then T is quasi-nonexpansive. Example in [16] shows that T is a 1/2-nonspreading multivalued mapping which is not nonexpansive.

A mapping $B : C \to H$ is called *α-inverse strongly monotone*, if there exists $\alpha > 0$ such that

$$\langle x - y, Bx - By\rangle \ge \alpha\|Bx - By\|^2, \tag{22}$$

for all $x, y \in C$. We see that if B is *α-inverse strongly monotone*, then $\langle x - y, Bx - By\rangle \ge 0$ and $\|Bx - By\| \ge (1/\alpha)\|x - y\|$ for all $x, y \in C$. Moreover, for any constant $r > 0$, it is easy to see that

$$\begin{aligned}&\|(I - rB)x - (I - rB)y\|^2 \\ &\quad \le \|x - y\|^2 - r(2\alpha - r)\|Bx - By\|^2,\end{aligned} \tag{23}$$

where I is identity mapping. In particular, if $r \in (0, 2\alpha)$, then $(I - rB)$ is a nonexpansive mapping. For more example of inverse-strongly monotone mappings, see [17, 18].

Let A be a mapping of H into 2^H; the effective domain of A is denoted by $\mathrm{dom}(A)$; that is, $\mathrm{dom}(A) = \{x \in H : Ax \neq 0\}$. A multivalued mapping A is said to be a monotone operator on H if

$$\langle x - y, u - v\rangle \ge 0 \tag{24}$$

for all $x, y \in \mathrm{dom}(A)$, $u \in Ax$, and $v \in Ay$. A monotone operator A on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For maximal monotone operator A on H and $r > 0$, we may define a single-valued operator $J_r^A : H \to \mathrm{dom}(A)$ by $J_r^A = (I + rA)^{-1}$, which is called the resolvent of A for $r > 0$. If we let $B : C \to H$ be a single value operator and let A be a maximal monotone operator in H with $D(A) \supset C$ and $D(B) \supset C$, then, using the concept by [19], for $r > 0$,

$$\mathrm{Fix}\left(J_r^A(I - rB)\right) = (A + B)^{-1}(0). \tag{25}$$

Lemma 2 (see [16]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to K(C)$ be a k-nonspreading multivalued mapping with $k \in (0, 1/2]$. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

3. Main Results

Theorem 3. *Let H be a real Hilbert space and C be a nonempty closed convex subset of Hilbert spaces H. Let $A : D(A) \subseteq C \to 2^H$ be a maximal monotone operator, $B : C \to H$ be an α-inverse strongly monotone operator, and $T : C \to K(C)$ be a 1/2-nonspreading multivalued mapping. Assume that $\Theta = \mathrm{Fix}(T) \cap (A + B)^{-1}(0) \neq \emptyset$ and $\{x_n\}$ is a sequence defined by*

$$\begin{aligned}&x_0 \in C \text{ arbitrarily},\\ &u_n = J_{r_n}^A(I - r_n B)x_n,\\ &y_n \in a_n x_n + (1 - a_n)Tu_n,\\ &D_{n+1} = \left\{x \in D_n : \|y_n - x\|^2 \le \|x_n - x\|^2\right\},\\ &x_{n+1} = P_{D_{n+1}}x_0, \quad \forall n \ge 0,\ D_0 = C \subset H\end{aligned} \tag{26}$$

for all $n \ge 0$, where $\{a_n\} \subset (0, 1)$ and $\{r_n\}$ is a real number sequence in $(0, 2\alpha)$. Suppose that the following conditions hold:

(a) $\lim_{n\to\infty} a_n = 0$, *and* $\sum_{n=1}^{\infty} a_n = \infty$,

(b) $\lim_{n\to\infty} r_n = r$ *and* $r \in (0, \alpha)$.

Then, the sequence $\{x_n\}$ *converges strongly to a point* $x^* \in \Theta$.

