Research Article

Quasilinear Evolution Equations in L^p_μ -Spaces with Lower Regular Initial Data

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We study the Cauchy problem of the quasilinear evolution equations in L^p_μ -spaces. Based on the theories of maximal L^p -regularity of sectorial operators, interpolation spaces, and time-weighted L^p -spaces, we establish the local posedness for a class of abstract quasilinear evolution equations with lower regular initial data. To illustrate our results, we also deal with the second-order parabolic equations and the Navier-Stokes equations in $L^{p,q}$ -spaces with temporal weights.

1. Introduction

In this paper, we will deal with the following quasilinear evolution equation:

$$\begin{aligned} \dot{u} + A(t, u)u &= F(t, u), \quad t > 0, \\ u(0) &= u_0 \end{aligned} \quad (1)$$

in a Banach space couple (X, X_1) with $X_1 \hookrightarrow X$ densely. Take $1 < p < \infty$ and $\mu \in (1/p, 1]$, and denote by $X_{\mu-1/p, p} = (X, X_1)_{\mu-1/p, p}$ the real interpolation space between X and X_1 (see [1, Ch. 3] or [2, Ch. 1]). Given $T_0 > 0$ or $T_0 = +\infty$, let $J_0 = [0, T_0)$. And, for each $0 < T < T_0$, let $J_T = [0, T]$. Suppose that $A : J_0 \times X_{\mu-1/p, p} \rightarrow \mathcal{B}(X_1, X)$ is a continuous map, $F : J_0 \times X_{\mu-1/p, p} \rightarrow X$ is a Caratheodory map, and $u_0 \in X_{\mu-1/p, p}$. We will deal with this problem in the framework of maximal regularity of the sectorial operators in L^p -spaces with temporal weights.

Suppose that A_0 is a sectorial operator defined in X with the domain $\mathcal{D}(A_0) = X_1$ (refer to [3–5] for the definition of sectorial operator), we say A_0 has the maximal L^p -regularity on J_0 , or $A_0 \in \mathcal{MR}_p(J_0, X, X_1)$ in symbol, and we mean that, for each $f \in L^p(J_0; X)$, the following Cauchy problem

$$\begin{aligned} \dot{u} + A_0 u &= f(t), \quad t > 0, \\ u(0) &= 0 \end{aligned} \quad (2)$$

has a unique solution in $W^{1,p}_0(J_0, X) \cap L^p(J_0, X_1)$. And there is a constant $C_0 > 0$, such that

$$\|u'\|_{L^p(J_0, X)} + \|u\|_{L^p(J_0, X_1)} \leq C_0 \|f\|_{L^p(J_0, X)} \quad (3)$$

for $u = (\partial_t + \widehat{A}_0)^{-1} f$ with $f \in L^p(J_0, X)$. It is easy to see that if $A_0 \in \mathcal{MR}_p(J_0, X, X_1)$, then $A_0 \in \mathcal{MR}_p(J_T, X, X_1)$ for arbitrary $T < T_0$. Especially, if $A_0 \in \mathcal{MR}_p(\mathbb{R}_+, X, X_1)$, then the constant C_0 in (4) is independent of T .

In general, a sectorial operator A_0 does not have the property of L^p -maximal regularity, even though its negative $-A$ generates an exponentially decaying analytic semigroup. However, if X is a $\mathcal{UMD}$ space, and A_0 admits bounded imaginary powers A_0^{is} ($|s| \leq \varepsilon$ for some $\varepsilon > 0$), or $A_0 \in \mathcal{BIP}$ for short (see [3, Ch. 3], [5, 6]), then A_0 has the L^p -maximal regularity. Further investigation, involving the vector-valued multiplier theorem and H^∞ -functional calculus of sectorial operators, showed that, in a $\mathcal{UMD}$ space, L^p -maximal regularity equals $\mathcal{R}$ -sectoriality for a sectorial operator (see [7], [8, Ch. 3]). Recall that, given $\Omega \subseteq \mathbb{R}^n$ a bounded or exterior domain with compact C^1 type boundary, $\Omega = \mathbb{R}^n$ or a half space $\mathbb{R}^n_+$, for each $1 < q < \infty$, the Lebesgue space $L^q(\Omega)$ is of $\mathcal{UMD}$ type, on which the elliptic differential operators with some boundary conditions are $\mathcal{R}$ -sectorial (cf. [7, 9]). And the Stokes operator A_p is of $\mathcal{BIP}$ type with the power angle $\theta_{A_p} < \pi/2$ (see [10, 11]). Hence, we can make

$L^p - L^q$ estimates for the parabolic evolution equations (see [12, 13]) and the Navier-Stokes equations (see [14, 15]).

As for the maximal regularity of a sectorial operator A_0 in L^p_μ -setting, or equivalently $A_0 \in \mathcal{MR}_p^\mu(J_0, X, X_1)$, it means that, for each $f \in L^p_\mu(J_0, X)$, Pr. (2) has a unique solution $u \in {}_0W_\mu^{1,p}(J_0, X) \cap L^p_\mu(J_0, X_1)$ with the estimate

$$\|u'\|_{L^p_\mu(J_0, X)} + \|u\|_{L^p_\mu(J_0, X_1)} \leq C_0 \|f\|_{L^p_\mu(J_0, X)} \quad (4)$$

associated with a μ -dependent constant C_0 . Prüss-Simonett in [16] showed that $A_0 \in \mathcal{MR}_p(J_0, X, X_1)$ is equivalent to $A_0 \in \mathcal{MR}_p^\mu(J_0, X, X_1)$ for all $\mu \in (1/p, 1]$. After that, Köhne-Prüss-Wilke in [17] studied Pr. (1) of autonomous type (i.e., $A(t, u) = A(u)$ and $F(t, u) = F(u)$), including the local well posedness and long time behavior of the solutions, in L^p_μ -spaces.

Motivated by the theory of ε -regular solutions (cf. [18, 19]), this paper will deal with Pr. (1) of nonautonomous type in weighted L^p -spaces. Unlike [17], here the regularity of the initial value is lower than that of the space where the nonlinearity F acts; more precisely, u_0 lies in the space $X_{\mu-1/p, p}$ for some $\mu \in (1/p, 1)$ and F maps $J_0 \times X_{1-1/p, p}$ into X .

This paper is organized as follows: as preliminaries, in Section 2, we give some definitions and properties of the time-weighted Lebesgue and Sobolev spaces of abstract valued functions, and the maximal regular space $\mathbb{E}_{1,p}^\mu(J_T)$ and its closed subspace ${}_0\mathbb{E}_{1,p}^\mu(J_T)$. In Section 3, we establish the local and global existence of $\mathbb{E}_{1,p}^\mu(J_T)$ and $Y_{\mu,T}$ solutions in subcritical and critical cases, respectively, using estimate (4), together with the embedding properties of ${}_0\mathbb{E}_{1,p}^\mu(J_T)$ and the smoothing action of the operator $\mathcal{R}$. To illustrate the obtained results, in Section 4, we make $L^p_\mu - L^q$ estimates for the parabolic evolution equations and the Navier-Stokes equations, using the fractional order Sobolev space $W^{2(\mu-1/p), p}(\Omega)$ and Besov space $B_q^{2(\mu-1/p), p}(\Omega)^n$ as the state spaces.

Framework of our study can be incorporated into the theory of sectorial operators and analytic semigroups, together with time-weighted L^p -spaces. Results obtained here have their meaning in the study of quasilinear evolution spaces driven by the sectorial operators. For the relative investigations on the quasilinear evolutions in L^p or L^p_μ spaces with different focuses, please refer to [5, 16, 17, 20], with the references therein.

2. Preliminaries

Let $1 < p < \infty$ and $\mu \in (1/p, 1]$; define

$$L^p_\mu(J_T, X) = \{u \in L^p_{\text{loc}}(J_T, X) : t^{1-\mu} \|u(t)\| \in L^p(J_T)\} \quad (5)$$

with the norm $\|u\|_{L^p_\mu(J_T, X)} = \|t^{1-\mu} u\|_{L^p(J_T, X)}$, and

$$\begin{aligned} & W_\mu^{1,p}(J_T, X) \\ &= \{u \in W_{\text{loc}}^{1,p}(J_T, X) : t^{1-\mu} \|u'(t)\| \in L^p(J_T)\} \end{aligned} \quad (6)$$

with the norm $\|u\|_{W_\mu^{1,p}(J_T, X)} = \|t^{1-\mu} u\|_{W^{1,p}(J_T, X)}$, and its closed subspace with zero trace

$${}_0W_\mu^{1,p}(J_T, X) = \{u \in W_\mu^{1,p}(J_T, X) : \gamma u = 0\}, \quad (7)$$

where γu denotes the trace of u . We can also define the maximal regularity spaces

$$\begin{aligned} \mathbb{E}_{1,p}^\mu(J_T) &= W_\mu^{1,p}(J_T, X) \cap L^p_\mu(J_T, X_1), \\ {}_0\mathbb{E}_{1,p}^\mu(J_T) &= {}_0W_\mu^{1,p}(J_T, X) \cap L^p_\mu(J_T, X_1). \end{aligned} \quad (8)$$

Here are some interpolating and embedding properties of the L^p_μ and $W_\mu^{1,p}$ spaces, all of which can be found in [21, Ch. 1] or [22].

