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Research Article

Commutators of Square Functions Related to Fractional Differentiation for Second-Order Elliptic Operators

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Let $L = -\mathrm{div}(A\nabla)$ be a second-order divergence form elliptic operator, where A is an accretive $n\times n$ matrix with bounded measurable complex coefficients in $\mathbb{R}^n$. In this paper, we mainly establish the L^p boundedness for the commutators generated by $b \in I_\alpha(BMO)$ and the square function related to fractional differentiation for second-order elliptic operators.

1. Introduction

Let T be a linear operator in a measurable function space; then, the commutator formed by T and $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$[b,T]\,f(x) := b(x)\,Tf(x) - T(bf)(x). \tag{1}$$

For $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, set

$$M(b,Q) = \frac{1}{|Q|}\int_Q |b(y) - b_Q|\,dy, \tag{2}$$

where Q is a cube in $\mathbb{R}^n$ and $b_Q = |Q|^{-1}\int_Q b(y)dy$. Then, the BMO space is defined as

$$BMO = \left\{ b \in L_{\mathrm{loc}}(\mathbb{R}^n) : \|b\|_* := \sup_{Q\subset\mathbb{R}^n} M(b,Q) < \infty \right\}. \tag{3}$$

Let $0 < \alpha < 1$, and consider the fractional differentiation operators of even and odd parities, defined for tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$, by $\widehat{D^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$. Let I_α be the Riesz potential operator of order α and be defined in the space of tempered distribution modulo polynomials by setting $\widehat{I_\alpha f}(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$. The BMO Sobolev space $I_\alpha(BMO)$ is the image of BMO under I_α. Equivalently, $b \in I_\alpha(BMO)$ if and only if $D^\alpha b \in BMO$. Let $b \in \mathrm{Lip}_\alpha$, subsequently, if b satisfies $\|b\|_{\mathrm{Lip}_\alpha} = \sup |b(x) - b(y)|/|x-y|^\alpha < \infty$, in which the supremum is taken over all $x, y \in \mathbb{R}^n$ and $x \neq y$. For $\alpha \in (0,1)$, $I_\alpha(BMO)$ is a space of functions modulo constants that is properly contained in Lip_α (see [1, 2]).

Before presenting our main theorem, we introduce the second-order elliptic operator L as follows: For $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{C}^n$, denote its complex conjugate $(\overline{\xi}_1,\ldots,\overline{\xi}_n)$ by $\overline{\xi}$. Let $A = A(x)$ be an $n\times n$ matrix of complex L^∞ coefficients defined on $\mathbb{R}^n$ that satisfy the ellipticity condition:

$$\begin{gathered} \lambda|\xi|^2 \le \mathrm{Re}\, A\xi\cdot\overline{\xi}, \\ \left|A\xi\cdot\overline{\zeta}\right| \le \Lambda|\xi||\zeta|, \end{gathered} \tag{4}$$

for $\xi, \zeta \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \le \Lambda < \infty$. Here, the inner product notation $u\cdot v = u_1v_1 + \cdots + u_nv_n$. Therefore, $A\xi\cdot\overline{\zeta} \equiv \sum_{j,k} a_{j,k}(x)\xi_k\overline{\zeta}_j$. Associated with such a matrix A, we define a second-order divergence form operator:

$$Lf = -\mathrm{div}(A\nabla f) = -\sum_{j=1}^n \partial_j\left((A\nabla f)_j\right), \tag{5}$$

which we interpret in the usual weak sense via a sesquilinear form. The operator $-L$ generates a semigroup $(e^{-tL})_{t>0}$ and the gradient of the semigroup $(\nabla t^{1/2}e^{-tL})_{t>0}$.

In this paper, we first present a general criterion for weak-type (p,p) boundedness of commutators with square functions and $b \in I_\alpha(BMO)$.

Theorem 1. *Let $1 < p < 2$, $0 < \alpha \le 1$, $b \in I_\alpha(BMO)$, and m be an integer greater than 1. Let E and F be two closed subsets of $\mathbb{R}^n$ with a Euclidean distance $d(E,F)$ between each other, and let $\{T_s\}_{s>0}$ be a family of sublinear operators acting on $L^2(\mathbb{R}^n)$. Assume that, for $t > 0$ and $2 < p_1 < \infty$,*

$$\left\|\left(\int_0^\infty \left|\left[b, T_s\left(\mathscr{I} - e^{-tL}\right)^m\right] f\right|^2 \frac{ds}{s}\right)^{1/2}\right\|_{L^{p_1}(F)} \le Ct^\eta \left(\frac{d(E,F)^2}{t}\right)^{-(m-\eta)} \|f\|_{L^p(E)}, \tag{6}$$

where $\eta = (1/2)(n/p_1 - n/p)$. Furthermore, if

$$\left\|\left(\int_0^\infty |T_s f|^2 \frac{ds}{s}\right)^{1/2}\right\|_{L^2} \le C\|D^\alpha f\|_{L^2}, \tag{7}$$

$$\left\|\left(\int_0^\infty |[b, T_s] f|^2 \frac{ds}{s}\right)^{1/2}\right\|_{L^2} \le C\|f\|_{L^2}. \tag{8}$$

Then, we have

$$\left|\left\{x \in \mathbb{R}^n : \left|\left(\int_0^\infty |[b, T_s] f(x)|^2 \frac{ds}{s}\right)^{1/2}\right| > \lambda\right\}\right| \le C\lambda^{-p}\|f\|_{L^p}^p. \tag{9}$$

We recall a square function, which is representative of larger classes of square functions associated with L, given as follows [3]:

$$g_L f(x) = \left(\int_0^\infty \left|t^{1/2}\nabla e^{-tL} f(x)\right|^2 \frac{dt}{t}\right)^{1/2}. \tag{10}$$

In 2007, Aushcer [3] proved that $\|g_L f\|_{L^p} \sim \|f\|_{L^p}$ for $q_L < p < \tilde{q}_L$. The interval $(q_L, \tilde{q}_L)$ is the maximal open interval required for the semigroup $(\sqrt{L}\nabla e^{-tL})_{t>0}$ to be L^p bounded. We recall that $(p_L, \tilde{p}_L)$ is the maximal open interval required for the semigroup $(e^{-tL})_{t>0}$ to be L^p bounded. In [3], the author has shown in general that $p_L = 1$, $\tilde{p}_L = \infty$ if $n = 1, 2$; $p_L < 2n/(n+2)$, $\tilde{p}_L > 2n/(n-2)$ if $n \ge 3$ and $p_L = q_L$.

Many researchers have contributed to the commutators associated with the second-order elliptic operator, and among the numerous studies, some related to development and applications have been cited herein [4–9]. In particular, commutators with fractional differentiations associated with L play an important role in the theory of linear partial differential equations and harmonic analysis [10–14]. Naturally, the case of the commutators of square functions being related to fractional differentials associated with L is worth studying.