Proof. First, we will show that $\Theta \subset D_n, \forall n \geq 0$ by using by the mathematical induction. Clearly, $\Theta \subset C = D_0$ and assume that $\Theta \subset D_n$ for some $n \geq 0$. Let $p \in \Theta$ be fixed. So, we can obtain that $p \in Tp$ and $p = J_{r_n}^A(I - r_n B)p$ and since $J_{r_n}^A$ and $(I - r_n B)$ are nonexpansive mappings, we have

$$\begin{aligned} \|u_n - p\|^2 &= \left\| J_{r_n}^A (I - r_n B) x_n - J_{r_n}^A (I - r_n B) p \right\|^2 \\ &\leq \|(I - r_n B) x_n - (I - r_n B) p\|^2 \\ &\leq \|x_n - p\|^2 . \end{aligned} \tag{27}$$

Since $y_n \in a_n x_n + (1 - a_n) T u_n$, there is $z_n \in T u_n$ such that $y_n = a_n x_n + (1 - a_n) z_n$ and then we get

$$\begin{aligned} \|y_n - p\|^2 &= \|a_n x_n + (1 - a_n) z_n - p\|^2 \\ &= a_n \|x_n - p\|^2 + (1 - a_n) \|z_n - p\|^2 \\ &\quad - a_n (1 - a_n) \|x_n - z_n\|^2 . \end{aligned} \tag{28}$$

From (27), (28), and Condition (I), it follows that

$$\begin{aligned} \|y_n - p\|^2 &\leq a_n \|x_n - p\|^2 + (1 - a_n) \|z_n - p\|^2 \\ &= a_n \|x_n - p\|^2 + (1 - a_n) d(z_n, Tp)^2 \\ &\leq a_n \|x_n - p\|^2 + (1 - a_n) \mathscr{H}(Tu_n, Tp)^2 \\ &\leq a_n \|x_n - p\|^2 + (1 - a_n) \|u_n - p\|^2 \\ &\leq a_n \|x_n - p\|^2 + (1 - a_n) \|x_n - p\|^2 \\ &= \|x_n - p\|^2 . \end{aligned} \tag{29}$$

That is, $p \in D_{n+1}$ and so $\Theta \subset D_{n+1}$. Therefore, $\Theta \subset D_n$ for all $n \geq 0$.

By the assumptions, we can conclude that D_n is nonempty closed convex subset of H and then $\Theta \subset D_{n+1} \subset D_n$, $\forall n \geq 0$. For fixed $p \in \Theta$ and from $x_{n+1} = P_{D_{n+1}} x_0$, we obtain that

$$\|x_{n+1} - x_0\| = \left\| P_{D_{n+1}} x_0 - x_0 \right\| \leq \|p - x_0\| . \tag{30}$$

This implies that the sequence $\{x_n\}$ is bounded. Since $x_n = P_{D_n} x_0$ and $D_{n+1} \subset D_n$, $\forall n \geq 0$, by the properties of the metric projection, we have

$$\langle x_0 - x_m, x_m - x_n \rangle \geq 0, \tag{31}$$

for any $n \geq m \geq 0$. Next, we want to show that $\{x_n\}$ is a Cauchy sequence. We compute

$$\begin{aligned} \|x_n - x_0\|^2 &= \langle x_n - x_0, x_n - x_0 \rangle \\ &= \langle x_n - x_0, x_n - x_{n+1} + x_{n+1} - x_0 \rangle \\ &= \langle x_n - x_0, x_n - x_{n+1} \rangle \\ &\quad + \langle x_n - x_0, x_{n+1} - x_0 \rangle . \end{aligned} \tag{32}$$

This implies that

$$\begin{aligned} \langle x_0 - x_n, x_n - x_{n+1} \rangle &= - \|x_n - x_0\|^2 \\ &\quad + \langle x_n - x_0, x_{n+1} - x_0 \rangle \\ &= - \|x_n - x_0\|^2 \\ &\quad + \|x_n - x_0\| \|x_{n+1} - x_0\| . \end{aligned} \tag{33}$$

By (31) and (33), we get that

$$0 \leq - \|x_n - x_0\|^2 + \|x_n - x_0\| \|x_{n+1} - x_0\| . \tag{34}$$

Therefore,

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| , \quad \forall n \geq 0. \tag{35}$$

We have from (30) and (35) that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. For any $n \geq m \geq 0$, by using (31) again, we get

$$\begin{aligned} \|x_n - x_0\|^2 &= \|x_n - x_m + x_m - x_0\|^2 \\ &= \|x_n - x_m\|^2 + \|x_m - x_0\|^2 \\ &\quad + 2 \langle x_n - x_m, x_m - x_0 \rangle \\ &\geq \|x_n - x_m\|^2 + \|x_m - x_0\|^2 . \end{aligned} \tag{36}$$