- (i) $L^p_\mu(J_T, X) \hookrightarrow L^q(J_T, X)$ whenever $1 \leq q < p$, and $\max\{1/p, 1 - 1/p + 1/q\} < \mu < 1$, and

$$\begin{aligned} & \|u\|_{L^q(J_T, X)} \\ & \leq \frac{T^{(p-q)/p - (1-\mu)q}}{[1 - (1-\mu)pq/(p-q)]^{(p-q)/p}} \|u\|_{L^p_\mu(J_T, X)}. \end{aligned} \quad (9)$$

- (ii) For each $\eta > 1 - \mu + 1/p$,

$$\begin{aligned} & W_\mu^{\eta,p}(J_T, X) \hookrightarrow \\ & L^\infty(J_T, X), \end{aligned} \quad (10)$$

- (iii) For the indexes $\sigma > 0$, $q \in (1, \infty)$, and $\theta \in (0, 1)$,

$$\begin{aligned} & W_\mu^{1,p}(J_T, L^q(\Omega)) \cap L^p_\mu(J_T, W^{\sigma,q}(\Omega)) \hookrightarrow \\ & W_\mu^{\theta,p}(J_T, B_q^{\sigma(1-\theta), p}(\Omega)). \end{aligned} \quad (11)$$

where Ω is a domain in $\mathbb{R}^n$ and $W^{\sigma,q}(\Omega)$ and $B_q^{\sigma(1-\theta), p}(\Omega) = (L^q(\Omega), W^{\sigma,q}(\Omega))_{1-\theta, p}$ denote the Sobolev and Besov spaces with real indexes, respectively.

Suppose that A_0 is a sectorial operator with $\mathcal{D}(A_0) = X_1$, and $-A_0$ generates a uniformly bounded analytic semigroup e^{-tA_0} . From [4, Ch. 5], we know that, for each $\alpha \in (0, 1)$ and all $x \in X_{\alpha, p}$, the norm $\|x\|_{X_{\alpha, p}}$ is equivalent to

$$\|x\|_X + \left(\int_0^\infty (t^{1-\alpha} \|A_0 e^{-tA_0} x\|_X)^p \frac{dt}{t} \right)^{1/p}. \quad (12)$$

Introduce the continuous function space with time weight

$$\begin{aligned} & C_\mu(0, T; X_{1-1/p, p}) = \left\{ u \right. \\ & \left. \in C((0, T], X_{1-1/p, p}) : \sup_{t \in (0, T]} t^{1-\mu} \|u(t)\|_{X_{1-1/p, p}} \right. \\ & \left. < \infty \right\} \end{aligned} \quad (13)$$

endowed with the norm

$$\|u\|_{C_\mu(0,T;X_{1-1/p,p})} = \sup_{t \in (0,T]} \|u(t)\|_{X_{1-1/p,p}} \quad (14)$$

and its closed subspace

$$\begin{aligned} {}_0C_\mu(0,T;X_{1-1/p,p}) &= \left\{ u \right. \\ &\left. \in C_\mu(0,T;X_{1-1/p,p}) : \lim_{t \rightarrow 0^+} t^{1-\mu} \|u(t)\|_{X_{1-1/p,p}} = 0 \right\}. \end{aligned} \quad (15)$$

Proposition 1. ${}_0E_{1,p}^\mu(J_T) \hookrightarrow {}_0C_\mu(0,T;X_{1-1/p,p}) \cap C_0([0,T];X_{\mu-1/p,p})$ with the imbedding constant independent of T .

Proof. We will only prove ${}_0E_{1,p}^\mu(J_T) \hookrightarrow {}_0C_\mu(0,T;X_{1-1/p,p})$; for the other inclusion, please refer to [16]. Taking any $u \in {}_0E_{1,p}^\mu(J_T)$, we have that $\Phi_\mu u \in {}_0E_{1,p}(J_T)$ and $\|\Phi_\mu u\|_{{}_0E_{1,p}(J_T)} \leq C\|u\|_{{}_0E_{1,p}^\mu(J_T)}$, where $(\Phi_\mu u)(t) = t^{1-\mu}u(t)$ and the constant C is independent of T (see [21, Ch. 1]). Notice that ${}_0E_{1,p}(J_T) \hookrightarrow C([0,T];X_{1-1/p,p})$ with the imbedding norm independent of T , and we then have $t^{1-\mu}u \in C([0,T];X_{1-1/p,p})$ with $\lim_{t \rightarrow 0^+} t^{1-\mu}\|u(t)\|_{X_{1-1/p,p}} = 0$, and the desired result comes. $\square$

Suppose that, for all $u \in {}_0E_{1,p}^\mu(J_T)$,

$$\max \left\{ \|u\|_{C(J_T, X_{\mu-1/p})}, \|u\|_{C_\mu(0,T;X_{1-1/p,p})} \right\} \leq C_1 \|u\|_{{}_0E_{1,p}^\mu(J_T)}, \quad (16)$$

where $C_1 > 0$ is independent of T .

Proposition 2. For each $u_0 \in X_{\mu-1/p,p}$, define $(\mathcal{R}u_0)(t) = e^{-tA_0}u_0$. Then $\mathcal{R}u_0 \in {}_0C_\nu(0,T;X_{1-1/p,p})$ for all $\nu \in (1/p, \mu]$ together with the estimate

$$\|\mathcal{R}u_0\|_{C_\nu(0,T;X_{1-1/p,p})} \leq C_T T^{\mu-\nu} \|u_0\|_{X_{\mu-1/p,p}}, \quad (17)$$

where $C_T = \max\{T^{1-\mu}M, 1\}$ and $M \geq 1$ is the upper bounds of the semigroup e^{-tA_0} on the infinite interval $[0, \infty)$.

Proof. Firstly, for each $\nu \in (1/p, \mu]$, we can deduce that

$$\begin{aligned} t^{1-\nu} \|e^{-tA_0}u_0\|_{X_{1-1/p,p}} &= t^{1-\nu} \left(\|e^{-tA_0}u_0\|_X \right. \\ &\quad \left. + \left(\int_0^\infty \|A_0 e^{-(s+t)A_0}u_0\|_X^p ds \right)^{1/p} \right) \\ &\leq t^{\mu-\nu} \left(T^{1-\mu}M \|u_0\|_X \right. \\ &\quad \left. + \left(\int_t^\infty s^{(1-\mu)p} \|A_0 e^{-sA_0}u_0\|_X^p ds \right)^{1/p} \right) \\ &\leq C_T t^{\mu-\nu} \|u_0\|_{X_{\mu-1/p,p}}. \end{aligned} \quad (18)$$

Secondly, for each $\eta \in (\mu, 1]$ and $\nu \in X_{\eta-1/p,p}$, we have that

$$\lim_{t \rightarrow 0^+} t^{1-\mu} \|e^{-tA_0}v\|_{X_{1-1/p,p}} = 0. \quad (19)$$

This relation, together with the dense imbedding $X_{1-1/p,p} \hookrightarrow X_{\mu-1/p,p}$ and the upper bounds $\sup_{t \in (0,T]} t^{1-\mu} \|e^{-tA_0}u_0\|_{X_{1-1/p,p}} \leq \|u_0\|_{X_{\mu-1/p,p}}$, leads to the desired inclusion $\mathcal{R}u_0 \in {}_0C_\mu(0,T;X_{1-1/p,p})$. $\square$

Analogously, we can derive the following.

Proposition 3. $\mathcal{R}u_0 \in {}_0E_{1,p}^\nu(J_T)$ and $\|\mathcal{R}u_0\|_{{}_0E_{1,p}^\nu(J_T)} \leq C_T T^{\mu-\nu} \|u_0\|_{X_{\mu-1/p,p}}$ for all $\nu \in (1/p, \mu]$ and $u_0 \in X_{\mu-1/p,p}$, where $C_T = \max\{2^{1/p}, M[(1-\mu)p+1]^{-1/p}T^{1-\mu+1/p}\}$.

Remark 4. From [5], we know that, for every sectorial operator A_0 lying in $\mathcal{MR}_p(\mathbb{R}_+, X, X_1)$, the corresponding semigroup e^{-tA_0} decays exponentially, and consequently the constant C_T appearing in Propositions 1 and 2 is independent of T , whence the subscript T can be erased.

The following lemma tells us that the property of L^p -maximal regularity can be preserved under small perturbations (refer to [5]).

Lemma 5. Given a sectorial operator $A_0 \in \mathcal{MR}_p(J_0, X, X_1)$ with the constant C_0 as in (4), suppose that $A_1 \in \mathcal{B}(X_1, X)$ is closed in X satisfying $\|A_1 - A_0\|_{\mathcal{B}(X_1, X)} \leq \kappa/C_0$ for some $\kappa \in [0, 1)$. Then $A_1 \in \mathcal{MR}_p(J_0, X, X_1)$ with the bounds $C_0/(1-\kappa)$.

3. Main Results and Proofs

Firstly, we give some hypotheses on A and F used in this paper.