In this paper, we define a square function related to the fractional differential operator associated with L as follows:

$$G_L^\alpha f(x) = \left(\int_0^\infty \left|t^{1/2}\nabla e^{-tL} f(x)\right|^2 \frac{dt}{t^{1+\alpha}}\right)^{1/2}, \quad 0 < \alpha < 1. \tag{11}$$

Moreover, for $0 < \alpha < 1$ and $b \in I_\alpha(BMO)$, the commutator of G_L^α can be defined by

$$G_{L;b}^\alpha f(x) = \left(\int_0^\infty \left|\left[b, t^{1/2}\nabla e^{-tL}\right] f(x)\right|^2 \frac{dt}{t^{1+\alpha}}\right)^{1/2}. \tag{12}$$

In this paper, we also establish the L^p boundedness for $G_{L;b}^\alpha$.

Theorem 2. *Let L be a second-order elliptic operator in divergence form defined by (5), $0 < \alpha < 1$, and $b \in I_\alpha(BMO)$. Then, for $p_L < p \le 2$, we have*

$$\left\|G_{L;b}^\alpha f\right\|_{L^p} \le C\|D^\alpha b\|_{BMO}\|f\|_{L^p}. \tag{13}$$

The remainder of this paper is organised as follows: in Section 2, we present some lemmas that play an important role in the proof of the main results; in Section 3, we prove Theorem 1; in Section 4, we prove Theorem 2. For $p \ge 1$, p' denotes the dual exponent of p, i.e., $p' = p/(p-1)$. Throughout this paper, the letter "C" will stand for a positive constant that is independent of the essential variables but will not necessarily have the same value for each occurrence.

2. Preliminary Lemmas

The second-order elliptic operator L in divergence form is defined by (5) and has the following off-diagonal estimates (see [3, 4, 9] and references therein).

Lemma 3 (see [3, 4, 9]). *Let L be a second-order elliptic operator defined by (5), let E and F be two closed sets of $\mathbb{R}^n$, and let set $d(E,F)$ denote the distance between E and F. Then, for $t > 0$, the complex-valued function f, and vector-valued functions $\vec{f}$, the following statements hold:*

(i) Let $q_L < p \le q < \tilde{q}_L$ and $f \in L^p(\mathbb{R}^n)$ supported in E,

$$\left\|t^{1/2}\nabla e^{-tL} f\right\|_{L^q(F)} \le Ct^{-(n/2)(1/p-1/q)}e^{-c(d(E,F)^2/t)}\|f\|_{L^p(E)}; \tag{14}$$

(ii) Let $p_L < p \le q < \tilde{p}_L$, $f \in L^p(\mathbb{R}^n)$ supported in E and $k \in \mathbb{N}$,

$$\left\|(tL)^k e^{-tL} f\right\|_{L^q(F)} \le Ct^{-(n/2)(1/p-1/q)}e^{-c(d(E,F)^2/t)}\|f\|_{L^p(E)}; \tag{15}$$

(iii) Let $p_L < p \le q < \tilde{p}_L$ and $\vec{f} \in \dot{L}_1^p(\mathbb{R}^n)$ supported in E,

$$\left\|t\nabla e^{-tL}\operatorname{div}\vec{f}\right\|_{L^q(F)} \le Ct^{-(n/2)(1/p-1/q)}e^{-c(d(E,F)^2/t)}\left\|\vec{f}\right\|_{L^p(E)}. \tag{16}$$

In particular, if we choose $E = F = \mathbb{R}^n$, the abovementioned $L^p - L^q$ off-diagonal estimates become $L^p - L^q$ estimates.

Another very useful and well-known lemma for off-diagonal estimates is introduced here, which could be proved by using a similar argument for the proof of a previous lemma [9, lemma 2.3].

Lemma 4 ([3, 9]). *Let $1 < p \le q < \infty$ and E, F be arbitrary closed subsets of $\mathbb{R}^n$. Assume that the two families of the operators $\{A_t\}_{t>0}$ and $\{B_s\}_{s>0}$ satisfy the following off-diagonal estimates:*

$$\begin{aligned} \|A_t(g\chi_E)\|_{L^q(F)} &\le Ce^{-d(E,F)^2/ct}\|g\|_{L^q(E)}, \\ \|B_s(g\chi_E)\|_{L^q(F)} &\le Cs^{(n/2)(1/q-1/p)}e^{-d(E,F)^2/cs}\|g\|_{L^p(E)}. \end{aligned} \tag{17}$$

Then, for $t, s > 0$ and $f \in L^p(\mathbb{R}^n)$ supported in E, we have

$$\begin{aligned} &\|A_tB_sf\|_{L^q(F)} \\ &\quad \le Cs^{(n/2)(1/q-1/p)}e^{-d(E,F)^2/c\max\{t,s\}}\|f\|_{L^p(E)}. \end{aligned} \tag{18}$$

Next, let us introduce a criterion that deals with the boundedness of the commutators of the operators satisfying $L^p \longrightarrow L^q$ off-diagonal estimates, which can be proved in [15].

Lemma 5 (see [15]). *Let E and F be two closed subsets of $\mathbb{R}^n$ with a Euclidean distance $d(E,F)$, and let $\{T_s\}_{s>0}$ be a family of sublinear operators acting on $L^2(\mathbb{R}^n)$. Assume that, for $1 \le p \le q \le \infty$ and $f \in C_0^\infty(\mathbb{R}^n)$ with* supp $f \subset E$,

$$\|T_sf\|_{L^q(E)} \le Cs^{(1/2)(n/q-n/p)}e^{-d(E,F)^2/cs}\|f\|_{L^p(E)}. \tag{19}$$

If $\max\{diamE, diamF\} \le Cd(E,F)$, then for $0 < \alpha < 1$ and $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, we have

$$\begin{aligned} &\left\|\left[b, s^{-\alpha/2}T_s\right]f\right\|_{L^q(F)} \\ &\quad \le Cs^{(1/2)(n/q-n/p)}e^{-d(E,F)^2/cs}\|b\|_{\mathrm{Lip}_\alpha}\|f\|_{L^p(E)}, \end{aligned} \tag{20}$$

where C is independent of s, b, and f.

The following two lemmas are about the $L^p - L^q$ off-diagonal estimates related to some commutators of the Lipschitz function and semigroups for second-order elliptic operators.

