Research Article

Commutators of Square Functions Related to Fractional Differentiation for Second-Order Elliptic Operators

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Let \( L = -\text{div}(AV) \) be a second-order divergence form elliptic operator, where \( A \) is an accretive \( n \times n \) matrix with bounded measurable complex coefficients in \( \mathbb{R}^n \). In this paper, we mainly establish the \( L^p \) boundedness for the commutators generated by \( b \in I_\alpha(BMO) \) and the square function related to fractional differentiation for second-order elliptic operators.

1. Introduction

Let \( T \) be a linear operator in a measurable function space; then, the commutator formed by \( T \) and \( b \in L_{\text{loc}}(\mathbb{R}^n) \) is defined by

\[
[b, T] f(x) = b(x) T f(x) - T(bf)(x).
\]

For \( b \in L_{\text{loc}}(\mathbb{R}^n) \), set

\[
M(b,Q) = \frac{1}{|Q|} \int_Q |b(y) - b_Q| \, dy,
\]

where \( Q \) is a cube in \( \mathbb{R}^n \) and \( b_Q = |Q|^{-1} \int_Q b(y) \, dy \). Then, the \( BMO \) space is defined as

\[
BMO = \left\{ b \in L_{\text{loc}}(\mathbb{R}^n) : \|b\|_* = \sup_{Q \subset \mathbb{R}^n} M(b,Q) < \infty \right\}.
\]

Let \( 0 < \alpha < 1 \), and consider the fractional differentiation operators of even and odd parities, defined for tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \), by \( D^\alpha f(\xi) = |\xi|^{1-\alpha} \hat{f}(\xi) \). Let \( I_\alpha \) be the Riesz potential operator of order \( \alpha \) and be defined in the space of tempered distribution modulo polynomials by setting \( \hat{I}_\alpha f(\xi) = |\xi|^{-n} \hat{f}(\xi) \). The \( BMO \) Sobolev space \( I_\alpha(BMO) \) is the image of \( BMO \) under \( I_\alpha \). Equivalent, \( b \in I_\alpha(BMO) \) if and only if \( D^\alpha b \in BMO \). Let \( b \in \text{Lip}_\alpha \), subsequently, if \( b \) satisfies \( \|b\|_{\text{Lip}_\alpha} = \sup_{x,y \in \mathbb{R}^n, x \neq y} |b(x) - b(y)|/|x - y|^\alpha < \infty \), in which the supremum is taken over all \( x, y \in \mathbb{R}^n \) and \( x \neq y \). For \( \alpha \in (0, 1) \), \( I_\alpha(BMO) \) is a space of functions modulo constants that is properly contained in \( \text{Lip}_\alpha \) (see [1, 2]).

Before presenting our main theorem, we introduce the second-order elliptic operator \( L \) as follows: For \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \), denote its complex conjugate \( (\overline{\xi}_1, \ldots, \overline{\xi}_n) \) by \( \bar{\xi} \). Let \( A = A(x) \) be an \( n \times n \) matrix of complex \( L^\infty \) coefficients defined on \( \mathbb{R}^n \) that satisfy the ellipticity condition:

\[
\lambda |\xi|^2 \leq \text{Re} A\xi \cdot \overline{\xi},
\]

\[
|A\xi \cdot \overline{\xi}| \leq \Lambda |\xi||\xi|,
\]

for \( \xi, \zeta \in \mathbb{C}^n \) and for some \( \lambda, \Lambda \) such that \( 0 < \lambda \leq \Lambda < \infty \). Here, the inner product notation \( u \cdot v = u_1 v_1 + \cdots + u_n v_n \). Therefore, \( A\xi \cdot \overline{\xi} \equiv \sum_{j,k} a_{jk}(x) \xi_j \overline{\xi}_k \). Associated with such a matrix \( A \), we define a second-order divergence form operator:

\[
Lf = -\text{div}(AVf) = -\sum_{j=1}^n \partial_j \left( (AVf)_j \right),
\]
which we interpret in the usual weak sense via a sesquilinear form. The operator \(-L\) generates a semigroup \((e^{-tL})_{t \geq 0}\) and the gradient of the semigroup \((\nabla t^{1/2} e^{-tL})_{t > 0}\).

In this paper, we first present a general criterion for weak-type \((p, p)\) boundedness of commutators with square functions and \(b \in I_p(BMO)\).

**Theorem 1.** Let \(1 < p < 2, 0 < \alpha \leq 1, b \in I_p(BMO)\), and \(m\) be an integer greater than 1. Let \(E\) and \(F\) be two closed subsets of \(\mathbb{R}^n\) with a Euclidean distance \(d(E, F)\) between each other, and let \(\{T_t\}_{t>0}\) be a family of sublinear operators acting on \(L^2(\mathbb{R}^n)\). Assume that, for \(t > 0\) and \(2 < p_t < \infty\),

\[
\left\| \left( \int_0^\infty \left[ [b, T_t] (F - e^{-tL})^m \right] f \, ds \right)^{1/2} \right\|_{L^p(F)} \leq C p_t \left( \frac{d(E, F)^2}{t} \right)^{(m-n)/2} \|f\|_{L^p(F)},
\]

where \(\eta = (1/2)(n/p_t - n/p)\). Furthermore, if

\[
\left\| \left( \int_0^\infty |T_t f|^2 \, ds \right)^{1/2} \right\|_{L^2} \leq C \|D^\alpha f\|_{L^p(F)},
\]

\[
\left\| \left( \int_0^\infty |[b, T_t] f|^2 \, ds \right)^{1/2} \right\|_{L^2} \leq C \|f\|_{L^p(E)}.
\]