$H(A)$: (i) $A : J_0 \times X_{\mu-1/p} \rightarrow \mathcal{B}(X_1, X)$ is continuous. And, for each $R > 0$, there is a constant $L(R) > 0$ such that

$$\|A(t, u) - A(t, v)\|_{\mathcal{B}(X_1, X)} \leq L(R) \|u - v\|_{X_{\mu-1/p,p}} \quad (20)$$

for all $t \in J_0$ and all $u, v \in X_{\mu-1/p,p}$ with $\|u\|_{X_{\mu-1/p,p}}, \|v\|_{X_{\mu-1/p,p}} \leq R$.

(ii) $A_0 = A(0, u_0) \in \mathcal{MR}_p(J_0, X, X_1)$ for some $u_0 \in X_{\mu-1/p,p}$.

$H(F)$: $F : J_0 \times X_{1-1/p,p} \rightarrow X$ is an operator of Caratheodory type; that is,

(i) for each $u \in X_{1-1/p,p}$, $F(\cdot, u)$ is strongly measurable,
(ii) for a.a. $t \in J_0$, $F(t, \cdot)$ is locally Lipschitz, and there is a number $\rho > 0$ and a function $\phi \in L^\infty(J_0, \mathbb{R}_+)$ for which

$$\begin{aligned} \|F(t, u) - F(t, v)\|_X \\ \leq \phi(t) \|u - v\|_{X_{1-1/p,p}} \left(1 + \|u\|_{X_{1-1/p,p}}^{\rho-1} + \|v\|_{X_{1-1/p,p}}^{\rho-1} \right) \end{aligned} \quad (21)$$

holds for a.a. $t \in J_0$ and all $u, v \in X_{1-1/p,p}$,

(iii) $F(\cdot, 0) \in L_\mu^p(0, T; X)$.

It is easy to show that, under all the hypotheses listed above, the following functions associated with T : $\|F(\cdot, 0)\|_{L_\mu^p(0,T;X)}$, $\max_{t \in J_T} \|A(t, u_0) - A(0, u_0)\|_{\mathcal{B}(X_1, X)}$, $\|\mathcal{R}u_0\|_{C_\mu(0,T;X_{1-1/p,p})}$, $\max_{t \in [0,T]} \|\mathcal{R}u_0(t) - u_0\|_{X_{\mu-1/p,p}}$, and $\|\mathcal{R}u_0\|_{{}_0E_{1,p}^\mu(J_T)}$ are all infinitely small as $T \rightarrow 0$. Without any confusion arising, in the coming discussions, the first small

quantity is denoted by $\beta(T)$, and other ones related to u_0 are denoted by $\delta(T)$ uniformly.

We shall now discuss the local well posedness of Pr. (1) in two cases.

Case 1 ($\mu > 1 - p^{-1}(\rho - 1)^{-1}$). For each $v \in \mathbb{E}_{1,p}^\mu(J_T)$, consider the auxiliary problem

$$\begin{aligned} u + A_0 u &= F(t, v) - (A(0, v_0) - A(t, v))v, \quad t > 0, \\ u(0) &= u_1. \end{aligned} \quad (22)$$

Introduce the map $\mathcal{G} : \mathbb{E}_{1,p}^\mu(J_T) \rightarrow L_\mu^p(J_T, X)$:

$$\mathcal{G}v = F(\cdot, v) - (A(0, u_0) - A(\cdot, v))v. \quad (23)$$

Then solution u of the auxiliary problem (22) can be represented by

$$u = \mathcal{R}u_1 + (\partial_t + \widehat{A}_0)^{-1} \mathcal{G}v := \mathcal{F}v, \quad (24)$$

and consequently solution of the original problem (1) is exactly the fixed point of $\mathcal{F}$.

For any $u_1 \in B_{u_0}^\mu(\varepsilon) = \{u \in X_{\mu-1/p,p} : \|u - u_0\|_{X_{\mu-1/p,p}} \leq \varepsilon\}$, let

$$\begin{aligned} \mathbb{B}_{u_0}^\mu(r) &= \left\{ u \in \mathbb{E}_{1,p}^\mu(J_T) : u(0) \right. \\ &= u_1, \left. \|u - \mathcal{R}u_1\|_{\mathbb{E}_{1,p}^\mu(J_T)} \leq r \right\}, \end{aligned} \quad (25)$$

where $\varepsilon, r \in (0, 1)$ are small positive numbers. Taking any $v \in \mathbb{B}_{u_0}^\mu(r)$, using (16) and Proposition 2, we can derive that

$$\begin{aligned} \|v\|_{C(J, X_{\mu-1/p,p})} &\leq \|v - \mathcal{R}u_1\|_{C(J, X_{\mu-1/p,p})} \\ &\quad + \|\mathcal{R}u_1 - \mathcal{R}u_0\|_{C(J, X_{\mu-1/p,p})} \\ &\quad + \|\mathcal{R}u_0\|_{C(J, X_{\mu-1/p,p})} \\ &\leq C_1 \|v - \mathcal{R}u_1\|_{\mathbb{E}_{1,p}^\mu(J_T)} \\ &\quad + M \|u_1 - u_0\|_{X_{\mu-1/p,p}} \\ &\quad + C_T \|u_0\|_{X_{\mu-1/p,p}} \\ &\leq C_1 r + M\varepsilon + C_T \|u_0\|_{X_{\mu-1/p,p}} \\ &\leq C_1 + \max\{M, C_{T_0}\} + C_{T_0} \|u_0\|_{X_{\mu-1/p,p}} \\ &=: R, \end{aligned} \quad (26)$$

and $\|v\|_{C_\mu(0,T;X_{1-1/p,p})} \leq R$ in the same way. Now we can give the a priori estimates for $F(\cdot, v(\cdot))$ and $(A(0, u_0) - A(\cdot, v))v$ in $L_\mu^p(0, T; X)$ as follows:

$$\begin{aligned} \|F(\cdot, v(\cdot))\|_{L_\mu^p(0,T;X)} &\leq \|F(\cdot, 0)\|_{L_\mu^p(0,T;X)} + \|F(\cdot, v(\cdot)) \\ &\quad - F(\cdot, 0)\|_{L_\mu^p(0,T;X)} \leq \beta(T) + \|\phi\|_\infty \\ &\quad \cdot \left[\int_0^T t^{(1-\mu)p} \|v(t)\|_{X_{1-1/p,p}}^p \cdot \left(1 + \|v(t)\|_{X_{1-1/p,p}}^{\rho-1}\right)^p dt \right]^{1/p} \\ &\leq \beta(T) + \|\phi\|_\infty \left[T^{1/p} \|v\|_{C_\mu(0,T;X_{1-1/p,p})} \right. \\ &\quad \left. + \frac{T^{1/p-(1-\mu)(\rho-1)}}{[1 - (1-\mu)p(\rho-1)]^{1/p}} \|v\|_{C_\mu(0,T;X_{1-1/p,p})}^\rho \right] \leq \beta(T) \\ &\quad + C_\phi [T^{1/p} R + T^{1/p-(1-\mu)(\rho-1)} R^\rho], \end{aligned} \quad (27)$$

where $C_\phi = \|\phi\|_\infty \max\{1, [1 - (1-\mu)p(\rho-1)]^{-1/p}\}$ and

$$\begin{aligned} \|(A(0, u_0) - A(\cdot, v))v\|_{L_\mu^p(0,T;X)} &\leq \max_{t \in [0,T]} \|A(0, u_0) - A(t, u_0)\|_{B(X_1, X)} \|v\|_{L_\mu^p(0,T;X_1)} \\ &\quad + \|(A(\cdot, u_0) - A(\cdot, v))v\|_{L_\mu^p(0,T;X)} \\ &\leq \left[\delta(T) + L(R) \max_{t \in [0,T]} \|v(t) - u_0\|_{X_{\mu-1/p,p}} \right] \\ &\quad \cdot \|v\|_{L_\mu^p(0,T;X_1)} \\ &\leq [\delta(T) + L(R)(C_1 r + M\varepsilon + \delta(T))] \|v\|_{\mathbb{E}_{1,p}^\mu(J_T)} \\ &\leq [\delta(T) + L(R)(C_1 r + M\varepsilon + \delta(T))] \\ &\quad \cdot (r + C_{T_0} \varepsilon + \delta(T)). \end{aligned} \quad (28)$$

Hence

$$\begin{aligned} \|\mathcal{F}v - \mathcal{R}u_1\|_{\mathbb{E}_{1,p}^\mu(J_T)} &\leq C_0 \left\{ \beta(T) \right. \\ &\quad + C_\phi [T^{1/p} R + T^{1/p-(1-\mu)(\rho-1)} R^\rho] \\ &\quad + [\delta(T) + L(R)(C_1 r + M\varepsilon + \delta(T))] \\ &\quad \left. \cdot (r + C_{T_0} \varepsilon + \delta(T)) \right\}. \end{aligned} \quad (29)$$