Consequently, we obtain that

$$\|x_n - x_m\|^2 \leq \|x_n - x_0\|^2 - \|x_m - x_0\|^2 . \tag{37}$$

Hence, as $m \to \infty$ and $n \to \infty$, we have

$$\|x_n - x_m\| \longrightarrow 0. \tag{38}$$

Therefore the sequence $\{x_n\}$ is a Cauchy sequence. Without loss of generality, we can assume that

$$x_n \longrightarrow x^* \text{ as } n \longrightarrow \infty. \tag{39}$$

Next, we will prove that $\lim_{n\to\infty} \|z_n - u_n\| = 0$ for some $z_n \in T u_n$ by dividing the proof into 4 steps.

Step I. We will prove that $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Since $x_{n+1} = P_{D_{n+1}} x_0 \in D_{n+1} \subset D_n$, from (26), we have

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 \tag{40}$$

and we obtain

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| . \tag{41}$$

By (38), we conclude that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{42}$$

Consider

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| . \tag{43}$$

Then, by (38) and (42), we obtain that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{44}$$

Step II. We will show that $\lim_{n\to\infty}\|Bx_n - Bp\| = 0$.

Note that

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|a_n x_n + (1-a_n) TJ_{r_n}^A (I - r_n B) x_n - p\right\|^2 \\
&= a_n \|x_n - p\|^2 + (1-a_n)\left\|TJ_{r_n}^A (I - r_n B) x_n - p\right\|^2 \\
&\quad - a_n (1-a_n)\left\|x_n - TJ_{r_n}^A (I - r_n B) x_n\right\|^2 \\
&\le a_n \|x_n - p\|^2 + (1-a_n)\left\|TJ_{r_n}^A (I - r_n B) x_n - p\right\|^2 \\
&= a_n \|x_n - p\|^2 + (1-a_n)\|Tu_n - p\|^2 \\
&= a_n \|x_n - p\|^2 + (1-a_n)\|z_n - p\|^2 \\
&\qquad\qquad \text{for some } z_n \in Tu_n \\
&= a_n \|x_n - p\|^2 + (1-a_n) d(z_n, Tp)^2 \\
&\le a_n \|x_n - p\|^2 + (1-a_n)\mathscr{H}(Tu_n, Tp)^2 \\
&\le a_n \|x_n - p\|^2 + (1-a_n)\|u_n - p\|^2 \\
&= a_n \|x_n - p\|^2 \\
&\quad + (1-a_n)\left\|J_{r_n}^A (I - r_n B) x_n - J_{r_n} A (I - r_n B) p\right\|^2.
\end{aligned} \tag{45}$$

It follows that

$$\begin{aligned}
\|y_n - p\|^2 &\le a_n \|x_n - p\|^2 \\
&\quad + (1-a_n)\|(I - r_n B) x_n - (I - r_n B) p\|^2 \\
&\le a_n \|x_n - p\|^2 \\
&\quad + (1-a_n)\left\{\|x_n - p\|^2 - 2r_n(2\alpha - r_n)\|Bx_n - Bp\|\right\} \\
&= a_n \|x_n - p\|^2 + (1-a_n)\|x_n - p\|^2 \\
&\quad - 2r_n(2\alpha - r_n)(1-a_n)\|Bx_n - Bp\|^2.
\end{aligned} \tag{46}$$

This implies that

$$\begin{aligned}
&2r_n(2\alpha - r_n)(1-a_n)\|Bx_n - Bp\|^2 \\
&\quad \le \|x_n - p\|^2 - \|y_n - p\|^2.
\end{aligned} \tag{47}$$

By using (44), $r_n \in (0, 2\alpha)$, and $a_n \in (0,1)$, then we conclude that

$$\lim_{n\to\infty}\|Bx_n - Bp\| = 0. \tag{48}$$

Step III. We will show that $\lim_{n\to\infty}\|J_{r_n}^A(I - r_n B)x_n - x_n\| = \lim_{n\to\infty}\|u_n - x_n\| = 0$.