Lemma 6 (see [16]). *Let E and F be two closed subsets of $\mathbb{R}^n$ with a Euclidean distance $d(E,F)$. Assume that $0 < \alpha < 1$ and $b \in Lip_\alpha(\mathbb{R}^n)$ and $0 \le \gamma \le 1$. If $\max\{diamE, diamF\} \le Cd(E,F)$, then for $s > 0$ and $f \in L^p(\mathbb{R}^n)$ supported in E, we obtain the following for some $\varepsilon > 0$:*

$$\begin{aligned} &\left\|s^{-\alpha/2}\left[b, s^{\gamma/2}L^{\gamma/2}e^{-sL}\right]f\right\|_{L^q(F)} \\ &\quad \le Cs^{(1/2)(n/q-n/p)}e^{-d(E,F)^2/cs}\|b\|_{Lip_\alpha}\|f\|_{L^p(E)}, \\ &\qquad\qquad p_L < p \le q < 2+\varepsilon. \end{aligned} \tag{21}$$

Lemma 7 (see [16]). *Let L be the second-order elliptic operator in divergence form defined by (5), E and F be two closed sets of $\mathbb{R}^n$, and $d(E,F)$ express the distance between E and F. Assume that $0 < \alpha < 1$, $b \in Lip_\alpha(\mathbb{R}^n)$, and $0 \le \gamma \le 1$. Then, for $s > 0$, $f \in L^p(\mathbb{R}^n)$ supported in E, we obtain*

$$\begin{aligned} &\left\|\left[b, s^{(1-\alpha)/2}\nabla e^{-tL}\right]f\right\|_{L^q(F)} \\ &\quad \le C\|b\|_{Lip_\alpha}s^{-(n/2)(1/p-1/q)}e^{-d(E,F)^2/cs}\|f\|_{L^p(E)}, \\ &\qquad\qquad p_L < p \le q < \widetilde{p}_L, \end{aligned} \tag{22}$$

where C is independent of s, b, f.

3. Proof of Theorem 1

For any fixed $\lambda > 0$, without loss of generality, we may assume that $f \in L^p(\mathbb{R}^n)$ is nonnegative. Let us write M for the Hardy-Littlewood maximal function. We use the Calderón-Zygmund decomposition for $f(x)^p$ at height λ^p. Then, there exists a collection of pairwise disjoint cubes $\{Q_j\}_j$ such that

$$\left\{x \in \mathbb{R}^n : M(f^p)(x)^{1/p} > \lambda\right\} = \bigcup_j Q_j, \tag{23}$$

and they satisfy the following property:

$$\lambda \le \left(\frac{1}{|Q_j|}\int_{Q_j} f(x)^p\,dx\right)^{1/p} \le C\lambda. \tag{24}$$

Then, we write $f = g + h = g + \sum_j h_j$, where

$$\begin{aligned} g(x) &= f(x)\chi_{\mathbb{R}^n\setminus\bigcup_j Q_j} \\ &\quad + \sum_j\left(\frac{1}{|Q_j|}\int_{Q_j} f(y)\,dy\right)\chi_{Q_j}(x), \\ h_j(x) &= \left(f(x) - \frac{1}{|Q_j|}\int_{Q_j} f(y)\,dy\right)\chi_{Q_j}(x). \end{aligned} \tag{25}$$

After estimating (24), $p > 1$ and the standard arguments yield $0 \le g(x) \le C\lambda$ for almost every $x \in \mathbb{R}^n$. Then,

$$\int_{Q_j} h_j(x)\,dx = 0,$$

and

$$\left(\frac{1}{|Q_j|}\int_{Q_j}|h_j(x)|^p\,dx\right)^{1/p} \le C\lambda,$$

hence

$$\left|\left\{x \in \mathbb{R}^n : \left|\left(\int_0^\infty |[b, T_s]f(x)|^2\frac{ds}{s}\right)^{1/2}\right| > 3\lambda\right\}\right|$$

$$\begin{aligned}
&\leq \left|\left\{x \in \mathbb{R}^n : \left|\left(\int_0^\infty \left|[b, T_s]\, g(x)\right|^2 \frac{ds}{s}\right)^{1/2}\right| > \lambda\right\}\right| \\
&+ \left|\left\{x \in \mathbb{R}^n : \left|\left(\int_0^\infty \left|[b, T_s]\, h(x)\right|^2 \frac{ds}{s}\right)^{1/2}\right| > 2\lambda\right\}\right| \\
&:= I + II.
\end{aligned} \tag{26}$$

We estimate every term separately. For I, we use (8) and the properties of g to obtain

$$\begin{aligned}
I &\leq \frac{1}{\lambda^2} \int_{\mathbb{R}^n} \int_0^\infty \left|[b, T_s]\, g(x)\right|^2 \frac{ds}{s} dx \\
&\leq \frac{C}{\lambda^2} \int_{\mathbb{R}^n} |g(x)|^2 dx \leq \frac{C\lambda^{2-p}}{\lambda^2} \int_{\mathbb{R}^n} |g(x)|^p dx \\
&\leq C\lambda^{-p} \|f\|_{L^p}^p .
\end{aligned} \tag{27}$$

Now, we proceed with II. Let us fix an integer $m \geq 1$. We write $t_j = \ell(Q_j)^2$, where $\ell(Q_j)$ stands for the side length of the cube Q_j. We use the notation $Q_j^* = 2Q_j$, where, in general, we write ρQ for the ρ-dilated Q, i.e., for the cube with the same centre as Q and with the side length $\rho\ell(Q)$. Let $E^* = \mathbb{R}^n \setminus \bigcup_j Q_j^*$. Because

$$\begin{aligned}
&\left(\int_0^\infty \left|[b, T_s]\, h(x)\right|^2 \frac{ds}{s}\right)^{1/2} \\
&= \left(\int_0^\infty \left|\sum_j [b, T_s]\, h_j(x)\right|^2 \frac{ds}{s}\right)^{1/2} \\
&\leq \left(\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I} - e^{-t_j L}\right)^m\right] h_j(x)\right|^2 \frac{ds}{s}\right)^{1/2} \\
&+ \left(\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I} - \left(\mathcal{I} - e^{-t_j L}\right)^m\right)\right] h_j(x)\right|^2 \frac{ds}{s}\right)^{1/2},
\end{aligned} \tag{28}$$

we obtain

$$\begin{aligned}
II \leq &\left|\left\{x : \left(\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I} - e^{-t_j L}\right)^m\right] h_j(x)\right|^2 \frac{ds}{s}\right)^{1/2} > \lambda\right\}\right| \\
&+ \left|\left\{x : \left(\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I} - \left(\mathcal{I} - e^{-t_j L}\right)^m\right)\right] h_j(x)\right|^2 \frac{ds}{s}\right)^{1/2} > \lambda\right\}\right| \leq \left|\bigcup_j Q_j^*\right| + \left|\left\{x \in E^* : \left(\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I} - e^{-t_j L}\right)^m\right] h_j(x)\right|^2 \frac{ds}{s}\right)^{1/2} > \lambda\right\}\right| \\
&+ \left|\left\{x : \left(\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I} - \left(\mathcal{I} - e^{-t_j L}\right)^m\right)\right] h_j(x)\right|^2 \frac{ds}{s}\right)^{1/2} > \lambda\right\}\right| := II_1 + II_2 + II_3.
\end{aligned} \tag{29}$$