In the next step, we will show the local Lipschitz property of $\mathcal{F}$. For this purpose, take any $v, \bar{v} \in \mathbb{B}_{u_0}^\mu(r)$ and derive that

$$\begin{aligned} \|F(\cdot, v(\cdot)) - F(\cdot, \bar{v}(\cdot))\|_{L_\mu^p(0,T;X)} &\leq \|\phi\|_\infty \left[\int_0^T t^{(1-\mu)p} \|v(t) - \bar{v}(t)\|_{X_{1-1/p,p}}^p \left(1 + \|v(t)\|_{X_{1-1/p,p}}^{\rho-1} + \|\bar{v}(t)\|_{X_{1-1/p,p}}^{\rho-1}\right)^p dt \right]^{1/p} \\ &\leq \|\phi\|_\infty \left[T^{1/p} + \frac{T^{1/p-(1-\mu)(\rho-1)}}{[1 - (1-\mu)p(\rho-1)]^{1/p}} \left(\|v\|_{C_\mu(0,T;X_{1-1/p,p})}^{\rho-1} + \|\bar{v}\|_{C_\mu(0,T;X_{1-1/p,p})}^{\rho-1} \right) \right] \cdot \|v - \bar{v}\|_{C_\mu(0,T;X_{1-1/p,p})} \end{aligned}$$

$$\begin{aligned}
&\leq C_1 C_\phi \left[T^{1/p} + 2T^{1/p-(1-\mu)(\rho-1)} R^{\rho-1} \right] \|v - \bar{v}\|_{\mathbb{E}_{1,p}^\mu(J_T)}, \\
&\|(A(0, u_0) - A(\cdot, v))v - (A(0, u_0) - A(\cdot, \bar{v}))\bar{v}\|_{L_\mu^p(0,T;X)} \leq \|(A(0, u_0) - A(\cdot, v))(v - \bar{v})\|_{L_\mu^p(0,T;X)} \\
&\quad + \|(A(\cdot, v) - A(\cdot, \bar{v}))\bar{v}\|_{L_\mu^p(0,T;X)} \leq \delta(T) \|v - \bar{v}\|_{L_\mu^p(0,T;X)} + \|(A(\cdot, u_0) - A(\cdot, v))(v - \bar{v})\|_{L_\mu^p(0,T;X)} \\
&\quad + L(R) \|\bar{v}\|_{L_\mu^p(0,T;X)} \max_{t \in [0,T]} \|v(t) - \bar{v}(t)\|_{X_{\mu-1/p,p}} \leq \left[\delta(T) + L(R) \max_{t \in [0,T]} \|v(t) - u_0\|_{X_{\mu-1/p,p}} \right] \|v - \bar{v}\|_{\mathbb{E}_{1,p}^\mu(J_T)} \\
&\quad + L(R) \|\bar{v}\|_{\mathbb{E}_{1,p}^\mu(J_T)} \max_{t \in [0,T]} \|v(t) - \bar{v}(t)\|_{X_{\mu-1/p,p}} \\
&\leq \left[\delta(T) + 2C_1 L(R) r + (M + C_1 C_{T_0}) L(R) \varepsilon + (1 + C_1) L(R) \delta(T) \right] \|v - \bar{v}\|_{\mathbb{E}_{1,p}^\mu(J_T)}.
\end{aligned} \tag{30}$$

Consequently

$$\begin{aligned}
\|\mathcal{F}v - \mathcal{F}\bar{v}\|_{\mathbb{E}_{1,p}^\mu(J_T)} &\leq C_0 \left\{ C_1 C_\phi \left[T^{1/p} + T^{1/p-(1-\mu)(\rho-1)} R^{\rho-1} \right] + \left[\delta(T) + 2C_1 L(R) r \right. \right. \\
&\quad \left. \left. + (M + C_1 C_{T_0}) L(R) \varepsilon + (1 + C_1) L(R) \delta(T) \right] \right\} \|v - \bar{v}\|_{\mathbb{E}_{1,p}^\mu(J_T)}.
\end{aligned} \tag{31}$$

Take

$$r \leq \min \left\{ \left(2(2 + C_0)(1 + L(R)(C_1 + M + 1)) \right)^{-1}, \left(4C_0 \left(1 + (3C_1 + M + C_{T_0} C_1 + 1) L(R) \right) \right)^{-1} \right\} \tag{32}$$

and $\varepsilon \leq r$, select $T_1 \in (0, T_0)$ such that $|\beta(T_1)| \leq r/4C_0$ and $|\delta(T_1)| \leq r$, and let

$$T = \min \left\{ T_1, \left(\frac{r}{8C_0 C_\phi R^p} \right)^{1/(1/p-(1-\mu)(\rho-1))} \right\}, \tag{33}$$

and then, from (29) and (31), we can deduce that, for all $u_1 \in B_{u_0}^\mu(\varepsilon)$ and $v, \bar{v} \in \mathbb{B}_{u_0}^\mu(r)$, $\|\mathcal{F}v - \mathcal{R}_0 u_1\|_{\mathbb{E}_{1,p}^\mu(J_T)} \leq r$ and $\|\mathcal{F}v - \mathcal{F}\bar{v}\|_{\mathbb{E}_{1,p}^\mu(J_T)} \leq 2^{-1} \|v - \bar{v}\|_{\mathbb{E}_{1,p}^\mu(J_T)}$. Thus by Banach's contraction theorem, the nonlinear map $\mathcal{F}$ has a unique fixed point, which is the unique local solution of Pr. (1) simultaneously.

Theorem 6. Suppose all the hypotheses upon F and A together with $\mu > 1 - p^{-1}(\rho - 1)^{-1}$ are satisfied, and then there exist $\varepsilon > 0$ and $T > 0$ such that, for each $u_1 \in B_{u_0}^\mu(\varepsilon)$, Pr. (1) has a unique solution in the space $\mathbb{E}_{1,p}^\mu(J_T)$.

Now, we turn to deal with the continuation of the local solutions. Suppose that $H(A)$ (ii) is strengthened as follows:

(ii') For all $(t, u) \in J_0 \times X_{\mu-1/p,p}$, $A(t, u) \in \mathcal{MR}_p(\mathbb{R}_+)$.

Under this situation, by invoking Lemma 5, together with the continuity of A , we can deduce that, for each precompact subset $\mathcal{K} \subset J_0 \times X_{\mu-1/p,p}$, there is a common constant C_0 , for which Ine. (4) is satisfied for all $A(t, u)$ with $(t, u) \in J_0 \times \mathcal{K}$. We can also assume that all the semigroups $e^{-sA(t,u)}$ share the same bounds M .

Taking any $u_0 \in X_{\mu-1/p,p}$, and using Theorem 6, we can find a finite interval $J_T \subseteq J_0$, on which Pr. (1) has a unique solution $\tilde{u} \in \mathbb{E}_{1,p}^\mu(J_T)$ with the initial value $\tilde{u}(0) = u_0$. Denote by $\bar{A}(t, u) = A(t + T, u)$ and $\bar{F}(t, u) = F(t + T, u)$. Evidently,

$\bar{A}, \bar{F}$ verify hypotheses $H(A)$ and $H(F)$, respectively. Notice that $\tilde{u} \in \mathbb{E}_{1,p}^\mu([t, T])$ for arbitrary $0 < t < T$, and then, following the same process as in [17], we can find a time $\tau \in (T, T_0]$ for which there is a unique solution $\bar{u}$ surviving on the maximal existing interval $[0, \tau - T]$ solving the following problem:

$$\begin{aligned}
\dot{\bar{u}} + \bar{A}(t, \bar{u}) \bar{u} &= \bar{F}(t, \bar{u}), \quad t > 0, \\
\bar{u}(0) &= \tilde{u}(T).
\end{aligned} \tag{34}$$

Define

$$u(t) = \begin{cases} \tilde{u}(t), & t \in [0, T], \\ \bar{u}(t - T), & t \in (T, \tau). \end{cases} \tag{35}$$

Then $u \in \mathbb{E}_{1,p}^\mu(\tilde{T})$ for any $0 < \tilde{T} < \tau$ and solves (1) on the interval $[0, \tau)$. We can also conclude that u could not be extended beyond τ anymore. Thus the function u constructed above is exactly the solution of (1) arising out of $u_0 \in X_{\mu-1/p,p}$ with the maximal existing interval $[0, \tau)$. If $\tau = T_0$, then u is called globally existing. If $\tau < T_0$, then we obtain

$$\int_0^\tau t^{(1-\mu)p} (\| \dot{u}(t) \|_X^p + \| u(t) \|_{X_1}^p) dt = \infty, \tag{36}$$

and $\lim_{t \rightarrow \tau^-} u(t)$ does not exist in $X_{\mu-1/p,p}$. In this case, we say that u blows up at time τ .

Theorem 7. Suppose that $X_1 \hookrightarrow X$, and there exists $\nu \in (1 - p^{-1}(\rho - 1)^{-1}, \mu)$ such that $H(A)$ (i) is also satisfied with

v . Suppose also, on its maximal interval of existence $[0, \tau)$, solution u of Pr. (1) verifies

$$\sup_{t \in [0, \tau)} \|u(t)\|_{X_{\mu-1/p, p}} < \infty. \quad (37)$$

Then $\tau = T_0$, and u survives globally.