Then, we have

\[
\left\{ x \in \mathbb{R}^n : \left( \int_0^\infty |[b, T_t] f(x)|^2 \, ds \right)^{1/2} > \lambda \right\} \leq C \lambda^{-p} \|f\|_{L^p(F)}^p.
\]

We recall a square function, which is representative of larger classes of square functions associated with \(L\), given as follows [3]:

\[
g_L f(x) = \left( \int_0^\infty \|\nabla e^{-tL} f(x)\|^2 \, dt \right)^{1/2}.
\]

In this paper, we define a square function related to the fractional differential operator associated with \(L\) as follows:

\[
G^\alpha_L f(x) = \left( \int_0^\infty \left| t^{1/2} \nabla e^{-tL} f(x) \right|^2 \frac{dt}{t^{1+\alpha}} \right)^{1/2},
\]

\(0 < \alpha < 1\).

Moreover, for \(0 < \alpha < 1\) and \(b \in I_p(BMO)\), the commutator of \(G^\alpha_L\) can be defined by

\[
G^\alpha_L f(x) = \left( \int_0^\infty \left| [b, t^{1/2} \nabla e^{-tL}] f(x) \right|^2 \frac{dt}{t^{1+\alpha}} \right)^{1/2}.
\]

In this paper, we also establish the \(L^p\) boundedness for \(G^\alpha_L\).

**Theorem 2.** Let \(L\) be a second-order elliptic operator in divergence form defined by (5), \(0 < \alpha < 1\), and \(b \in I_p(BMO)\). Then, for \(p_t < p \leq 2\), we have

\[
\|G^\alpha_L f\|_{L^p} \leq C \|D^\alpha b\|_{BMO} \|f\|_{L^p}.
\]

The remainder of this paper is organised as follows: in Section 2, we present some lemmas that play an important role in the proof of the main results; in Section 3, we prove Theorem 1; in Section 4, we prove Theorem 2. For \(p \geq 1\), \(p'\) denotes the dual exponent of \(p\), i.e., \(1/p + 1/p' = 1\). Throughout this paper, the letter “C” will stand for a positive constant that is independent of the essential variables but will not necessarily have the same value for each occurrence.

### 2. Preliminary Lemmas

The second-order elliptic operator \(L\) in divergence form is defined by (5) and has the following off-diagonal estimates (see [3, 4, 9] and references therein).

**Lemma 3** (see [3, 4, 9]). Let \(L\) be a second-order elliptic operator defined by (5), let \(E\) and \(F\) be two closed sets of \(\mathbb{R}^n\), and let \(d(E, F)\) denote the distance between \(E\) and \(F\). Then, for \(t > 0\), the complex-valued function \(f\) and vector-valued functions \(\vec{f}\), the following statements hold:

(i) Let \(q_L < p \leq q < \tilde{q}_L\) and \(f \in L^q(\mathbb{R}^n)\) supported in \(E\),

\[
\left\| t^{1/2} \nabla e^{-tL} f \right\|_{L^q(F)} \leq C t^{-(n/2)(1/p - 1/q)} e^{-c(d(E,F)^2)t^2/8} \|f\|_{L^q(F)}.
\]

(ii) Let \(p_L < p \leq q < \tilde{p}_L\), \(f \in L^p(\mathbb{R}^n)\) supported in \(E\) and \(k \in \mathbb{N}_0\),

\[
\left\| t L^k e^{-tL} f \right\|_{L^p(F)} \leq C t^{-(n/2)(1/p - 1/q)} e^{-c(d(E,F)^2)t^2/8} \|f\|_{L^p(F)}.
\]

(iii) Let \(p_L < p \leq q < \tilde{p}_L\), \(\vec{f} \in L^q(\mathbb{R}^n)\) supported in \(E\),

\[
\left\| \nabla e^{-tL} \nabla \vec{f} \right\|_{L^q(F)} \leq C t^{-(n/2)(1/p - 1/q)} e^{-c(d(E,F)^2)t^2/8} \|\vec{f}\|_{L^q(F)}.
\]
In particular, if we choose $E = F = \mathbb{R}^n$, the abovementioned $L^p - L^q$ off-diagonal estimates become $L^p - L^q$ estimates.

Another very useful and well-known lemma for off-diagonal estimates is introduced here, which could be proved by using a similar argument for the proof of a previous lemma [9, lemma 2.3].

**Lemma 4** ([3, 9]). Let $1 < p \leq q < \infty$ and $E, F$ be arbitrary closed subsets of $\mathbb{R}^n$. Assume that the two families of the operators $\{A_j\}_{j \geq 0}$ and $\{B_j\}_{j \geq 0}$ satisfy the following off-diagonal estimates:

\[
\begin{align*}
\|A_1(g\chi_E)\|_{L^p(E)} &\leq Ce^{-d(E,F)\gamma/ct}\|g\|_{L^q(E)}, \\
\|B_1(g\chi_E)\|_{L^p(E)} &\leq Cs^{\pi/2/(1-q/p)}e^{-d(E,F)\gamma/ct}\|g\|_{L^q(E)}.
\end{align*}
\]

Then, for $t, s > 0$ and $f \in L^p(\mathbb{R}^n)$ supported in $E$, we have

\[
\begin{align*}
\|A_1B_sf\|_{L^p(E)} &\leq Cs^{\pi/2/(1-q/p)}e^{-d(E,F)\gamma/ct}\max[t,s]\|f\|_{L^p(E)}.
\end{align*}
\]

Next, let us introduce a criterion that deals with the boundedness of the commutators of the operators satisfying $L^p \rightarrow L^q$ off-diagonal estimates, which can be proved in [15].