Note that

$$\begin{aligned}
&\left\|J_{r_n}^A (I - r_n B) x_n - p\right\|^2 = \left\|J_{r_n}^A (I - r_n B) x_n \right. \\
&\quad \left. - J_{r_n}^A (I - r_n B) p\right\|^2 \le \left\langle (I - r_n B) x_n \right. \\
&\quad \left. - (I - r_n B) p, J_{r_n}^A (I - r_n B) x_n - J_{r_n}^A (I - r_n B) p \right\rangle \\
&\quad = \left\langle J_{r_n}^A (I - r_n B) x_n - p, (I - r_n B) x_n \right. \\
&\quad \left. - (I - r_n B) p \right\rangle.
\end{aligned} \tag{49}$$

Consider

$$\begin{aligned}
&\left\|\left(J_{r_n}^A (I - r_n B) x_n - p\right) \right. \\
&\quad \left. - ((I - r_n B) x_n - (I - r_n B) p)\right\|^2 \\
&\quad = \left\|J_{r_n}^A (I - r_n B) x_n - p\right\|^2 - 2\left\langle J_{r_n}^A (I - r_n B) x_n \right. \\
&\quad \left. - p, (I - r_n B) x_n - (I - r_n B) p \right\rangle + \|(I - r_n B) x_n \\
&\quad - (I - r_n B) p\|^2.
\end{aligned} \tag{50}$$

Then, we get

$$\begin{aligned}
&\left\langle J_{r_n}^A (I - r_n B) x_n - p, (I - r_n B) x_n - (I - r_n B) p \right\rangle \\
&\quad = \frac{1}{2}\left\{\left\|J_{r_n}^A (I - r_n B) x_n - p\right\|^2 + \|(I - r_n B) x_n \right. \\
&\quad - (I - r_n B) p\|^2 - \left\|\left(J_{r_n}^A (I - r_n B) x_n - p\right) \right. \\
&\quad \left. \left. - ((I - r_n B) x_n - (I - r_n B) p)\right\|^2 \right\}.
\end{aligned} \tag{51}$$

From (49) and (51), we obtain that

$$\begin{aligned}
&\left\|J_{r_n}^A (I - r_n B) x_n - p\right\|^2 \le \frac{1}{2}\left\{\left\|J_{r_n}^A (I - r_n B) x_n - p\right\|^2 \right. \\
&\quad + \|(I - r_n B) x_n - (I - r_n B) p\|^2 \\
&\quad - \left\|\left(J_{r_n}^A (I - r_n B) x_n - p\right) \right. \\
&\quad \left. \left. - ((I - r_n B) x_n - (I - r_n B) p)\right\|^2 \right\} \\
&\quad = \frac{1}{2}\left\{\left\|J_{r_n}^A (I - r_n B) x_n - p\right\|^2 + \|(I - r_n B) x_n \right. \\
&\quad - (I - r_n B) p\|^2 - \left\|\left(J_{r_n}^A (I - r_n B) x_n - x_n\right) \right. \\
&\quad \left. - r_n (Bp - Bx_n)\right\|^2 \Big\} \le \frac{1}{2}\left\{\left\|J_{r_n}^A (I - r_n B) x_n - p\right\|^2 \right. \\
&\quad + \|x_n - p\|^2 - \left\|J_{r_n}^A (I - r_n B) x_n - x_n\right\|^2 \\
&\quad + 2r_n \left\|J_{r_n}^A (I - r_n B) x_n - x_n\right\| \|Bx_n - Bp\| - \|r_n\|^2 \\
&\quad \left. \cdot \|Bp - Bx_n\|^2 \right\}.
\end{aligned} \tag{52}$$

Form (52), we have

$$\begin{aligned}&\left\|J_{r_n}^A(I-r_nB)x_n-p\right\|^2\\&\quad\le\|x_n-p\|^2-\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|^2\\&\qquad-\|r_n\|^2\|Bx_n-Bp\|^2\\&\qquad+2r_n\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|\|Bx_n-Bp\|.\end{aligned}\tag{53}$$

Then, from (45) and (53), we obtain that

$$\begin{aligned}&\|y_n-p\|^2\le a_n\|x_n-p\|^2+(1-a_n)\left\|J_{r_n}^A(I-r_nB)x_n\right.\\&\left.-p\right\|^2\le a_n\|x_n-p\|^2+(1-a_n)\left\{\|x_n-p\|^2\right.\\&-\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|^2\\&+2r_n\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|\|Bx_n-Bp\|\\&\left.-\|r_n\|^2\|Bx_n-Bp\|^2\right\}=\|x_n-p\|^2-(1-a_n)\\&\cdot\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|^2+(1-a_n)\\&\cdot\left\{2r_n\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|\|Bx_n-Bp\|\right.\\&\left.+\|r_n\|^2\|Bx_n-Bp\|^2\right\}.\end{aligned}\tag{54}$$