Proof. We will prove it by contradiction. Suppose that $\tau < T_0$, and then the set $\mathcal{K} = \{(t, u(t)) : t \in [0, \tau)\}$ is bounded in $J_0 \times X_{\mu-1/p, p}$ and therefore precompact in $J_0 \times X_{\nu-1/p, p}$ since $X_1 \hookrightarrow X$. Thus $u_\tau = \lim_{t \rightarrow \tau^-} u(t)$ exists in $X_{\nu-1/p, p}$. Define $u(\tau) = u_\tau$, and then u can be extended continuously onto the whole interval $[0, \tau]$. Assume that $\sup_{t \in [0, \tau]} \|u(t)\|_{X_{\nu-1/p, p}} = R_\nu$, and then

$$\begin{aligned} & \|A(\bar{t} + \delta, u(\bar{t} + \delta)) - A(\bar{t} + \delta + t, u(\bar{t} + \delta))\|_{B(X_1, X)} \\ & \leq 2L(R_\nu) \|u(\bar{t} + \delta) - u(\bar{t})\|_{X_{\nu-1/p, p}} \\ & + \|A(\bar{t}, u(\bar{t})) - A(\bar{t} + t, u(\bar{t}))\|_{B(X_1, X)}. \end{aligned} \quad (38)$$

Thus, by the continuity of u and A , we can deduce that

$$\lim_{T \rightarrow 0} \sup_{t \in [0, T]} \|A(\bar{t} + t, u(\bar{t})) - A(\bar{t}, u(\bar{t}))\|_{B(X_1, X)} = 0 \quad (39)$$

uniformly for $\bar{t} \in [0, \tau]$.

In addition, for arbitrary $\varepsilon > 0$, there is a partition of $[0, \tau]$: $0 = t_0 < t_1 < \dots < t_m = \tau$ such that

$$\left| \int_0^T t^{(1-\nu)p} \|F(t + \bar{t}, 0) - F(t + \bar{s}, 0)\|_X^p dt \right| < \varepsilon \quad (40)$$

provided $\bar{t}, \bar{s} \in [t_{k-1}, t_k]$ for all $k \in \{1, 2, \dots, m\}$. Notice that

$$\lim_{T \rightarrow 0} \int_0^T t^{(1-\nu)p} \|F(t + t_k, 0)\|_X^p dt = 0 \quad (41)$$

for all $k \in \{1, 2, \dots, m\}$; we obtain

$$\lim_{T \rightarrow 0} \int_0^T t^{(1-\nu)p} \|F(t + \bar{t}, 0)\|_X^p dt = 0 \quad (42)$$

uniformly for $\bar{t} \in [0, \tau]$.

Now we can take an arbitrary sequence $\{t_n\}$ converging to τ from the left side, and consider the following problem:

$$\begin{aligned} \dot{u}_n + \bar{A}_n(t, u_n) u_n &= \bar{F}_n(t, u_n), \quad t > 0, \\ u_n(0) &= u(t_n), \end{aligned} \quad (43)$$

where $\bar{A}_n(t, u) = A(t + t_n, u)$ and $\bar{F}_n(t, u) = F(t + t_n, u)$. Assume that (16) holds also with ν , and then, by repeating the reasoning process as in Theorem 6 n times with the same C_0 , C_1 , $\beta(T)$, and $\delta(T)$, we obtain a small number $T > 0$, such that, on the common interval $[0, T]$, Pr. (43) has a unique solution u_n in $\mathbb{E}_{1, p}^\nu(J_T)$. By the uniqueness, we can find that $u(t) = u_n(t - t_n)$ for $t \in [t_n, t_n + T]$, which means that u can be extended beyond the singular point τ . This contradicts to the maximality of τ since $t_n + T > \tau$ for n large enough. Thus the proof has been completed. $\square$

Case 2 ($\mu = 1 - p^{-1}(\rho - 1)^{-1}((\rho - 1)(\rho - 1) > 1)$). In this case, we only deal with the Cauchy problem of the semilinear parabolic equation; namely,

$$\begin{aligned} \dot{u} + A_0 u &= F(t, u), \quad t > 0, \\ u(0) &= u_0. \end{aligned} \quad (44)$$

Here, A_0 is a sectorial operator lying in $\mathcal{MR}_p(\mathbb{R}_+)$.

Fix $\nu \in (1/p, \mu)$ and $0 < T \leq 1$, and suppose that inequalities (4) and (16) also hold for A with μ replaced by ν . Take

$$Y_{\mu, T} = C([0, T]; X_{\mu-1/p, p}) \cap C_\mu(0, T; X_{1-1/p, p}) \quad (45)$$

as our work space, which is a Banach space endowed with the norm

$$\|v\|_{Y_{\mu, T}} = \max \left\{ \|v\|_{C([0, T]; X_{\mu-1/p, p})}, \|v\|_{C_\mu(0, T; X_{1-1/p, p})} \right\}, \quad (46)$$

and let ${}_0Y_{\mu, T} = C_0([0, T]; X_{\mu-1/p, p}) \cap {}_0C_\mu(0, T; X_{1-1/p, p})$. Consider the ball in $Y_{\mu, T}$:

$$\begin{aligned} \mathcal{B}_{u_1}^\mu(r) \\ = \left\{ v \in Y_{\mu, T} : v - \mathcal{R}u_1 \in {}_0Y_{\mu, T}, \|v - \mathcal{R}u_1\|_{Y_{\mu, T}} \leq r \right\}. \end{aligned} \quad (47)$$

For all $u_1 \in B_{u_0}^\mu(\varepsilon)$, $v, \bar{v} \in \mathcal{B}_{u_1}^\mu(r)$, and $t \in (0, T]$, by $H(F)$, we can deduce that

$$\begin{aligned} \|F(\cdot, v(\cdot))\|_{L_\nu^p(0, t; X)} &\leq t^{\mu-\nu} \|F(\cdot, 0)\|_{L_\mu^p(0, t; X)} + \|\phi\|_\infty \\ &\cdot \left[\|v\|_{L_\nu^p(0, t; X_{1-1/p, p})} \right. \\ &+ \left(\int_0^t s^{(1-\nu)p} \|v(s)\|_{X_{1-1/p, p}}^{pp} ds \right)^{1/p} \Big] \leq t^{\mu-\nu} \beta(t) \\ &+ \|\phi\|_\infty \left\{ \frac{t^{(\mu-\nu)+1/p}}{[(\mu-\nu)p+1]^{1/p}} \|v\|_{C_\mu(0, t; X_{1-1/p, p})} \right. \\ &+ \left(\int_0^t t^{(1-\nu)p-(1-\mu)pp} dt \right)^{1/p} \|v\|_{C_\mu(0, t; X_{1-1/p, p})}^p \Big\} \\ &\leq t^{\mu-\nu} \left\{ \beta(t) + C_{\phi, \nu} \left[t^{1/p} \left(\|v - \mathcal{R}u_1\|_{C_\mu(0, t; X_{1-1/p, p})} \right. \right. \right. \\ &+ \left. \left. \|\mathcal{R}u_1\|_{C_\mu(0, t; X_{1-1/p, p})} \right) + \left(\|v - \mathcal{R}u_1\|_{C_\mu(0, t; X_{1-1/p, p})} \right. \right. \\ &+ \left. \left. \|\mathcal{R}u_1\|_{C_\mu(0, t; X_{1-1/p, p})} \right)^p \right] \Big\}, \end{aligned} \quad (48)$$

where $C_{\phi,\nu} = \|\phi\|_{\infty} [p(\mu - \nu)]^{-1/p}$. And

$$\begin{aligned} & \|F(\cdot, \nu(\cdot)) - F(\cdot, \bar{\nu}(\cdot))\|_{L^p_{\nu}(0,t;X)} \leq \|\phi\|_{\infty} \\ & \cdot \left\{ \frac{t^{1/p+\mu-\nu}}{[(\mu - \nu)p + 1]^{1/p}} \right. \\ & + \frac{t^{\mu-\nu}}{p(\mu - \nu)^{1/p}} \left(\|v\|_{C_{\mu}(0,t;X_{1-1/p,p})}^{\rho-1} \right. \\ & \left. \left. + \|\bar{v}\|_{C_{\mu}(0,t;X_{1-1/p,p})}^{\rho-1} \right) \right\} \cdot \|v - \bar{v}\|_{C_{\mu}(0,t;X_{1-1/p,p})} \\ & \leq C_{\phi,\nu} t^{\mu-\nu} \left[t^{1/p} + 2(r + C\varepsilon + \delta(t))^{\rho-1} \right] \|v - \bar{v}\|_{Y_{\mu,t}}. \end{aligned} \quad (49)$$

On the other hand, for each $w \in {}_0E_{1,p}^{\nu}(J_T)$, we have

$$\begin{aligned} & \|w(t)\|_{X_{\mu-1/p,p}}^p \\ & = \|w(t)\|_X^p + \int_0^{\infty} s^{(1-\mu)p} \|A_0 e^{-sA_0} w(t)\|_X^p ds \\ & \leq \|w(t)\|_X^p + t^{(1-\mu)p} \int_0^t \|A_0 e^{-sA_0} w(t)\|_X^p ds \\ & \quad + t^{(\nu-\mu)p} \int_t^{\infty} s^{(1-\nu)p} \|A_0 e^{-sA_0} w(t)\|_X^p ds \\ & \leq t^{(\nu-\mu)p} \left[t^{(1-\nu)p} \|w(t)\|_{X_{1-1/p,p}}^p + \|w(t)\|_{X_{\nu-1/p,p}}^p \right] \\ & \leq 2C_1^p t^{(\nu-\mu)p} \|w\|_{E_{1,p}^{\nu}(J_T)}^p. \end{aligned} \quad (50)$$