**Lemma 5** (see [15]). Let $E$ and $F$ be two closed subsets of $\mathbb{R}^n$ with a Euclidean distance $d(E,F)$, and let $\{T_j\}_{j \geq 0}$ be a family of sublinear operators acting on $L^q(\mathbb{R}^n)$. Assume that, for $1 \leq p \leq q \leq \infty$ and $f \in C_0^\infty(\mathbb{R}^n)$ with supp $f \subset E$,

\[
\|T_jf\|_{L^q(E)} \leq Cs^{(\pi/2)(n/q-n/p)}e^{-d(E,F)\gamma/ct}\|f\|_{L^p(E)},
\]

If $\max\{\text{diam}E, \text{diam}F\} \leq Cd(E,F)$, then for $0 < \alpha < 1$ and $b \in \text{Lip}_c(\mathbb{R}^n)$, we have

\[
\begin{align*}
\|b, s^{-\alpha/2}T_jf\|_{L^p(E)} &\leq Cs^{(\pi/2)(n/q-n/p)}e^{-d(E,F)\gamma/ct}\|b\|_{Lip_e}\|f\|_{L^p(E)},
\end{align*}
\]

where $C$ is independent of $s, b$, and $f$.

The following two lemmas are about the $L^p - L^q$ off-diagonal estimates related to some commutators of the Lipschitz function and semigroups for second-order elliptic operators.

**Lemma 6** (see [16]). Let $E$ and $F$ be two closed subsets of $\mathbb{R}^n$ with a Euclidean distance $d(E,F)$. Assume that $0 < \alpha < 1$ and $b \in \text{Lip}_c(\mathbb{R}^n)$ and $0 \leq \gamma \leq 1$. If $\max\{\text{diam}E, \text{diam}F\} \leq Cd(E,F)$, then for $s > 0$ and $f \in L^p(\mathbb{R}^n)$ supported in $E$, we obtain the following for some $\epsilon > 0$:

\[
\begin{align*}
\|s^{-\alpha/2}[b, s^{\alpha/2}f]e^{-\epsilon L}\|_{L^p(E)} &\leq Cs^{(\pi/2)(n/q-n/p)}e^{-d(E,F)\gamma/ct}\|b\|_{Lip_e}\|f\|_{L^p(E)},
\end{align*}
\]

where $C$ is independent of $s, b$, and $f$.

**Lemma 7** (see [16]). Let $L$ be the second-order elliptic operator in divergence form defined by (5), $E$ and $F$ be two closed sets of $\mathbb{R}^n$, and $d(E, F)$ express the distance between $E$ and $F$. Assume that $0 < \alpha < 1, b \in \text{Lip}_c(\mathbb{R}^n)$, and $0 \leq \gamma \leq 1$. Then, for $s > 0$, $f \in L^p(\mathbb{R}^n)$ supported in $E$, we obtain

\[
\begin{align*}
\|\{b, s^{(1-\alpha)/2}f\}e^{-\epsilon L}\|_{L^p(E)} &\leq C\|b\|_{Lip_e}s^{(\pi/2)(1-1/p)}e^{-d(E,F)\gamma/ct}\|f\|_{L^p(E)},
\end{align*}
\]

where $C$ is independent of $s, b, f$. 

3. **Proof of Theorem 1**

For any fixed $\lambda > 0$, without loss of generality, we may assume that $f \in L^p(\mathbb{R}^n)$ is nonnegative. Let us write $M$ for the Hardy-Littlewood maximal function. We use the Calderón-Zygmund decomposition for $f(x)^p$ at height $\lambda^p$. Then, there exists a collection of pairwise disjoint cubes $\{Q_j\}_{j}$ such that

\[
\{x \in \mathbb{R}^n : M(f^p)(x)^{1/p} > \lambda\} = \bigcup_j Q_j,
\]

and they satisfy the following property:

\[
\lambda \leq \left(\frac{1}{|Q_j|}\int_{Q_j} f(x)^p \, dx\right)^{1/p} \leq C\lambda.
\]

Then, we write $f = g + h = g + \sum_j h_j$, where

\[
g(x) = f(x)\chi_{\mathbb{R}^n \setminus \bigcup_j Q_j},
\]

\[
+ \sum_j \left(\frac{1}{|Q_j|}\int_{Q_j} f(y) \, dy\right)\chi_{Q_j}(x),
\]

\[
h_j(x) = \left(f(x) - \frac{1}{|Q_j|}\int_{Q_j} f(y) \, dy\right)\chi_{Q_j}(x).
\]