The first term can be estimated as follows:

$$II_1 \leq \sum_j \left|Q_j^*\right| \leq \frac{C}{\lambda^p} \|f\|_{L^p}^p . \tag{30}$$

Now, we complete the estimate of II_2. By Chebychev's inequality, we obtain

$$\begin{aligned}
II_2 &\leq \frac{1}{\lambda^{p_1}} \left\|\left(\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I} - e^{-t_j L}\right)^m\right] h_j(x)\right|^2 \frac{ds}{s}\right)^{1/2}\right\|_{L^{p_1}(E^*)}^{p_1} \\
&= \frac{1}{\lambda^{p_1}} \left|\sup_v \int_{\mathbb{R}^n} \int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I} - e^{-t_j L}\right)^m\right] h_j(x)\right|^2 \frac{ds}{s} v(x)\, dx\right|^{p_1/2},
\end{aligned} \tag{31}$$

where the supremum is taken over all the functions $v \in L^{(p_1/2)'}(E^*)$ with $\|v\|_{L^{(p_1/2)'}(E^*)} = 1$. We set

$$\begin{aligned}
&S_0(Q_j) = 2Q_j; \\
&S_l(Q_j) = 2^{l+1}Q_j \setminus 2^l Q_j, \quad l = 1, 2, \cdots .
\end{aligned} \tag{32}$$

Let us recall that $E^* = \mathbb{R}^n \setminus \bigcup_j Q_j^*$. Because supp $v \subset (2Q_j)^c$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I}-e^{-t_jL}\right)^m\right] h_j(x)\right|^2 \frac{ds}{s} v(x)\,dx\right|^{1/2} \le \left|\int_{\mathbb{R}^n}\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I}-e^{-t_jL}\right)^m\right] h_j(x)\, v(x)^{1/2}\right|^2 \frac{ds}{s}dx\right|^{1/2} \\
&= \sum_j \left(\sum_{l=1}^\infty \int_{S_l(Q_j)}\int_0^\infty \left|\left[b, T_s\left(\mathcal{I}-e^{-t_jL}\right)^m\right] h_j(x)\, v(x)^{1/2}\right|^2 \frac{ds}{s}dx\right)^{1/2} \\
&\le C\sum_j \left(\sum_{l=1}^\infty \left\|\left(\int_0^\infty \left|\left[b, T_s\left(\mathcal{I}-e^{-t_jL}\right)^m\right] h_j\right|^2 \frac{ds}{s}\right)^{1/2}\right\|^2_{L^{p_1}(S_l(Q_j))} \|v\|_{L^{(p_1/2)'}(S_l(Q_j))}\right)^{1/2} \\
&\le C\sum_j \left(\sum_{l=1}^\infty t_j^{2\eta}\left(\frac{d\left(S_l\left(Q_j\right), Q_j\right)^2}{t_j}\right)^{-2(m-\eta)} \left\|h_j\right\|^2_{L^p(Q_j)} \|v\|_{L^{(p_1/2)'}(S_l(Q_j))}\right)^{1/2},
\end{aligned} \tag{33}$$

Where, in the last inequality, we used (6). Because $t_j = \ell^2(Q_j)$, for $l \ge 1$, we obtain $d(S_l(Q_j), Q_j) \ge 2^{l-2}\ell(Q_j)$. Recall that $\eta = (1/2)(n/p_1 - n/p)$. Subsequently, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I}-e^{-t_jL}\right)^m\right] h_j(x)\right|^2 \frac{ds}{s} v(x)\,dx\right|^{1/2} \\
&\le C\lambda\sum_j \left(\sum_{l=1}^\infty \ell\left(Q_j\right)^{2(n/p_1-n/p)} 2^{-4(m-\eta)l} \left|Q_j\right|^{2/p} \|v\|_{L^{(p_1/2)'}(S_l(Q_j))}\right)^{1/2} \\
&\le C\lambda\sum_j \left(\sum_{l=1}^\infty \ell\left(Q_j\right)^{2(n/p_1-n/p)} 2^{-4(m-\eta)l} \left|Q_j\right|^{2/p} \left|2^{l+1}Q_j\right|^{1/(p_1/2)'} \left(\frac{1}{\left|2^{l+1}Q_j\right|}\int_{2^{l+1}Q_j} |v(y)|^{(p_1/2)'}\,dy\right)^{1/(p_1/2)'}\right)^{1/2} \\
&\le C\lambda\sum_j \left(\sum_{l=1}^\infty 2^{-4(m-\eta)l+ln\cdot 1/(p_1/2)'} \left|Q_j\right| \operatorname*{ess\,inf}_{y\in Q_j} M\left(|v|^{(p_1/2)'}\right)(y)^{1/(p_1/2)'}\right)^{1/2} \\
&\le C\lambda\sum_j \left(\left|Q_j\right| \operatorname*{ess\,inf}_{y\in Q_j} M\left(|v|^{(p_1/2)'}\right)(y)^{1/(p_1/2)'}\right)^{1/2} \le C\lambda\sum_j \left|Q_j\right|^{1/2} \operatorname*{ess\,inf}_{y\in Q_j} M\left(|v|^{(p_1/2)'}\right)(y)^{1/2(p_1/2)'}.
\end{aligned} \tag{34}$$

Then, using $|Q_j| \sim \lambda^{-p}\|f\|^p_{L^p(Q_j)}$, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_0^\infty \left|\sum_j \left[b, T_s\left(\mathcal{I}-e^{-t_jL}\right)^m\right] h_j(x)\right|^2 \frac{ds}{s} v(x)\,dx\right|^{1/2} \\
&\le C\lambda^{1+(1/2)p} \|f\|^{-p/2}_{L^p} \\
&\cdot \sum_j \left|Q_j\right| \operatorname*{ess\,inf}_{y\in Q_j} M\left(|v|^{(p_1/2)'}\right)(y)^{1/2(p_1/2)'} \\
&\le C\lambda^{1+(1/2)p} \|f\|^{-p/2}_{L^p} \\
&\cdot \sum_j \int_{Q_j} M\left(|v|^{(p_1/2)'}\right)(x)^{1/2(p_1/2)'}\,dx \\
&\le C\lambda^{1+(1/2)p} \|f\|^{-p/2}_{L^p} \\
&\cdot \int_{\bigcup_j Q_j} M\left(|v|^{(p_1/2)'}\right)(x)^{1/2(p_1/2)'}\,dx.
\end{aligned} \tag{35}$$