Therefore, for the corresponding function $u = \mathcal{R}u_1 + (\partial_t + A_0)^{-1} \circ \mathcal{G}v$, we obtain

$$\begin{aligned} & \max \left\{ \|u(t) - \mathcal{R}u_1(t)\|_{X_{\mu-1/p,p}}, t^{1-\mu} \|u(t) \right. \\ & \left. - \mathcal{R}u_1(t)\|_{X_{1-1/p,p}} \right\} \leq 2C_0C_1 \left\{ \beta(t) \right. \\ & + C_{\phi,\nu} \left[t^{1/p} \left(\|v - \mathcal{R}u_1\|_{C_{\mu}(0,t;X_{1-1/p,p})} \right. \right. \\ & \left. \left. + \|\mathcal{R}u_1\|_{C_{\mu}(0,t;X_{1-1/p,p})} \right) + \left(\|v - \mathcal{R}u_1\|_{C_{\mu}(0,t;X_{1-1/p,p})} \right. \right. \\ & \left. \left. + \|\mathcal{R}u_1\|_{C_{\mu}(0,t;X_{1-1/p,p})} \right)^{\rho} \right] \right\} \end{aligned} \quad (51)$$

for all $t \in (0, T]$, which means that $u - \mathcal{R}u_1 \in {}_0Y_{\mu,T}$, and

$$\begin{aligned} & \|u - \mathcal{R}u_1\|_{Y_{\mu,T}} \leq 2C_0C_1 \left\{ \beta(T) \right. \\ & \left. + C_{\phi,\nu} \left[T^{1/p} (r + \varepsilon + \delta(T)) + (r + \varepsilon + \delta(T))^{\rho} \right] \right\}. \end{aligned} \quad (52)$$

Furthermore, for another function $\bar{u} = (\partial_t + A_0)^{-1} \bar{v}$, we have

$$\begin{aligned} & \|u - \bar{u}\|_{Y_{\mu,T}} \leq 2C_0C_1C_{\phi,\nu} \left[T^{1/p} + 2(r + \varepsilon + \delta(T))^{\rho-1} \right] \\ & \cdot \|v - \bar{v}\|_{Y_{\mu,T}}. \end{aligned} \quad (53)$$

Take

$$r = \left[8C_0C_1C_{\phi,\nu} (C + 2)^{\rho} \right]^{-1/(\rho-1)} \quad (54)$$

and $\varepsilon \leq r$, select $T_1 \in (0, T_0)$ such that $|\beta(T_1)| \leq r/4C_0C_1$ and $|\delta(T_1)| \leq r$, and let

$$T = \min \left\{ T_1, \left[8C_0C_1C_{\phi,\nu} (C + 2)^{-\rho} \right]^{-1/p} \right\}, \quad (55)$$

and then, from the a priori estimates (52) and (53), we can derive that, for all $u_1 \in B_{u_0}^{\mu}(\varepsilon)$ and $v, \bar{v} \in \mathcal{B}_{u_0}^{\mu}(r)$,

$$\begin{aligned} & \left\| (\partial_t + \widehat{A}_0)^{-1} \circ \mathcal{G}v \right\|_{Y_{\mu,T}} \leq r, \\ & \left\| (\partial_t + \widehat{A}_0)^{-1} \circ \mathcal{G}v - (\partial_t + \widehat{A}_0)^{-1} \circ \mathcal{G}\bar{v} \right\|_{Y_{\mu,T}} \\ & \leq 2^{-1} \|v - \bar{v}\|_{Y_{\mu,T}}. \end{aligned} \quad (56)$$

Thus using Banach's contraction theorem, we can conclude that the multiple map $\mathcal{R}u_1 + (\partial_t + \widehat{A}_0)^{-1} \circ \mathcal{G}$ has a unique fixed point, which is the unique local solution of Pr. (44) exactly. That is, consider the following theorem.

Theorem 8. Assume that hypotheses $H(F)$ are satisfied; then there exist $\varepsilon > 0$ and $T > 0$, such that, for each $u_1 \in B_{u_0}^{\mu}(\varepsilon)$, Pr. (44) has a unique solution in the space $Y_{\mu,T}$.

Remark 9. Like Case 1, each $Y_{\mu,T}$ solution of Pr. (44) can be extended onto a maximal interval $[0, \tau)$ such that $\lim_{t \rightarrow \tau} u(t)$ does not exist in $X_{\mu-1/p,p}$ provided $\tau < \infty$. Moreover if $X_1 \hookrightarrow X$, and there exists $\mu_1 > 1 - 1 - p^{-1}(\rho - 1)^{-1}$ such that

$$\sup_{t \in [0, \tau)} \|u(t)\|_{X_{\mu_1-1/p,p}} < \infty, \quad (57)$$

then this solution exists globally.

4. Examples

Example 10. Consider the second-order parabolic equation

$$\begin{aligned} & \partial_t u - \sum_{i,j=1}^n a_{ij}(t, x, u) D_{ij} u = f(t, u, \nabla u), \\ & t > 0, \quad x \in \Omega, \end{aligned} \quad (58)$$

$$u|_{\partial\Omega} = 0, \quad t > 0,$$

$$u(0) = u_0, \quad x \in \Omega.$$

Suppose that $\Omega \subseteq \mathbb{R}^n$ ($n \geq 3$) is bounded domain with C^2 boundary, $1 < p < \infty$, $J_0 = [0, T_0]$ with $T_0 < \infty$, and $a_{ij} \in C(J_0 \times \overline{\Omega} \times \mathbb{R})$, $i, j = 1, 2, \dots, n$, and $(a_{ij}(t, x, u))_{n \times n}$ is a symmetric and positive definite matrix for every $(t, x, u) \in J_0 \times \overline{\Omega} \times \mathbb{R}$. Suppose also $f : J_0 \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function of Caratheodory type; that is, $t \mapsto f(t, u, \xi)$ is measurable for each $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and $(u, \xi) \mapsto f(t, u, \xi)$ is locally

Lipschitz for a.e. $t \in J_0$. Moreover, there are numbers $\rho_0, \rho \geq 1$ and a function $\phi \in L^\infty(J_0)$, such that

$$\begin{aligned} & |f(t, u_1, \xi_1) - f(t, u_2, \xi_2)| \leq \phi(t) \\ & \cdot \left[|u_1 - u_2| \left(1 + |u_1|^{\rho_0-1} + |u_2|^{\rho_0-1} \right) \right. \\ & \left. + |\xi_1 - \xi_2| \left(1 + |\xi_1|^{\rho-1} + |\xi_2|^{\rho-1} \right) \right] \end{aligned} \quad (59)$$

for a.e. $t \in J_0$, all $u_i \in \mathbb{R}$, and all $\xi_i \in \mathbb{R}^n$, $i = 1, 2$.

Let $X = L^p(\Omega)$ and $X_1 = \{u \in W^{2,p}(\Omega) : u|_{\partial\Omega} = 0\}$, where $u|_{\partial\Omega}$ is the trace of u on $\partial\Omega$. Then the interpolate space $X_{\mu-1/p,p}$ can be described as

$$\begin{aligned} X_{\mu-1/p,p} &= \begin{cases} \{u \in W^{2\mu-2/p,p}(\Omega) : u|_{\partial\Omega} = 0\}, & \mu p > 3/2, \\ W^{2\mu-2/p,p}(\Omega), & 1 < \mu p < 3/2. \end{cases} \end{aligned} \quad (60)$$

By Nirenberg-Gagliardo's inequality, we have that $X_{\mu-1/p,p} \hookrightarrow C(\bar{\Omega})$ provided

$$\begin{aligned} p &> \frac{n+2}{2}, \\ \mu &> \frac{n+2}{2p}, \end{aligned} \quad (61)$$

and $W^{1-2/p,p}(\Omega) \hookrightarrow L^{p(\rho-1)}(\Omega)$ whenever

$$\frac{1}{\rho-1} + \frac{p-2}{n} \geq 1. \quad (62)$$

Therefore, for each $R > 0$, there is correspondingly a constant $C_R > 0$ such that, for any $u_i \in X_{1-1/p,p}$ with $\|u_i\|_{C(\bar{\Omega})} \leq R$, $i = 1, 2$, we have

$$\begin{aligned} & \|F(t, u_1) - F(t, u_2)\|_X \leq \phi(t) \\ & \cdot \left(\|u_1 - u_2\|_{L^p(\Omega)} + \|\nabla u_1 - \nabla u_2\|_{L^p(\Omega)} \right) \\ & \cdot \left(1 + R^{\rho_0-1} + \|\nabla u_1\|_{L^{p(\rho-1)}(\Omega)}^{\rho-1} + \|\nabla u_2\|_{L^{p(\rho-1)}(\Omega)}^{\rho-1} \right) \\ & \leq C_R \phi(t) \|u_1 - u_2\|_{X_{1-1/p,p}} \\ & \cdot \left(1 + \|u_1\|_{X_{1-1/p,p}}^{\rho-1} + \|u_2\|_{X_{1-1/p,p}}^{\rho-1} \right), \end{aligned} \quad (63)$$

where $F(t, u_i) = f(t, u_i, \nabla u_i)$, $i = 1, 2$.