After estimating (24), $p > 1$ and the standard arguments yield $0 \leq g(x) \leq C\lambda$ for almost every $x \in \mathbb{R}^n$. Then,

\[
\int_{Q_j} h_j(x) \, dx = 0,
\]

and

\[
\left(\frac{1}{|Q_j|}\int_{Q_j} |h_j(x)|^p \, dx\right)^{1/p} \leq C\lambda,
\]

hence

\[
\left\{ x \in \mathbb{R}^n : \left(\int_0^\infty \left|\int_0^t |b, T_s| f(x) ds\right|^{1/2} \frac{ds}{s}\right)^{1/2} > 3\lambda \right\}.
\]
Because $\varrho Q$ stands for the side length of the cube $Q$, we use the notation $Q_i = 2Q_j$, where, in general, we write $\varrho Q$ for the $\varrho$-dilated $Q_i$, i.e., for the cube with the same centre as $Q$ and with the side length $\varrho \ell(Q)$. Let $E^* = \mathbb{R}^n \setminus \bigcup_j Q_j^*$. Because

$$
\left( \int_0^\infty \left| \sum_j [b, T_{\ell_j}] h_j(x) \right|^p \frac{ds}{s} \right)^{1/p} \leq \left( \int_0^\infty \sum_j [b, T_{\ell_j}] h_j(x) \right)^{1/p} \frac{ds}{s}
$$

we obtain

$$
II \leq \left\{ x : \left( \int_0^\infty \sum_j [b, T_{\ell_j}] (\mathcal{F} - e^{-tL_i})^m h_j(x) \right)^{1/2} \frac{ds}{s} > \lambda \right\}
$$

We estimate every term separately. For $I$, we use (8) and the properties of $g$ to obtain

$$
I \leq \frac{1}{\lambda^2} \int_{\mathbb{R}^n} \int_0^\infty |[b, T_{s_t}] g(x)|^2 \frac{ds}{s} dx \leq \frac{C\lambda^{-2-p}}{\lambda^2} \int_{\mathbb{R}^n} |g(x)|^p dx \leq C\lambda^{-p} \|f\|_{L^p}^p.
$$

Now, we proceed with $II$. Let us fix an integer $m \geq 1$. We write $t_j = \ell(Q_j)^2$, where $\ell(Q_j)$ stands for the side length of the cube $Q_j$. We use the notation $Q_j^* = Q_j$, where, in general, we write $\varrho Q$ for the $\varrho$-dilated $Q_i$, i.e., for the cube with the same centre as $Q$ and with the side length $\varrho \ell(Q)$. Let $E^* = \mathbb{R}^n \setminus \bigcup_j Q_j^*$. Because

$$
\left( \int_0^\infty \left| \sum_j [b, T_{\ell_j}] h_j(x) \right|^2 \frac{ds}{s} \right)^{1/2} = \left( \int_0^\infty \sum_j [b, T_{\ell_j}] h_j(x) \right)^{1/2} \frac{ds}{s}
$$

we obtain

$$
II \leq \left\{ x : \left( \int_0^\infty \sum_j [b, T_{\ell_j}] (\mathcal{F} - e^{-tL_i})^m h_j(x) \right)^{1/2} \frac{ds}{s} > \lambda \right\}
$$

The first term can be estimated as follows:

$$
II_1 \leq \sum_j |Q_j^*| \leq \frac{C}{\lambda^p} \|f\|_{L^p}^p.
$$

Now, we complete the estimate of $II_2$. By Chebychev's inequality, we obtain

$$
II_2 \leq \frac{1}{\lambda^p} \left\| \left( \int_0^\infty \sum_j [b, T_{\ell_j}] (\mathcal{F} - e^{-tL_i})^m h_j(x) \right)^{1/2} \frac{ds}{s} \right\|_{L^p(E^*)}^{p/2}
$$

where the supremum is taken over all the functions $v \in L^{(p/2)}(E^*)$ with $\|v\|_{L^{(p/2)}(E^*)} = 1$. We set

$$
S_0(Q_j) = 2Q_j;
$$

$$
S_l(Q_j) = 2^{l+1}Q_j \setminus 2^lQ_j,
$$

$l = 1, 2, \cdots$.
Let us recall that \( E^* = \mathbb{R}^n \setminus \bigcup_j Q_j \). Because \( \text{supp } \nu \subset (2Q_j)^c \), we have

\[
\left| \int_{\mathbb{R}^n} \int_0^\infty \left[ \sum_j \left[ b_j T_s \left( \mathcal{F} e^{-t \cdot L} \right)^m \right] h_j (x) \right] \frac{ds}{s} v(x) \, dx \right|^{1/2} \leq \left| \int_{\mathbb{R}^n} \int_0^\infty \left[ \sum_j \left[ b_j T_s \left( \mathcal{F} e^{-t \cdot L} \right)^m \right] h_j (x) v(x) \right] \frac{ds}{s} \, dx \right|^{1/2}
\]

\[
= \sum_j \left( \sum_{l=1}^\infty \int_{ \mathbb{R}^n (Q_j) } \left[ b_j T_s \left( \mathcal{F} e^{-t \cdot L} \right)^m \right] h_j (x) v(x) \right)^{1/2} \frac{ds}{s} \, dx \right)^{1/2}
\]

\[
\leq C \sum_j \left( \sum_{l=1}^\infty \left\| \int_0^\infty \left[ b_j T_s \left( \mathcal{F} e^{-t \cdot L} \right)^m \right] h_j (x) \left[ \frac{ds}{s} \right] \right\|_{L^p (\mathbb{R}^n (Q_j) )}^2 \right)^{1/2}
\]

\[
\leq C \sum_j \left( \sum_{l=1}^\infty \left( \frac{d (S_j (Q_j) )}{t_j} \right) ^{-2(m-\eta)} \left\| h_j \right\|_{L^p (\mathbb{R}^n (Q_j) )}^2 \right)^{1/2}
\]