Then, because the Hardy-Littlewood maximal function is of weak-type $(1,1)$, we use $\|v\|_{L^2} = 1$ and Kolmogorov's lemma to obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_0^\infty\left|\sum_j\left[b,T_s\left(\mathcal{I}-e^{-t_jL}\right)^m\right]h_j(x)\right|^2\frac{ds}{s}v(x)\,dx\right|^{1/2}\\
&\le C\lambda^{1+(1/2)p}\|f\|_{L^p(Q_j)}^{-p/2}\left|\bigcup_j Q_j\right|^{1-1/2(p_1/2)'}\left\||v|^{(p_1/2)'}\right\|_1^{1/2(p_1/2)'}\\
&\le C\lambda^{1+(1/2)p}\|f\|_{L^p}^{-p/2}\left|\bigcup_j Q_j\right|^{1/2+1/p_1}.
\end{aligned}\tag{36}$$

Then, we plug the estimate into (31) to obtain

$$\begin{aligned}
II_2&\le C\lambda^{pp_1/2}\|f\|_{L^p}^{-pp_1/2}\left|\bigcup_j Q_j\right|^{p_1(1/2+1/p_1)}\\
&\le C\lambda^{pp_1/2}\|f\|_{L^p}^{-pp_1/2}\\
&\quad\cdot\left|\left\{x\in\mathbb{R}^n: M\left(f^p\right)(x)^{1/p}>\lambda\right\}\right|^{(p_1/2+1)}.
\end{aligned}\tag{37}$$

Applying M of the weak-type $(1,1)$, we obtain

$$\begin{aligned}
II_2&\le C\lambda^{pp_1/2}\|f\|_{L^p}^{-pp_1/2}\|f\|_{L^p}^{p(p_1/2+1)}\lambda^{-p(p_1/2+1)}\\
&\le C\lambda^{-p}\|f\|_{L^p}^p.
\end{aligned}\tag{38}$$

We now examine II_3. Recall that

$$\begin{aligned}
II_3=\Bigg|\Bigg\{x:&\\
&\left(\int_0^\infty\left|\sum_j\left[b,T_s\left(\mathcal{I}-\left(\mathcal{I}-e^{-t_jL}\right)^m\right)\right]h_j(x)\right|^2\frac{ds}{s}\right)^{1/2}\\
&>\lambda\Bigg\}\Bigg|.
\end{aligned}\tag{39}$$

Then,

$$\begin{aligned}
\mathcal{I}-\left(\mathcal{I}-e^{-t_jL}\right)^m&=\mathcal{I}-\sum_{k=0}^m C_m^k e^{-kt_jL}\\
&=-\sum_{k=1}^m C_m^k e^{-kt_jL}.
\end{aligned}\tag{40}$$

Thus, from Chebychev's inequality,

$$\begin{aligned}
II_3&\le\frac{C}{\lambda^2}\sum_{k=1}^m\left\|\left(\int_0^\infty\left|\left[b,T_s\sum_j e^{-kt_jL}\right]h_j\right|^2\frac{ds}{s}\right)^{1/2}\right\|_{L^2}^2\\
&\le\frac{C}{\lambda^2}\sum_{k=1}^m\left\|\left(\int_0^\infty\left|[b,T_s]\sum_j e^{-kt_jL}h_j\right|^2\frac{ds}{s}\right)^{1/2}\right\|_{L^2}^2\\
&\quad+\frac{C}{\lambda^2}\sum_{k=1}^m\left\|\left(\int_0^\infty\left|T_s\left[b,\sum_j e^{-kt_jL}\right]h_j\right|^2\frac{ds}{s}\right)^{1/2}\right\|_{L^2}^2\\
&:=II_{3,1}+II_{3,2}.
\end{aligned}\tag{41}$$

We fix $1\le k\le m$. Then, for $II_{3,1}$, by (8), we obtain

$$II_{3,1}\le\frac{C}{\lambda^2}\sum_{k=1}^m\left\|\sum_j e^{-kt_jL}h_j\right\|_{L^2}^2.\tag{42}$$

Hence, by (see [9])

$$\left\|\sum_j e^{-kt_jL}h_j\right\|_{L^2}^2\le C\lambda^{2-p}\|f\|_{L^p}^p,\tag{43}$$

we obtain

$$II_{3,1}\le C\frac{\|f\|_{L^p}^p}{\lambda^p}.\tag{44}$$

For $II_{3,2}$, by (7) and $\|L^{\alpha/2}f\|_{L^2}\sim\|D^\alpha f\|_{L^2}$ (see [3]), we obtain

$$\begin{aligned}
II_{3,2}&\le\frac{C}{\lambda^2}\sum_{k=1}^m\left\|D^\alpha\sum_j\left[b,e^{-kt_jL}\right]h_j\right\|_{L^2}^2\\
&\le\frac{C}{\lambda^2}\sum_{k=1}^m\left\|L^{\alpha/2}\sum_j\left[b,e^{-kt_jL}\right]h_j\right\|_{L^2}^2\\
&\le\frac{C}{\lambda^2}\left(\sum_{k=1}^m\left\|\left[b,L^{\alpha/2}\right]\sum_j e^{-kt_jL}h_j\right\|_{L^2}^2\right.\\
&\quad\left.+\sum_{k=1}^m\left\|\sum_j\left[b,L^{\alpha/2}e^{-kt_jL}\right]h_j\right\|_{L^2}^2\right).
\end{aligned}\tag{45}$$

Then,

$$\begin{aligned}
II_{3,2}&\le\frac{C}{\lambda^2}\left(\sum_{k=1}^m\left\|\sum_j e^{-kt_jL}h_j\right\|_{L^2}^2\right.\\
&\quad\left.+\sum_{k=1}^m\left\|\sum_j\left[b,L^{\alpha/2}e^{-kt_jL}\right]h_j\right\|_{L^2}^2\right),
\end{aligned}\tag{46}$$

where we use the fact that $[b, L^{\alpha/2}]$ is bounded on L^2 with the bound $\|(-\Delta)^{\alpha/2} b\|_{BMO}$ (see [16]). Next, we estimate the abovementioned two norms, $\|\sum_j e^{-kt_jL} h_j\|_{L^2}$ and $\|\sum_j [b, L^{\alpha/2} e^{-kt_jL}] h_j\|_{L^2}$. Now, taking $v \in L^2(\mathbb{R}^n)$ with $\|v\|_{L^2} = 1$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \sum_j e^{-kt_jL} h_j(x) \overline{v(x)} dx\right| \\
&\quad = \left|\sum_j \sum_{l=0}^{\infty} \int_{S_l(Q_j)} e^{-kt_jL} h_j(x) \overline{v(x)} dx\right| \\
&\quad \le \sum_j \sum_{l=0}^{\infty} \left(\int_{S_l(Q_j)} \left|e^{-kt_jL} h_j(x)\right|^2 dx\right)^{1/2} \\
&\quad \cdot \left(\int_{S_l(Q_j)} |v(x)|^2 dx\right)^{1/2}.
\end{aligned} \tag{47}$$