Given $v \in X_{\mu-1/p,p}$ and $u \in X_1$, define

$$[A(t, v)u](x) = - \sum_{i,j=1}^n a_{ij}(t, x, v(x)) D_{ij}u(x). \quad (64)$$

It is easy to show that $A \in C(J_0 \times X_{\mu-1/p,p}, \mathcal{B}(X_1, X_0))$, and, from [7, 12], we know that $A(t, v) \in \mathcal{MR}_p(J_0, X_0)$ for all $(t, v) \in J_0 \times X_{\mu-1/p,p}$.

Now the numbers p, μ can be chosen to fit in our setting according to the variation of ρ . In fact, if $1 \leq \rho \leq 2$, then we take

$$\begin{aligned} \frac{n+2}{2} < p \leq \min \left\{ \frac{n+2}{2} + \frac{1}{\rho-1}, \theta n \right\}, \\ \mu &> \frac{n+2}{2p}; \end{aligned} \quad (65)$$

if $2 < \rho \leq 2n/(n+2) + 1$, then we take

$$\begin{aligned} \frac{n+2}{2} < p \leq \min \left\{ \frac{n+2}{2} + \frac{n-1}{\rho-1}, \theta n \right\}, \\ \mu &> \frac{n+2}{2p}; \end{aligned} \quad (66)$$

if $\rho > 2n/(n+2) + 1$, then we take

$$\begin{aligned} 2 + n \left(1 - \frac{1}{\rho-1} \right) \leq \min \left\{ p \leq n+2 + \frac{n-1}{\rho-1}, \theta n \right\}, \\ \mu &> \frac{n+2}{2p}, \end{aligned} \quad (67)$$

where $0 < \theta < 1$ is arbitrary. In this setting, all Conditions (61), (62), $\mu > 1 - 1/[p(\rho-1)]$, and $n/2p < 1/2$ are verified. Thus, using Theorems 6 and 7, we have the following.

Theorem 11. For the numbers p, μ chosen above, if the initial point $u_0 \in X_{\mu-1/p,p}$ (including the case $X_{1/2,p} = W_0^{1,p}(\Omega)$), there is a radius $\varepsilon > 0$ and a time $T > 0$ depending only on the $X_{\mu-1/p,p}$ -norm of u_0 , such that, for each $u_1 \in B_{u_0}^\mu(\varepsilon)$, there is a unique solution $u = u(\cdot, u_1)$ to Pr. (58) on the interval $[0, T]$ with $u(0) = u_1$ and satisfying

$$\begin{aligned} u &\in W_\mu^{1,p}(0, T; L^p(\Omega)) \cap L_\mu^p(0, T; W^{2,p}(\Omega)) \\ &\cap C([0, T]; C_0(\Omega)) \cap C_\mu(0, T; W^{2-2/p,p}(\Omega)). \end{aligned} \quad (68)$$

And, for any two points u_i , $i = 1, 2$, lying in the same ball $B_{u_0}^\mu(\varepsilon)$, and $w(t) = u(t, u_1) - u(t, u_2)$, we have the following estimates:

$$\begin{aligned} & \left(\int_0^T t^{(1-\mu)p} \|w(t)\|_{W^{2,p}(\Omega)}^p dt \right. \\ & \left. + \int_0^T t^{(1-\mu)p} \|\partial_t w(t)\|_{L^p(\Omega)}^p dt \right)^{1/p} \leq C \|u_1 \\ & - u_2\|_{X_{\mu-1/p,p}}, \end{aligned} \quad (69)$$

$$\max \left\{ \max_{t \in [0, T]} \|w(t)\|_{C(\Omega)}, \sup_{t \in (0, T]} t^{1-\mu} \|\nabla w(t)\|_{L^{p(\rho-1)}(\Omega)} \right\}$$

$$\leq C \|u_1 - u_2\|_{X_{\mu-1/p,p}}$$

for some $C > 0$.

Any such solution u can be extended onto a maximal interval $[0, T)$ uniquely. Furthermore, if

$$\left(\int_0^\tau t^{(1-\mu)p} \|u(t)\|_{W^{2,p}(\Omega)}^p dt + \int_0^\tau t^{(1-\mu)p} \|\partial_t u(t)\|_{L^p(\Omega)}^p dt \right)^{1/p} < \infty \quad (70)$$

or weakly $\sup_{t \in [0, \tau)} \|u(t)\|_{X_{\mu-1/p, p}} < \infty$, then $\tau = T_0$, and the solution u survives globally.

Remark 12. Since the regularity of the initial value is allowed to be lower than that of the space where the nonlinear map $F(t, \cdot)$ acts, here we need not require $W^{1-2/p, p}(\Omega) \hookrightarrow C(\overline{\Omega})$, and, consequently, condition $p > n+2$ which was used in [17] is replaced by the weaker one: $p > (n+2)/2$ in this paper.

Example 13. Let us consider the homogeneous Navier-Stokes equation

$$\begin{aligned} u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f(t), \quad t > 0, \quad x \in \Omega, \\ \operatorname{div} u &= 0, \quad t > 0, \quad x \in \Omega, \\ u|_{\partial\Omega} &= 0, \quad t > 0, \\ u(0) &= u_0, \quad x \in \Omega. \end{aligned} \quad (71)$$

Here $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with $C^{2+\eta}$ ($\eta > 0$) boundary. The vector-valued function u denotes the velocity of the flow, and the scalar function p denotes the pressure, ν is the viscous coefficient, and f is the body force. Denote

$$\mathcal{D}_\sigma = \{\Psi \in C_0^\infty(\Omega)^3 : \operatorname{div} \Psi = 0\}, \quad (72)$$

and let $\mathcal{H}_q$ be the complement of $\mathcal{D}_\sigma$ in the space $L^q(\Omega)^3$. Recall that, under the assumptions upon Ω , we have that

$$\mathcal{H}_q = \{u \in L^q(\Omega)^n : \operatorname{div} u = 0, \gamma_n u = 0\}, \quad (73)$$

where $\operatorname{div} u$ is the divergence of u in the sense of distribution, and $\gamma_n u$ denotes the trace of u such that $\gamma_n u = u \cdot n$ for all $u \in C^1(\overline{\Omega})^3$, where n denotes the unit outer normal vector at $x \in \partial\Omega$. Moreover we have the Helmholtz decomposition (see [23, Ch. III])

$$\begin{aligned} L^q(\Omega)^3 &= \mathcal{H}_q \oplus G_q \\ \text{with } G_q &= \{\nabla p : p \in W^{1,q}(\Omega)\}, \end{aligned} \quad (74)$$

from which we obtain a continuous projection $P_q : L^q(\Omega)^3 \rightarrow \mathcal{H}_q$ which is called the Helmholtz projection. Define

$$\begin{aligned} \mathcal{D}(A_q) &= \{u \in W^{2,q}(\Omega)^3 : \operatorname{div} u = 0, \text{ in } \Omega, u|_{\partial\Omega} = 0\}, \\ A_q u &= -P_q \Delta u. \end{aligned} \quad (75)$$

Recall that, for each $1 < q < \infty$, A_q is a sectorial operator of $\mathcal{B}\mathcal{F}\mathcal{P}$ type in $\mathcal{H}_q$, and consequently it has the L^p -maximal regularity on the infinite interval $\mathbb{R}^+$ (cf. [10]).

Let $B(u, u) := P_q(u \cdot \nabla)u$; then (71) can be abstracted by

$$\begin{aligned} u_t + \nu A_q u + B(u, u) &= P_q f(t), \quad t > 0, \\ u(0) &= u_0. \end{aligned} \quad (76)$$

For each $\theta \in (0, 1)$, denote $D_q^{\theta, p} := (H_q, \mathcal{D}(A_q))_{\theta, p}$; then we obtain $D_q^{\mu-1/p, p} \hookrightarrow B_q^{2\mu-2/p, p}(\Omega)^3$. Notice that (cf. [24, Ch. 7]), for every $s > 0$,

$$\begin{aligned} B_q^{s, p}(\Omega) &\hookrightarrow \begin{cases} C(\overline{\Omega}), & \text{provided } sq > 3, \\ L^\alpha(\Omega), & \text{provided } sq = 3, \alpha \in (1, \infty), \\ L^{\eta, p}(\Omega) \hookrightarrow L^\eta(\Omega), & \text{provided } sq < 3, \eta = \frac{3q}{(3-sq)} > p, \end{cases} \end{aligned} \quad (77)$$

where $L^{\beta, p}(\Omega)$ denotes the Lorenz space. And

$$\begin{aligned} B_q^{s, p}(\Omega) &\hookrightarrow W^{1, q}(\Omega), \quad \text{provided } s > 1, \\ B_q^{1, 2}(\Omega) &\hookrightarrow W^{1, q}(\Omega), \quad \text{provided } 2 \leq q < \infty. \end{aligned} \quad (78)$$

From the above embedding relations, we can derive the a priori estimates for the nonlinear term $(u \cdot \nabla)u$ as follows:

(1) If $p \geq 2$, $q > 3[2(1 - 1/p)]^{-1}$, then

$$\begin{aligned} \|(u \cdot \nabla)u\|_{L^q(\Omega)} &\leq \|u\|_{L^\infty(\Omega)} \|\nabla u\|_{L^q(\Omega)} \\ &\leq C \|u\|_{B_q^{2-2/p, p}(\Omega)}^2. \end{aligned} \quad (79)$$