Where, in the last inequality, we used (6). Because \( t_j = \tilde{e}^l (Q_j) \), for \( l \geq 1 \), we obtain \( d (S_j (Q_j) ) \geq 2^{l-2} \tilde{e} (Q_j) \). Recall that \( \eta = (1/2)(n/p_1 - n/p) \). Subsequently, we obtain

\[
\left| \int_{\mathbb{R}^n} \int_0^\infty \left[ \sum_j \left[ b_j T_s \left( \mathcal{F} e^{-t \cdot L} \right)^m \right] h_j (x) \right] \frac{ds}{s} v(x) \, dx \right|^{1/2} \leq \left| \int_{\mathbb{R}^n} \int_0^\infty \left[ \sum_j \left[ b_j T_s \left( \mathcal{F} e^{-t \cdot L} \right)^m \right] h_j (x) v(x) \right] \frac{ds}{s} \, dx \right|^{1/2}
\]

\[
\leq C \sum_j \left( \sum_{l=1}^\infty \left( \frac{d (S_j (Q_j) )}{t_j} \right) ^{-2(m-\eta)} \left\| h_j \right\|_{L^p (\mathbb{R}^n (Q_j) )}^2 \right)^{1/2}
\]

Then, using \( |Q_j| \sim \lambda^{-p} \| f \|_p L (\mathbb{R}^n ) \), we obtain

\[
\left| \int_{\mathbb{R}^n} \int_0^\infty \left[ \sum_j \left[ b_j T_s \left( \mathcal{F} e^{-t \cdot L} \right)^m \right] h_j (x) \right] \frac{ds}{s} v(x) \, dx \right|^{1/2} \leq C \lambda^{1+12/p} \| f \|_p \cdot \sum_j \left| Q_j \right| \text{ess inf } M \left( \left| v \right|^{(p,2)'} \right) (y)^{1/2(p,2)'} \leq C \lambda^{1+12/p} \| f \|_p \cdot \sum_j \left| Q_j \right| \text{ess inf } M \left( \left| v \right|^{(p,2)'} \right) (y)^{1/2(p,2)'}
\]

Then, because the Hardy-Littlewood maximal function is of weak-type \((1,1)\), we use \( \| \nu \|_{L^1} = 1 \) and Kolmogorov’s lemma to obtain

\[
\sum_j \int_{Q_j} M \left( \left| v \right|^{(p,2)'} \right) (x)^{1/2(p,2)'} \, dx \leq C \lambda^{1+12/p} \| f \|_p \cdot \sum_j \int_{Q_j} \text{ess inf } M \left( \left| v \right|^{(p,2)'} \right) (y)^{1/2(p,2)'} \, dx.
\]
Applying Theorem 1 into (31) to obtain

$$\int_{\mathbb{R}^n} \left| \sum_{j} \left[ b, T_j \left( \mathcal{F} - e^{-\ell t} \right)^m \right] h_j(x) \right|^2 \frac{ds}{s}^{1/2} \, dx$$

$$\leq C \lambda^{1+(1/2)p} \left\| \int_{\mathbb{R}^{3/2}} \mathcal{Q}_j \right\|_{L^p}^{1/2} \left\| \mathcal{Q}_j \right\|_{L^{1/2+p/2}}^{1/2+(p/2)}$$

$$\leq C \lambda^{1+(1/2)p} \left\| f \right\|_{L^p}^{1/2+(p/2)}.$$ (36)

Then, we plug the estimate into (31) to obtain

$$II_2 \leq C \lambda^{p/12} \left\| f \right\|_{L^p}^{p/12} \left\| \sum_{j} \mathcal{Q}_j \right\|_{L^2}^{1/2+(p/2)}$$

$$\leq C \lambda^{p/12} \left\| f \right\|_{L^p}^{p/12} \left\| \sum_{j} \mathcal{Q}_j \right\|_{L^2}^{1/2+(p/2)} M(\lambda)^{p/2+1}.$$ (37)

Applying $M$ of the weak-type $(1, 1)$, we obtain

$$II_2 \leq C \lambda^{p/12} \left\| f \right\|_{L^p}^{p/12} \left\| \mathcal{F} - e^{-\ell t} \right\|^m \left( \frac{ds}{s} \right) \left\| h_j(x) \right\|^{1/2}$$

$$\leq C \lambda^{p/12} \left\| f \right\|_{L^p}^{p/12} \mathcal{F} - e^{-\ell t} \right\|^m \left( \frac{ds}{s} \right) \left\| h_j(x) \right\|^{1/2}.$$ (38)

We now examine $II_3$. Recall that

$$II_3 = \left\{ x : \left( \int_{0}^{\infty} \left| \sum_{j} \left[ b, T_j \left( \mathcal{F} - e^{-\ell t} \right)^m \right] h_j(x) \right|^2 \frac{ds}{s} \right)^{1/2} \right\}.$$ (39)

Then, $\mathcal{F} - (\mathcal{F} - e^{-\ell t})^m = \mathcal{F} - \sum_{k} c_k e^{-\ell t} \mathcal{F}$

$$= -\sum_{k=1}^{m} c_k e^{-\ell t} \mathcal{F}.$$ (40)