It follows that

$$\begin{aligned}&(1-a_n)\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|^2\le\|x_n-p\|^2-\|y_n\\&-p\|^2+(1-a_n)\\&\cdot\left\{2r_n\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|\|Bx_n-Bp\|\right.\\&\left.-\|r_n\|^2\|Bx_n-Bp\|^2\right\}.\end{aligned}\tag{55}$$

Form (55), in view of conditions (b), and (44), we conclude that

$$\lim_{n\to\infty}\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|=\lim_{n\to\infty}\|u_n-x_n\|=0.\tag{56}$$

Step IV. We will show that $\lim_{n\to\infty}\|u_n-z_n\|=0$, for some $z_n\in Tu_n$.

By using Condition (I) and (28), we obtain that

$$\begin{aligned}\|y_n-p\|^2&=\|a_n(x_n-p)+(1-a_n)(z_n-p)\|^2\\&\qquad\text{for some } z_n\in Tu_n\\&=a_n\|x_n-p\|^2+(1-a_n)\|z_n-p\|^2\\&\quad-a_n(1-a_n)\|x_n-z_n\|^2\\&=a_n\|x_n-p\|^2+(1-a_n)d(z_n,Tp)^2\\&\quad-a(1-a_n)\|x_n-z_n\|^2\\&\le a_n\|x_n-p\|^2+(1-a_n)\mathscr{H}(Tu_n,Tp)^2\\&\quad-a_n(1-a_n)\|x_n-z_n\|^2\\&\le a_n\|x_n-p\|^2+(1-a_n)\|u_n-p\|^2\\&\quad-a_n(1-a_n)\|x_n-z_n\|^2\\&\le\|x_n-p\|^2-a_n(1-a_n)\|x_n-z_n\|^2.\end{aligned}\tag{57}$$

This implies that

$$a_n(1-a_n)\|x_n-z_n\|^2\le\|x_n-p\|^2-\|y_n-p\|^2.\tag{58}$$

Form (44), we conclude that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.\tag{59}$$

Consider

$$\|z_n-u_n\|\le\|z_n-x_n\|+\|x_n-u_n\|.\tag{60}$$

Therefore, we get from (56) and (59)

$$\lim_{n\to\infty}\|z_n-u_n\|=0.\tag{61}$$

Finally, We will prove that $x^*\in\mathrm{Fix}(T)\cap(A+B)^{-1}(0)$. Since $x_n\to x^*$ and by (56), we get that $u_n\to x^*$ also. Since $z_n\in Tu_n$ and $\lim_{n\to\infty}\|u_n-z_n\|=0$, by using Lemma 2, we have $x^*\in\mathrm{Fix}(T)$.

Consider

$$\begin{aligned}&\left\|J_{r_n}^A(I-r_nB)x^*-x^*\right\|\\&\quad\le\left\|J_{r_n}^A(I-r_nB)x_n-J_{r_n}^A(I-r_nB)x^*\right\|\\&\qquad+\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|+\|x_n-x^*\|\\&\quad\le2\|x_n-x^*\|+\left\|J_{r_n}^A(I-r_nB)x_n-x_n\right\|.\end{aligned}\tag{62}$$

Since $x_n\to x^*$ and by (56), we get that

$$\lim_{n\to\infty}\left\|J_{r_n}^A(I-r_nB)x^*-x^*\right\|=0.\tag{63}$$

That is, $J_r^A(I-rB)x^*=x^*$ as $n\to\infty$, which implies that $x^*\in(A+B)^{-1}(0)$. Therefore, we conclude that $x^*\in\Theta$ which completes the proof. □

4. Numerical Example and Convergence Analysis

In this section, we give the following numerical example to confirm the convergence of Theorem 3 by using the algorithm (26).