Note that, for all $p_L < r \le 2$, e^{-kt_jL} satisfies the $L^r - L^2$ off-diagonal estimates (see Lemma 3(ii)); let $t_j = \ell(Q_j)^2$, and we obtain (47), which is controlled by

$$\begin{aligned}
&\sum_j \sum_{l=0}^{\infty} e^{-4^l} t_j^{(1/2)(n/2-n/r)} \|h_j\|_{L^r(Q_j)} \left|2^{l+1} Q_j\right|^{1/2} \\
&\quad \cdot \left(\frac{1}{\left|2^{l+1} Q_j\right|} \int_{S_l(Q_j)} |v(x)|^2 dx\right)^{1/2} \\
&\le C\lambda \sum_j \sum_{l=0}^{\infty} e^{-4^l} \ell(Q_j)^{(n/2-n/r)} |Q_j|^{1/r} \left|2^{l+1} Q_j\right|^{1/2} \\
&\quad \cdot \left(\frac{1}{\left|2^{l+1} Q_j\right|} \int_{S_l(Q_j)} |v(x)|^2 dx\right)^{1/2} \\
&\le C\lambda \sum_j \sum_{l=0}^{\infty} e^{-4^l} 2^{ln} |Q_j| \\
&\quad \cdot \left(\frac{1}{\left|2^{l+1} Q_j\right|} \int_{2^{l+1} Q_j} |v(x)|^2 dx\right)^{1/2} \\
&\le C\lambda \sum_j |Q_j| \operatorname*{ess\,inf}_{y \in Q_j} M\left(|v|^2\right)^{1/2}(y) \\
&\le C\lambda \int_{\cup_j Q_j} M\left(|v|^2\right)^{1/2}(x)\, dx. \\
&\le C\lambda \left|\bigcup_j Q_j\right|^{1/2} \left\||v|^2\right\|_{L^1}^{1/2} \\
&\le C\lambda \left|\left\{x \in \mathbb{R}^n : M\left(f^p\right)(x)^{1/p} > \lambda\right\}\right|^{1/2}
\end{aligned} \tag{48}$$

Thus, from (47), (48), and the fact that M is of weak-type $(1,1)$, we obtain

$$\left\|\sum_j e^{-kt_jL} h_j\right\|_{L^2} \le C\lambda \left(\frac{\|f\|_{L^p}}{\lambda}\right)^{p/2}. \tag{49}$$

For the second term of $II_{3,2}$, $\|\sum_j [b, L^{\alpha/2} e^{-kt_jL}] h_j\|_{L^2}$, by Lemma 6 and the same procedures performed previously (48):

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \sum_j \left[b, L^{\alpha/2} e^{-kt_jL}\right] h_j(x) \overline{v(x)} dx\right| \\
&\quad = \left|\sum_j \sum_{l=0}^{\infty} \int_{S_l(Q_j)} \left[b, L^{\alpha/2} e^{-kt_jL}\right] h_j(x) \overline{v(x)} dx\right| \\
&\le \sum_j \sum_{l=0}^{\infty} \left(\int_{S_l(Q_j)} \left|t_j^{-\alpha/2} \left[b, t_j^{\alpha/2} L^{\alpha/2} e^{-kt_jL}\right] h_j(x)\right|^2 dx\right)^{1/2} \\
&\quad \cdot \left(\int_{S_l(Q_j)} |v(x)|^2 dx\right)^{1/2} \\
&\le C \sum_j \sum_{l=0}^{\infty} e^{-4^l} t_j^{(1/2)(n/2-n/r)} \|h_j\|_{L^r} \left|2^{l+1} Q_j\right|^{1/2} \left(\frac{1}{\left|2^{l+1} Q_j\right|}\right. \\
&\quad \left. \cdot \int_{S_l(Q_j)} |v(x)|^2 dx\right)^{1/2} \\
&\le C\lambda \left|\left\{x \in \mathbb{R}^n : M\left(f^p\right)(x)^{1/p} > \lambda\right\}\right|^{1/2}.
\end{aligned} \tag{50}$$

Then, we have

$$\left\|\sum_j \left[b, L^{\alpha/2} e^{-kt_jL}\right] h_j\right\|_{L^2} \le C\lambda \left(\frac{\|f\|_{L^p}}{\lambda}\right)^{p/2}. \tag{51}$$

Combining (49) and (51), we obtain

$$II_{3,2} \le C\lambda^{-p} \|f\|_{L^p}^p. \tag{52}$$

Combining the estimates of $II_{3,1}$ and $II_{3,2}$, we obtain

$$II_3 \le C \frac{\|f\|_{L^p}^p}{\lambda^p}. \tag{53}$$

The proof of Theorem 1 is now completed by combining the estimates of I, II_1, and II_2 with II_3.

4. Proof of Theorem 2

First, we introduce a lemma that will be used to prove Theorem 2.

Lemma 8 (see [16]). *For $0 < \alpha < 1$, let $b \in I_\alpha(BMO)$. Suppose that $\{\theta_s\}_{s>0}$ is a family of operators satisfying*

$$\begin{aligned}
&\left\|\theta_s\left(f 1_{S_k(Q)}\right)\right\|_{L^2(Q)}^2 \\
&\quad \le C 2^{-nk} \left(\frac{\sqrt{s}}{2^k \ell(Q)}\right)^{2m+2} \|f\|_{L^2(S_k(Q))}^2
\end{aligned} \tag{54}$$

for an $m \geq 1$ and all $0 < s \leq C\ell(Q)$. Assume that

$$\sup_{s>0} \|\theta_s\|_{L^2 \longrightarrow L^2} \leq C, \tag{55}$$

$$\theta_s 1 = 0 \tag{56}$$

in the sense of L^2_{loc} for all $s > 0$. Then, we have

$$\int_{\mathbb{R}^n} \int_0^\infty \left|s^{-\alpha/2}\theta_s f(x)\right|^2 \frac{dxds}{s} \leq C \|D^\alpha f\|_{L^2}^2 . \tag{57}$$

If, in addition,

$$\|\theta_s \operatorname{div}\|_{L^2 \longrightarrow L^2} \leq \frac{C}{\sqrt{s}}, \tag{58}$$

then

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_0^\infty \left|\left[b, s^{-\alpha/2}\theta_s\right] f(x)\right|^2 \frac{dxds}{s} \\ &\quad \leq C \|D^\alpha b\|_{BMO}^2 \|f\|_{L^2}^2 . \end{aligned} \tag{59}$$

Recall that

$$G^\alpha_{L;b} f(x) = \left(\int_0^\infty \left|\left[b, s^{(1-\alpha)/2}\nabla e^{-sL}\right] f(x)\right|^2 \frac{ds}{s}\right)^{1/2} . \tag{60}$$

The proof of Theorem 2 is presented in two steps.