Thus, from Chebychev’s inequality,

$$II_3 \leq C \lambda^2 \sum_{k=1}^{m} \left\| \int_{0}^{\infty} \left| \sum_{j} \left[ b, T_j \left( \mathcal{F} - e^{-\ell t} \right)^m \right] h_j(x) \right|^2 \frac{ds}{s} \right\|_{L^2}^{1/2}$$

$$\leq C \lambda^2 \sum_{k=1}^{m} \left\| \int_{0}^{\infty} \left| \sum_{j} \left[ b, T_j \left( \mathcal{F} - e^{-\ell t} \right)^m \right] h_j(x) \right|^2 \frac{ds}{s} \right\|_{L^2}^{1/2}$$

$$+ C \lambda^2 \sum_{k=1}^{m} \left\| \int_{0}^{\infty} \left| \sum_{j} \left[ b, T_j \left( \mathcal{F} - e^{-\ell t} \right)^m \right] h_j(x) \right|^2 \frac{ds}{s} \right\|_{L^2}^{1/2}.$$ (41)

We fix $1 \leq k \leq m$. Then, for $II_3, \lambda$, by (8), we obtain

$$II_{3,1} \leq C \lambda^2 \sum_{k=1}^{m} \left\| \sum_{j} e^{-\ell t} h_j \right\|_{L^2}^{2}.$$ (42)

Hence, by (9)

$$\left\| \sum_{j} e^{-\ell t} h_j \right\|_{L^2}^{2} \leq C \lambda^2 \sum_{k=1}^{m} \left\| \sum_{j} e^{-\ell t} h_j \right\|_{L^2}^{2}.$$ (43)

we obtain

$$II_{3,1} \leq C \lambda^2 \sum_{k=1}^{m} \left\| \sum_{j} e^{-\ell t} h_j \right\|_{L^2}^{2}.$$ (44)

For $II_{3,2}$, by (7) and $\|L^{1/2} f\|_{L^2} \sim \|D^a f\|_{L^2}$ (see [3]), we obtain

$$II_{3,2} \leq C \lambda^2 \sum_{k=1}^{m} \left\| \sum_{j} e^{-\ell t} h_j \right\|_{L^2}^{2}$$

$$\leq C \lambda^2 \sum_{k=1}^{m} \left\| \sum_{j} e^{-\ell t} h_j \right\|_{L^2}^{2}.$$ (45)

Then,

$$II_{3,2} \leq C \lambda^2 \left( \sum_{k=1}^{m} \left\| \sum_{j} e^{-\ell t} h_j \right\|_{L^2}^{2} \right)$$

$$+ \sum_{k=1}^{m} \left\| \sum_{j} e^{-\ell t} h_j \right\|_{L^2}^{2}.$$ (46)
where we use the fact that $[b, L^{\alpha/2}]$ is bounded on $L^2$ with the bound $\|(-\Delta)^{\alpha/2} b\|_{BMO}$ (see [16]). Next, we estimate the abovementioned two norms, $\|\sum e^{-ktJ^L}h_j\|_{L^2}$ and $\|\sum [b, L^{\alpha/2}e^{-ktJ^L}]h_j\|_{L^2}$. Now, taking $v \in L^2(\mathbb{R}^n)$ with $\|v\|_{L^2} = 1$,

$$\left|\int_{\mathbb{R}^n} \sum e^{-ktJ^L}h_j(x) \overline{v(x)} dx \right| = \left| \sum_{j=0}^{\infty} \int_{S_j(Q_j)} e^{-ktJ^L}h_j(x) \overline{v(x)} dx \right| \leq \sum_{j=0}^{\infty} \left( \int_{S_j(Q_j)} |e^{-ktJ^L}h_j(x)|^2 \overline{v(x)}^2 dx \right)^{1/2} 
\cdot \left( \int_{S_j(Q_j)} |v(x)|^2 dx \right)^{1/2} \left( \int_{S_j(Q_j)} |v(x)|^2 dx \right)^{1/2} \leq C \sum_{j=0}^{\infty} \left| Q_j \right| e^{-k_0\ell(Q_j)^{1/2}} \left( \int_{S_j(Q_j)} |v(x)|^2 dx \right)^{1/2} \cdot \left( \int_{S_j(Q_j)} |v(x)|^2 dx \right)^{1/2} \leq C \sum_{j=0}^{\infty} \left| Q_j \right| \operatorname{ess inf}_{y \in Q_j} M \left( |\cdot|^2 \right)^{1/2}(y) \leq C \lambda \int_{S_j(Q_j)} M \left( |\cdot|^2 \right)^{1/2}(x) dx. \tag{50}$$

Note that, for all $p_k < r \leq 2$, $e^{-ktJ^L}$ satisfies the $L^r - L^2$ off-diagonal estimates (see Lemma 3(ii); let $t_j = \ell(Q_j)^2$, and we obtain (47), which is controlled by

$$\leq C \lambda \left| x \in \mathbb{R}^n : M(\mathcal{F}^p)(x)^{1/p} > \lambda \right|^{1/2}. \tag{51}$$

Then, we have

$$\left\| \sum_j [b, L^{\alpha/2}e^{-ktJ^L}]h_j \right\|_{L^2} \leq C \lambda \left( \frac{f \| f \|_{L_p^p}}{\lambda} \right)^{p/2}. \tag{52}$$

Combining (49) and (51), we obtain

$$II_{3,2} \leq C \lambda^{-p} \| f \|_{L_p^p}^p. \tag{53}$$

Combining the estimates of $II_{3,1}$ and $II_{3,2}$, we obtain

$$II_3 \leq C \frac{\| f \|_{L_p^p}^p}{\lambda^p}. \tag{54}$$

The proof of Theorem 1 is now completed by combining the estimates of $I, II_1$, and $II_2$ with $II_3$.