(2) If $p > 2$, $(1 - 4/(3p))^{-1} < q \leq 3[2(1 - 1/p)]^{-1}$, then

$$\begin{aligned} \|(u \cdot \nabla)u\|_{L^q(\Omega)} &\leq \|u\|_{L^{qr}(\Omega)} \|\nabla u\|_{L^{qr'}(\Omega)} \\ &\leq C \|u\|_{B_q^{2-2/p, p}(\Omega)} \|u\|_{B_q^{1-2/p, p}(\Omega)}. \end{aligned} \quad (80)$$

Here $1 - 2q(1 - 1/p)/3 < r < (1 - 2/p)/3$ and $1/r + 1/r' = 1$. Therefore the nonlinear map $B(\cdot, \cdot)$ verifies

$$\begin{aligned} \|B(u, u)\|_{\mathcal{H}_q} &\leq C \|u\|_{D_q^{1-1/p, p}}^2, \\ \|B(u_1, u_1) - B(u_2, u_2)\|_{\mathcal{H}_q} &\leq C \|u_1 - u_2\|_{D_q^{1-1/p, p}} \left(\|u_1\|_{D_q^{1-1/p, p}} + \|u_2\|_{D_q^{1-1/p, p}} \right) \end{aligned} \quad (81)$$

provided

$$\begin{aligned} p &> 2, \\ \frac{4}{3p} + \frac{1}{q} &< 1. \end{aligned} \quad (82)$$

Pick $\mu \in (1 - 1/p, 1]$; then all the assumptions of $H(F)$ for the operator $F(t, u) = B(u, u) + P_q f(t)$ are satisfied with $\rho = 2$ provided $f \in L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)$. Thus, using Theorems 6 and 7 for the semilinear equations, we can derive the following.

Theorem 14. For each $u_0 \in D_q^{\mu-1/p,p}$ and $f_0 \in L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)$, there exist $\varepsilon, T > 0$ depending on the norm of $\|u_0\|_{D_q^{\mu-1/p,p}}$ and $\|f_0\|_{L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)}$, such that, for every $u_1 \in B_{u_0}^\mu(\varepsilon)$ and $f \in L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)$ with $\|f - f_0\|_{L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)} \leq \varepsilon$, Pr. (76) has a unique solution $u = u(\cdot, u_1, f)$ on the interval $[0, T]$ with the initial condition $u(0) = u_1$ verified. This solution satisfies

$$\begin{aligned} u &\in W_\mu^{1,p}(0, T; L^q(\Omega)^3) \cap L_\mu^p(0, T; W^{2,q}(\Omega)^3) \\ &\cap C([0, T]; B_q^{2\mu-2/p,p}(\Omega)^3) \\ &\cap C_\mu(0, T; B_q^{2-2/p,p}(\Omega)^3). \end{aligned} \quad (83)$$

And for all $u_i \in B_{u_0}^\mu(\varepsilon)$ and $f_i \in L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)$ with $\|f_i - f_0\|_{L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)} \leq \varepsilon$, $i = 1, 2$, by denoting $w(t) = u(t, u_1, f_1) - u(t, u_2, f_2)$, we have that

$$\begin{aligned} &\left(\int_0^T t^{(1-\mu)p} \|w(t)\|_{W^{2,q}(\Omega)}^p dt \right. \\ &\quad \left. + \int_0^T t^{(1-\mu)p} \|\partial_t w(t)\|_{L^q(\Omega)}^p dt \right)^{1/p} \leq C \left(\|u_1 \right. \\ &\quad \left. - u_2\|_{D_q^{\mu-1/p,p}} + \|f_1 - f_2\|_{L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)} \right), \\ &\max \left\{ \max_{t \in [0, T]} \|w(t)\|_{B_q^{2\mu-2/p,p}(\Omega)}, \sup_{t \in (0, T]} t^{1-\mu} \|\nabla w(t)\|_{L^q(\Omega)}, \right. \\ &\quad \left. \sup_{t \in (0, T]} t^{1-\mu} \|w(t)\|_{C(\Omega)} \right\} \leq C \left(\|u_1 - u_2\|_{D_q^{\mu-1/p,p}} + \|f_1 \right. \\ &\quad \left. - f_2\|_{L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)} \right) \end{aligned} \quad (84)$$

for some $C > 0$.

If we denote by $[0, \tau)$ the maximal interval of existence of the solution $u = u(\cdot, u_0, f)$ to Pr. (76) extended uniquely by the local solution, then $\tau < \infty$ with

$$\begin{aligned} &\left(\int_0^\tau t^{(1-\mu)p} \|u(t)\|_{W^{2,q}(\Omega)}^p dt \right. \\ &\quad \left. + \int_0^\tau t^{(1-\mu)p} \|\partial_t u(t)\|_{L^q(\Omega)}^p dt \right)^{1/p} = \infty \end{aligned} \quad (85)$$

and $\sup_{t \in [0, \tau)} \|u(t)\|_{D_q^{\mu-1/p,p}} = \infty$, or $\tau = \infty$ and the solution u exists globally.

Furthermore, for each $T \in (0, \tau)$, we have the following estimate:

$$\begin{aligned} &\left(\int_0^T t^{(1-\mu)p} \|u(t)\|_{W^{2,q}(\Omega)}^p dt \right. \\ &\quad \left. + \int_0^T t^{(1-\mu)p} \|\partial_t u(t)\|_{L^q(\Omega)}^p dt \right)^{1/p} \\ &\leq C_T \left(\|u_0\|_{D_q^{\mu-1/p,p}} + \|f\|_{L_\mu^p(\mathbb{R}_+, L^q(\Omega)^3)} \right) \end{aligned} \quad (86)$$

for some $C_T > 0$.

Remark 15. If $f \in L_\nu^p(\mathbb{R}_+, L^q(\Omega)^3)$ for some $1 - 1/p < \nu < \mu$, then we can also derive that $u \in {}_0C_\mu(0, T; D_q^{1-1/p,p})$; that is,

$$\lim_{t \rightarrow 0^+} t^{1-\mu} \|u(t)\|_{D_q^{1-1/p,p}} = 0. \quad (87)$$

Let us consider a special case. Let $q = 2, 8/3 < p < 4$, and $1 - 1/p < \mu < 1/2 + 1/p$, and then we have

$$\begin{aligned} \beta_1 &= \mu - \frac{1}{p} \in \left(\frac{1}{4}, \frac{1}{2} \right), \\ \gamma_1 &= 1 - \frac{1}{p} \in \left(\frac{5}{8}, \frac{3}{4} \right), \\ \varepsilon &= 1 - \mu \in \left(\frac{1}{8}, \frac{3}{8} \right). \end{aligned} \quad (88)$$

So if we take $\beta_1 < \beta$ and $\gamma_1 > \gamma$, then we have $\mathcal{D}(A_2^\beta) \hookrightarrow D_2^{\beta\gamma_1,p}$ and $D_2^{\gamma_1,p} \hookrightarrow \mathcal{D}(A_2^\gamma)$. Thus, for $u_0 \in \mathcal{D}(A_2^\beta)$, we obtain $\lim_{t \rightarrow 0^+} t^\varepsilon \|u(t)\|_{\mathcal{D}(A_2^\gamma)} = 0$. Compared to $\lim_{t \rightarrow 0^+} t^\beta \|u(t)\|_{\mathcal{D}(A_2^\gamma)} = 0$ which was derived in [25], despite the fact that the upper bounds of the regularity of $\mathcal{D}(A_2^\gamma)$ is lower than $3/4$, but, as a compensation, the lower bounds of exponent ε are less than $1/4$, which means that, here, solution $u(t)$ exhibits a better time asymptotic property as $t \rightarrow 0$. Notice that $\varepsilon = 1 - \mu = (1 - 1/p) - (\mu - 1/p)$, and limit (87) turns to be $\lim_{t \rightarrow 0^+} t^\varepsilon \|u(t)\|_{D_q^{\beta+\varepsilon,p}} = 0$. This is an L^q -version of the limit $\lim_{t \rightarrow 0^+} t^\varepsilon \|u(t)\|_{\mathcal{D}(A_2^{1/4+\varepsilon})} = 0$ ($u_0 \in \mathcal{D}(A_2^{1/4})$, $\varepsilon \in [1/4, 3/8]$) which was derived in [19] by means of ε -regular solution. It is worth pointing out that, by using the maximal L_μ^p -regularity of the Stokes operator A_θ , the continuity assumptions of the forcing term f has been removed here. Moreover, the asymptotic regular assumption upon f as $t \rightarrow 0$: $\|P_2 f(t)\|_{\mathcal{D}(A_2^{2\varepsilon-3/4})} = o(t^{-2\varepsilon})$ which was used in [19] or $\|P_2 f(t)\|_{\mathcal{H}_2} = o(t^{-1+\beta})$ which was used in [25] are replaced by a much more relaxed one: $f \in L_\nu^p(\mathbb{R}_+, L^q(\Omega)^3)$. In this sense, results obtained here can be regarded as useful supplements of those in [19, 25] for the case $q \neq 2$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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