Step 1 (the L^2 boundedness). Let $\theta_s = s^{1/2}\nabla e^{-sL}$. By Lemma 8, we only need to prove that $s^{1/2}\nabla e^{-sL}$ satisfies (54)-(56) and (58). Because $\|s^{1/2}\nabla e^{-sL} \operatorname{div}\|_{L^2 \longrightarrow L^2} \leq C/s$, it is easy to verify that $s^{1/2}\nabla e^{-sL}$ satisfies (54), (56), and (58). Moreover, it is easy to verify that $s^{1/2}\nabla e^{-sL}$ satisfies (55) by the L^2 off-diagonal estimate of the operator $s^{1/2}\nabla e^{-sL}$ (see Lemma 3(i, iii)). We obtain

$$\|G^\alpha_L f(x)\|_{L^2} \leq C \|D^\alpha f\|_{L^2} , \tag{61}$$

$$\left\|G^\alpha_{L;b} f(x)\right\|_{L^2} \leq C \|D^\alpha b\|_{BMO} \|f\|_{L^2} . \tag{62}$$

Step 2 (weak-type (p, p) boundedness for $p_L < p < 2$). We first prove that the commutator is of the weak-type (p, p) for $p_L < p < 2$. We apply Theorem 1 with $T_s = s^{(1-\alpha)/2}\nabla e^{-sL}$ to prove this result. Because G^α_L and $G^\alpha_{L;b}$ are bounded on $L^2(\mathbb{R}^n)$, verification of $\{s^{(1-\alpha)/2}\nabla e^{-sL}\}_{s>0}$ satisfies (6). Taking $p < 2 < p_1$, by the Minkowski inequality, we obtain

$$\begin{aligned} &\left\|\left(\int_0^\infty \left|\left[b, s^{1/2}\nabla e^{-sL}\left(\mathcal{I} - e^{-tL}\right)^m\right] f\right|^2 \frac{ds}{s^{1+\alpha}}\right)^{1/2}\right\|_{L^{p_1}(F)} \\ &\leq \left(\int_0^\infty \left\|\left[b, s^{1/2}\nabla e^{-sL}\left(\mathcal{I} - e^{-tL}\right)^m\right] f\right\|_{L^{p_1}(F)}^2 \frac{ds}{s^{1+\alpha}}\right)^{1/2} \\ &= C\left(\int_0^t \cdots + \int_t^\infty \cdots\right)^{1/2} = C\left(I_t + II_t\right)^{1/2} . \end{aligned} \tag{63}$$

Now, we study each operator separately. For the first operator, we have

$$\left(\mathcal{I} - e^{-tL}\right)^m = \sum_{k=0}^m C_m^k (-1)^k e^{-ktL} = \mathcal{I} + \sum_{k=1}^m c_k e^{-ktL}, \tag{64}$$

and, then,

$$\begin{aligned} I_t &\leq \int_0^t \left\|\left[b, s^{1/2}\nabla e^{-sL}\right] f\right\|_{L^{p_1}(F)}^2 \frac{ds}{s^{1+\alpha}} \\ &\quad + \sum_{k=1}^m c_k \int_0^t \left\|\left[b, s^{1/2}\nabla e^{-sL} e^{-ktL}\right] f\right\|_{L^{p_1}(F)}^2 \frac{ds}{s^{1+\alpha}} \\ &= I_{t,0} + \sum_{k=1}^m c_k I_{t,k}. \end{aligned} \tag{65}$$

For $I_{t,0}$, by Lemma 7(i), we have

$$\begin{aligned} I_{t,0} &\leq \int_0^t \left\|\left[b, s^{\frac{1-\alpha}{2}}\nabla e^{-sL}\right] f\right\|_{L^{p_1}(F)}^2 \frac{ds}{s} \\ &\leq C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 \int_0^t e^{-d(E,F)^2/cs} s^{(n/p_1 - n/p)} \frac{ds}{s} \\ &= C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 \int_0^t e^{-d(E,F)^2/cs} s^{2\eta} \frac{ds}{s} \\ &= C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 t^{2\eta} \int_1^\infty e^{-(d(E,F)^2/ct)s} s^{-2\eta} \frac{ds}{s} \\ &\leq C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 \\ &\quad \cdot t^{2\eta} \int_1^\infty \left(\frac{d(E,F)^2}{ct} s\right)^{-2(m-\eta)} s^{-2\eta} \frac{ds}{s} \\ &\leq C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 t^{2\eta} \left(\frac{d(E,F)^2}{t}\right)^{-2(m-\eta)} , \end{aligned} \tag{66}$$

where $\eta = (1/2)(n/p_1 - n/p)$. Now, with $1 \leq k \leq m$, by the commutative property of the semigroup and Lemma 7(i), we obtain

$$\begin{aligned} I_{t,k} &\leq \int_0^t \left\|\left[b, (s+kt)^{(1-\alpha)/2}\nabla e^{-(s+kt)L}\right] f\right\|_{L^{p_1}(F)}^2 \\ &\quad \cdot s(s+kt)^{\alpha-1} \frac{ds}{s^{1+\alpha}} \\ &\leq C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 \\ &\quad \cdot \int_0^t e^{-d(E,F)^2/c(s+kt)} (s+kt)^{(n/p_1 - n/p)} (s+kt)^{\alpha-1} s \frac{ds}{s^{1+\alpha}} \\ &\leq C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 e^{-d(E,F)^2/ct} t^{(n/p_1 - n/p + \alpha - 1)} \int_0^t s^{-\alpha} ds \\ &\leq C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 e^{-d(E,F)^2/ct} t^{(n/p_1 - n/p)} \\ &\leq C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 \left(\frac{d(E,F)^2}{t}\right)^{-2(m-\eta)} t^{2\eta}. \end{aligned} \tag{67}$$

Collecting this estimate and the one proved for $I_{t,0}$, we obtain

$$\begin{aligned} I_t &\le I_{t,0} + \sum_{k=1}^{m} |c_k| I_{t,k} \\ &\le C \|b\|_{\mathrm{Lip}_\alpha}^2 t^{2\eta} \left(\frac{d(E,F)^2}{t} \right)^{-2(m-\eta)} \|f\|_{L^p(E)}^2 . \end{aligned} \tag{68}$$