4. Proof of Theorem 2

First, we introduce a lemma that will be used to prove Theorem 2.

Lemma 8 (see [16]). For $0 < \alpha < 1$, let $b \in \mathcal{L}_\alpha(BMO)$. Suppose that $\{\theta_j\}_{j=1}^{\infty}$ is a family of operators satisfying

$$\| \theta_j \mathcal{F}_{a_j}(Q_j) \|_{L^2(Q_j)} \leq C \frac{\sqrt{\xi}}{2^{m+2}} \| f \|_{L_p^p(S_j(Q_j))}^{2m+2}, \tag{55}$$

Thus, from (47), (48), and the fact that $M$ is of weak-type $(1, 1), we obtain
for an $m \geq 1$ and all $0 < s \leq C \varepsilon (Q)$. Assume that
\begin{equation}
\sup_{s > 0} \| \theta_i \|_{L^2 \rightarrow L^2} \leq C, \quad \theta_1 = 0
\end{equation}
in the sense of $L^2_{\text{loc}}$ for all $s > 0$. Then, we have
\begin{equation}
\int_{\mathbb{R}^n} \left( \frac{1}{s} \right)^{1/2} \theta_2 f(x) \, dx \leq C \| D^\alpha f \|_{L^2}.
\end{equation}
If, in addition,
\begin{equation}
\| \theta_s \, \text{div} \|_{L^2 \rightarrow L^2} \leq \frac{C}{\sqrt{s}},
\end{equation}
then
\begin{equation}
\int_{\mathbb{R}^n} \left( \frac{1}{s} \right)^{1/2} \theta_2 f(x) \, dx \leq C \| D^\alpha b \|_{LMO} \| f \|_{L^2}.
\end{equation}
Recall that
\begin{equation}
G_{L,b}^a f(x) = \left( \int_0^\infty \left[ b, s^{(1-a)/2} \nabla \epsilon^{-(s+kt)L} \right] f(x) \, ds \right)^{1/2}.
\end{equation}
The proof of Theorem 2 is presented in two steps.

**Step 1 (the $L^2$ boundedness).** Let $\theta_i = s^{(1-a)/2} \nabla \epsilon^{-(s+kt)L}$. By Lemma 8, we only need to prove that $s^{1/2} \nabla \epsilon^{-(s+kt)L}$ satisfies (54)-(56) and (58). Because $s^{1/2} \nabla \epsilon^{-(s+kt)L}$ satisfies (54), (56), and (58). Moreover, it is easy to verify that $s^{1/2} \nabla \epsilon^{-(s+kt)L}$ satisfies (55) by the $L^2$ off-diagonal estimate of the operator $s^{1/2} \nabla \epsilon^{-(s+kt)L}$ (see Lemma 3(i, iii)). We obtain
\begin{equation}
\| G_{L,b}^a f(x) \|_{L^2} \leq C \| D^\alpha f \|_{L^2},
\end{equation}
\begin{equation}
\| G_{L,b}^a f(x) \|_{L^2} \leq C \| D^\alpha b \|_{LMO} \| f \|_{L^2}.
\end{equation}

**Step 2 (weak-type $(p, p)$ boundedness for $p_L < p < 2$).** We first prove that the commutator is of the weak-type $(p, p)$ for $p_L < p < 2$. We apply Theorem 1 with $T_s = s^{(1-a)/2} \nabla \epsilon^{-(s+kt)L}$ to prove this result. Because $G_{L,b}^a$ and $G_{L,b}^a$ are bounded on $L^2(\mathbb{R}^n)$, verification of $s^{1/2} \nabla \epsilon^{-(s+kt)L}_{L^2}$ satisfies (6). Taking $p < 2 < p_1$, by the Minkowski inequality, we obtain
\begin{equation}
\left\| \left( \int_0^\infty \left[ b, s^{1/2} \nabla \epsilon^{-(s+kt)L} \right] f \, ds \right)^{1/2} \|_{L^p(\mathbb{R}^n)} \right\|_{L^p(\mathbb{R}^n)} \leq C (I + II)^{1/2}.
\end{equation}
Now, we study each operator separately. For the first operator, we have
\begin{equation}
\left( \mathcal{I} - e^{-(s+kt)L} \right)^m = \sum_{k=0}^m C_k (-1)^k e^{-(s+kt)L} = \mathcal{I} + \sum_{k=1}^m \xi_k e^{-(s+kt)L},
\end{equation}
and, then,
\begin{equation}
I_1 \leq \int_0^1 \left\| \left[ b, s^{1/2} \nabla \epsilon^{-(s+kt)L} \right] f \right\|_{L^p(\mathbb{R}^n)} \, ds \right\|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{Lp_{E}} \| f \|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{Lp_{E}} \| f \|_{L^p(\mathbb{R}^n)}.
\end{equation}
For $I_{1,0}$, by Lemma 7(i), we have
\begin{equation}
I_{1,0} \leq \int_0^1 \left\| \left[ b, s^{1/2} \nabla \epsilon^{-(s+kt)L} \right] f \right\|_{L^p(\mathbb{R}^n)} \, ds \right\|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{Lp_{E}} \| f \|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{Lp_{E}} \| f \|_{L^p(\mathbb{R}^n)}.
\end{equation}
where $\eta = (1/2)(n/p_1 - n/p)$. Now, with $1 \leq k \leq m$, by the commutative property of the semigroup and Lemma 7(i), we obtain
\begin{equation}
I_{1,k} \leq \int_0^1 \left\| \left[ b, s^{1/2} \nabla \epsilon^{-(s+kt)L} \right] f \right\|_{L^p(\mathbb{R}^n)} \, ds \right\|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{Lp_{E}} \| f \|_{L^p(\mathbb{R}^n)} \leq C \| b \|_{Lp_{E}} \| f \|_{L^p(\mathbb{R}^n)}.
\end{equation}
Collecting this estimate and the one proved for $I_{t,0}$, we obtain
\begin{equation}
I_t \leq I_{t,0} + \sum_{k=1}^{m} |c_k| I_{t,k} \leq C \|b\|^2_{L^p(E)} t^\eta \left( \frac{d(E,F)^2}{t} \right)^{-(m-\eta)} \|f\|^2_{L^p(E)}.
\end{equation}