Example 1. Let $H=\mathbb{R}$ and $C=[0,4]$. Define the mappings $A:D(A)\subseteq C\to2^H$, $B:C\to H$, and $T:C\to K(C)$ by the following:

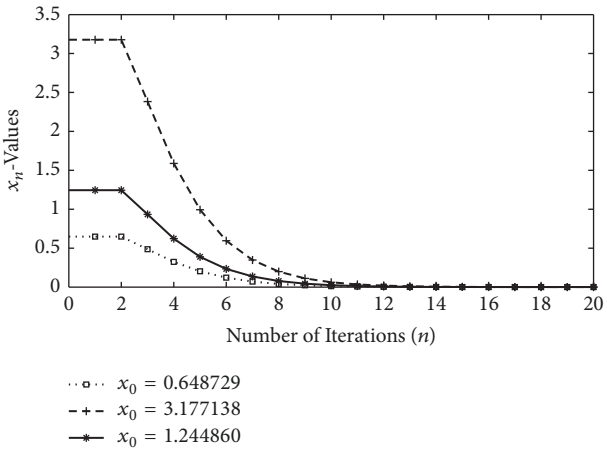

Figure 1: The comparison of convergent rate from 3 initial points.

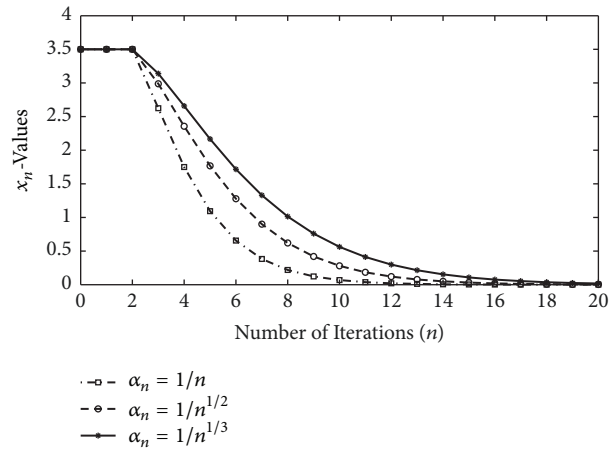

Figure 2: The comparison of convergent rate from 3 different alpha parameters.

$$A(x) := 2\left(x + \frac{1}{4}\right),$$
$$B(x) := \frac{1}{2}(x-1), \tag{64}$$
$$T(x) := \begin{cases} \{0\}, & x \in [0,2] \\ \left[0, e^{2-x}\right], & x \in (2,4]. \end{cases}$$

We see that the proposed mappings satisfy the assumptions in Theorem 3. For each $r_n > 0$, we obtain that $J_{r_n}^A(I - r_nB)(x) = x(2-r)/(2+4r)$. It is easy to see that a point 0 is in the fixed point sets of $J_{r_n}^A(I - r_nB)$ and T; that is, $0 \in \mathrm{Fix}(T) \cap (A+B)^{-1}(0)$.

In Figure 1, these initial points are randomly chosen from the set $D_0 = C$ and we find optimal solution in 20 steps. This indicates that the sequence x_n in algorithm (26) converges to the same point; that is, $0 \in \Theta$ as a solution of this example. In this experiment, Figure 1 indicates the behaviour of x_n for algorithm (26) that converges to the same solution; that is, $0 \in \mathrm{Fix}(T) \cap (A+B)^{-1}(0)$ as a solution of this example. Moreover, the decreasing on alpha function decreases rate of convergence to the optimal solutions which is shown in Figure 2. Figure 3 shows that $D_{n+1} \subset D_n \subset \cdots \subset D_2 \subset D_1 \subset D_0 = C$. This means that the iteration of C_n will squeeze the area until we obtain the approximated solution.

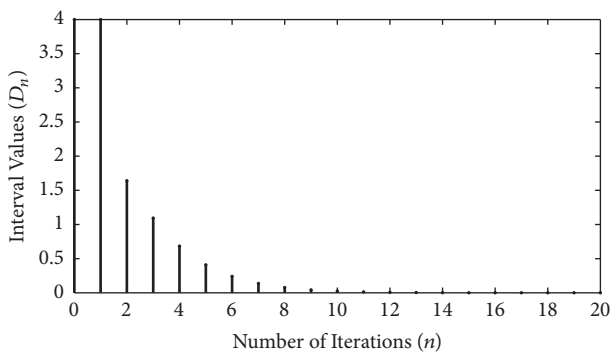

Figure 3: The behaviors of the set D_n.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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