Next, we proceed with the estimate of II_t:

$$\begin{aligned} II_t &= \int_t^\infty \left\| \left[b, s^{1/2} \nabla e^{-sL} \left(\mathcal{I} - e^{-tL} \right)^m \right] f \right\|_{L^{p_1}(F)}^2 \frac{ds}{s^{1+\alpha}} \\ &= C \int_t^\infty \left\| \left[b, s^{1/2} \nabla e^{-(m+1)sL} \left(\mathcal{I} - e^{-tL} \right)^m \right] \right. \\ &\quad \cdot \left. f \right\|_{L^{p_1}(F)}^2 \frac{ds}{s^{1+\alpha}} \\ &= C \int_t^\infty \left\| \left[b, s^{1/2} \nabla e^{-sL} \left(e^{-sL} - e^{-(s+t)L} \right)^m \right] \right. \\ &\quad \cdot \left. f \right\|_{L^{p_1}(F)}^2 \frac{ds}{s^{1+\alpha}} . \end{aligned} \tag{69}$$

Let E, F be two closed sets and g be such that supp $g \subset E$. For $t \le s$, by a similar previous argument [9], we can prove that

$$\begin{aligned} &\left\| \frac{s}{t} \left(e^{-sL} - e^{(s+t)L} \right) g \right\|_{L^{p_1}(F)} \\ &\quad \le C s^{(1/2)(n/p_1 - n/p_0)} e^{-d(E,F)^2/cs} \|g\|_{L^p(E)} \end{aligned} \tag{70}$$

uniformly on t. Then, by Lemma 4, we have

$$\begin{aligned} &\left\| \left(e^{-sL} - e^{(s+t)L} \right)^m g \right\|_{L^{p_1}(F)} \\ &\quad \le C \left(\frac{t}{s} \right)^m s^{(1/2)(n/p_1 - n/p_0)} e^{-d(E,F)^2/cs} \|g\|_{L^p(E)} . \end{aligned} \tag{71}$$

We write II_t as follows:

$$\begin{aligned} II_t &\le C \int_t^\infty \left\| \left[b, s^{1/2} \nabla e^{-sL} \right] \left(e^{-sL} - e^{-(s+t)L} \right)^m \right. \\ &\quad \cdot \left. f \right\|_{L^{p_1}(F)}^2 \frac{ds}{s^{1+\alpha}} \\ &\quad + C \int_t^\infty \left\| s^{1/2} \nabla e^{-sL} \left[b, \left(e^{-sL} - e^{-(s+t)L} \right)^m \right] \right. \\ &\quad \cdot \left. f \right\|_{L^{p_1}(F)}^2 \frac{ds}{s^{1+\alpha}} := G_1 + G_2 . \end{aligned} \tag{72}$$

First, we consider G_1:

$$\begin{aligned} G_1 &= \int_t^\infty \left\| \left(\left[b, s^{(1-\alpha)/2} \nabla e^{-sL} \right] \right) \right. \\ &\quad \circ \left. \left(e^{-sL} - e^{-(s+t)L} \right)^m f \right\|_{L^{p_1}(F)}^2 \frac{ds}{s} . \end{aligned} \tag{73}$$

Let us observe that, because of (71) and Lemma 7(i) in the composition of the operators, each of them verifies an off-diagonal estimate. This fact allows us to employ Lemma 4 to obtain

$$\begin{aligned} G_1 &\le C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 \\ &\quad \cdot \int_t^\infty e^{-d(E,F)^2/cs} \left(\frac{t}{s} \right)^{2m} s^{(n/p_1 - n/p)} \frac{ds}{s} \\ &= C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 \int_t^\infty e^{-d(E,F)^2/cs} \left(\frac{t}{s} \right)^{2m} s^{2\eta} \frac{ds}{s} \\ &= C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 t^{2\eta} \int_0^1 e^{-(d(E,F)^2/ct)s} s^{2m-2\eta} \frac{ds}{s} \\ &\le C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 t^{2\eta} \int_0^\infty e^{-(d(E,F)^2/ct)s} s^{2m-2\eta} \frac{ds}{s} \\ &= C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 t^{2\eta} \left(\frac{d(E,F)^2}{t} \right)^{-2(m-\eta)} \\ &\quad \cdot \int_0^\infty e^{-s} s^{2(m-\eta)} \frac{ds}{s} \\ &\le C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 t^{2\eta} \left(\frac{d(E,F)^2}{t} \right)^{-2(m-\eta)} , \end{aligned} \tag{74}$$

where $\eta = (1/2)(n/p_1 - n/p)$. From Lemma 5 and (71), it is easy to obtain

$$\begin{aligned} &\left\| \left[b, s^{-\alpha/2} \left(e^{-sL} - e^{(s+t)L} \right)^m \right] g \right\|_{L^{p_1}(F)} \\ &\quad \le C \|b\|_{\mathrm{Lip}_\alpha} \left(\frac{t}{s} \right)^m s^{(1/2)(n/p_1 - n/p_0)} e^{-d(E,F)^2/cs} \|g\|_{L^p(E)} . \end{aligned} \tag{75}$$

Finally, for G_2,

$$\begin{aligned} G_2 &= \int_t^\infty \left\| \left(s^{1/2} \nabla e^{-sL} \right) \right. \\ &\quad \circ \left. \left[b, s^{-\alpha/2} \left(e^{-sL} - e^{-(s+t)L} \right)^m \right] f \right\|_{L^{p_1}(F)}^2 \frac{ds}{s} . \end{aligned} \tag{76}$$

We know that the composition of the operators above (75) or Lemma 3 verifies an off-diagonal estimate. It follows from Lemma 4 that

$$\begin{aligned} G_2 &\le C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 \int_t^\infty e^{-d(E,F)^2/cs} \left(\frac{t}{s} \right)^{2m} s^{2\eta} \frac{ds}{s} \\ &\le C \|b\|_{\mathrm{Lip}_\alpha}^2 \|f\|_{L^p(E)}^2 t^{2\eta} \left(\frac{d(E,F)^2}{t} \right)^{-2(m-\eta)} . \end{aligned} \tag{77}$$

Collecting the estimates for G_1 and G_2, we obtain

$$II_t \le C \|b\|_{\mathrm{Lip}_\alpha}^2 t^{2\eta} \left(\frac{d(E,F)^2}{t} \right)^{-2(m-\eta)} \|f\|_{L^p(E)}^2 , \tag{78}$$

which combined with the estimate of I_t verify that $\{s^{(1-\alpha)/2} \nabla e^{-sL}\}_{s>0}$ satisfies (6). Thus, we have proven Theorem 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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