Next, we proceed with the estimate of $II_t$:
\begin{equation}
II_t = \int_t^\infty \left\| b, s^{1/2} \nabla e^{-tL} \left( e^{-tL} \right) \right\|^2_{L^p(E)} ds
= C \int_t^\infty \left\| b, s^{1/2} \nabla e^{-(m+1)sL} \left( e^{-tL} \right) \right\|^2_{L^p(E)} ds
\cdot f_{\|f\|_{L^p(E)}}^2 \frac{ds}{s^{1/\alpha}}.
\end{equation}

Let $E, F$ be two closed sets and $g$ be such that $\text{supp } g \subset E$. For $t \leq s$, by a similar previous argument [9], we can prove that
\begin{equation}
\left\| s \left( e^{-tL} - e^{-(s+t)L} \right) g \right\|_{L^p(E)} \leq C s^{(1/2)(n/p_1 - n/p)} e^{-d(E,F)^2/cs} \|g\|_{L^p(E)},
\end{equation}
uniformly on $t$. Then, by Lemma 4, we have
\begin{equation}
\left\| \left( e^{-tL} - e^{-(s+t)L} \right) \right\|_{L^p(E)} \leq C \left( \frac{t}{s} \right)^m s^{(1/2)(n/p_1 - n/p)} e^{-d(E,F)^2/cs} \|g\|_{L^p(E)}.
\end{equation}

We write $II_t$ as follows:
\begin{equation}
II_t \leq C \int_t^\infty \left\| b, s^{1/2} \nabla e^{-tL} \right\|^m \left( e^{-tL} - e^{-(s+t)L} \right) \right\|_{L^p(E)}^2 \frac{ds}{s^{1/\alpha}} \leq C \left( \frac{t}{s} \right)^m \left( e^{-d(E,F)^2/cs} \right) \|g\|_{L^p(E)}.
\end{equation}

First, we consider $G_1$:
\begin{equation}
G_1 = \int_t^\infty \left( b, s^{(1/\alpha)/2} \nabla e^{-tL} \right) \left( e^{-tL} - e^{-(s+t)L} \right) \right\|_{L^p(E)}^2 \frac{ds}{s}.
\end{equation}

Let us observe that, because of (71) and Lemma 7(i) in the composition of the operators, each of them verifies an off-diagonal estimate. This fact allows us to employ Lemma 4 to obtain
\begin{equation}
G_1 \leq C \|b\|^2_{L^p(E)} \left( \frac{d(E,F)^2}{t} \right)^{2\eta} \left( \frac{t}{s} \right)^{2m} \|g\|_{L^p(E)}.
\end{equation}

Finally, for $G_2$,
\begin{equation}
G_2 = \int_t^\infty \left( s^{1/\alpha/2} \nabla e^{-tL} \right) \left( e^{-tL} - e^{-(s+t)L} \right) \right\|_{L^p(E)}^2 \frac{ds}{s}.
\end{equation}

We know that the composition of the operators above (75) or Lemma 3 verifies an off-diagonal estimate. It follows from Lemma 4 that
\begin{equation}
G_2 \leq C \|b\|^2_{L^p(E)} \left( \frac{d(E,F)^2}{t} \right)^{2\eta} \left( \frac{t}{s} \right)^{2m} \|g\|_{L^p(E)}.
\end{equation}

First, we consider $G_1$:
\begin{equation}
I_t \leq I_{t,0} + \sum_{k=1}^{m} |c_k| I_{t,k} \leq C \|b\|^2_{L^p(E)} t^\eta \left( \frac{d(E,F)^2}{t} \right)^{-(m-\eta)} \|f\|^2_{L^p(E)}.
\end{equation}

which combined with the estimate of $I_t$ verify that $\left( s^{(1/\alpha)/2} \nabla e^{-tL} \right)_{t>0}$ satisfies (6). Thus, we have proven Theorem 2.
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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