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Research Article

Slant and Semi-Slant Submanifolds in Metallic Riemannian Manifolds

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The aim of our paper is to focus on some properties of slant and semi-slant submanifolds of metallic Riemannian manifolds. We give some characterizations for submanifolds to be slant or semi-slant submanifolds in metallic or Golden Riemannian manifolds and we obtain integrability conditions for the distributions involved in the semi-slant submanifolds of Riemannian manifolds endowed with metallic or Golden Riemannian structures. Examples of semi-slant submanifolds of the metallic and Golden Riemannian manifolds are given.

1. Introduction

Since B.Y. Chen defined slant submanifolds in complex manifolds ([1, 2]) in the early 1990s, the differential geometry of slant submanifolds has shown an increasing development. Then, many authors have studied slant submanifolds in different kind of manifolds, such as slant submanifolds in almost contact metric manifolds (A. Lotta ([3])), in Sasakian manifolds (J.L. Cabrerizo *et al.* ([4, 5])), in para-Hermitian manifold (P. Alegre, A. Carriazo ([6])), and in almost product Riemannian manifolds (B. Sahin ([7]), M. Atçeken ([8, 9])).

The notion of slant submanifold was generalized by semi-slant submanifold, pseudo-slant submanifold, and bi-slant submanifold, respectively, in different types of differentiable manifolds. The semi-slant submanifold of almost Hermitian manifold was introduced by N. Papagiuc ([10]). A. Cariazzo *et al.* ([11]) defined and studied bi-slant immersion in almost Hermitian manifolds and pseudo-slant submanifold in almost Hermitian manifolds. The pseudo-slant submanifolds in Kenmotsu or nearly Kenmotsu manifolds ([12, 13]), in LCS-manifolds ([14]), or in locally decomposable Riemannian manifolds ([15]) were studied by M. Atçeken *et al.* Moreover, many examples of semi-slant, pseudo-slant, and bi-slant submanifolds were built by most of the authors.

Semi-slant submanifolds are particular cases of bi-slant submanifolds, defined and studied by A. Cariazzo ([11]). The geometry of slant and semi-slant submanifolds in metallic Riemannian manifolds is related by the properties of slant and semi-slant submanifolds in almost product Riemannian manifolds, studied in ([7, 8, 16]).

The notion of Golden structure on a Riemannian manifold was introduced for the first time by C.E. Hretcanu and M. Crasmareanu in ([17]). Moreover, the authors investigated the properties of a Golden structure related to the almost product structure and of submanifolds in Golden Riemannian manifolds ([18, 19]). Examples of Golden and product-shaped hypersurfaces in real space forms were given in ([20]). The Golden structure was generalized as metallic structures, defined on Riemannian manifolds in ([21]). A.M. Blaga studied the properties of the conjugate connections by a Golden structure and expressed their virtual and structural tensor fields and their behavior on invariant distributions. Also, she studied the impact of the duality between the Golden and almost product structures on Golden and product conjugate connections ([22]). The properties of the metallic conjugate connections were studied by A.M. Blaga and C.E. Hretcanu in ([23]) where the virtual and structural tensor fields were expressed and their behavior on invariant distributions was analyzed.

Recently, the connection adapted on the almost Golden Riemannian structure was studied by F. Etayo *et al.* in ([24]). Some properties regarding the integrability of the Golden Riemannian structures were investigated by A. Gezer *et al.* in ([25]).

The metallic structure J is a polynomial structure, which was generally defined by S.I. Goldberg *et al.* in ([26, 27]), inspired by the metallic number given by $\sigma_{p,q} = (p + \sqrt{p^2 + 4q})/2$, which is the positive solution of the equation $x^2 - px - q = 0$, for positive integer values of p and q. These $\sigma_{p,q}$ numbers are members of the *metallic means family* or *metallic proportions* (as generalizations of the Golden number $\phi = (1 + \sqrt{5})/2 = 1.618\ldots$), introduced by Vera W. de Spinadel ([28]). Some examples of the members of the metallic means family are *the Silver mean, the Bronze mean, the Copper mean, the Nickel mean,* and many others.

The purpose of the present paper is to investigate the properties of slant and semi-slant submanifolds in metallic (or Golden) Riemannian manifolds. We have found a relation between the slant angles θ of a submanifold M in a Riemannian manifold $(\overline{M}, \overline{g})$ endowed with a metallic (or Golden) structure J and the slant angle ϑ of the same submanifold M of the almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$. Moreover, we have found some integrability conditions for the distributions which are involved in such types of submanifolds in metallic and Golden Riemannian manifolds. We have also given some examples of semi-slant submanifolds in metallic and Golden Riemannian manifolds.

2. Preliminaries

First of all we review some basic formulas and definitions for the metallic and Golden structures defined on a Riemannian manifold.

Let $\overline{M}$ be an m-dimensional manifold endowed with a tensor field J of type $(1,1)$. We say that the structure J is a *metallic structure* if it verifies

$$J^2 = pJ + qI, \tag{1}$$

for $p, q \in \mathbb{N}^*$, where I is the identity operator on the Lie algebra $\Gamma(T\overline{M})$ of vector fields on $\overline{M}$. In this situation, the pair $(\overline{M}, J)$ is called *metallic manifold*.

If $p = q = 1$ one obtains the *Golden structure* ([17]) determined by a $(1,1)$-tensor field J which verifies $J^2 = J + I$. In this case, $(\overline{M}, J)$ is called *Golden manifold*.

Moreover, if $(\overline{M}, \overline{g})$ is a Riemannian manifold endowed with a metallic (or a Golden) structure J, such that the Riemannian metric $\overline{g}$ is J-compatible, i.e.,

$$\overline{g}(JX, Y) = \overline{g}(X, JY), \tag{2}$$

for any $X, Y \in \Gamma(T\overline{M})$, then $(\overline{g}, J)$ is called a *metallic* (or a *Golden*) *Riemannian structure* and $(\overline{M}, \overline{g}, J)$ is a *metallic* (or a *Golden*) *Riemannian manifold*.

We can remark that

$$\overline{g}(JX, JY) = \overline{g}\left(J^2X, Y\right) = p\overline{g}(JX, Y) + q\overline{g}(X, Y), \tag{3}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Any metallic structure J on $\overline{M}$ induces two almost product structures on this manifold ([21]):

$$\begin{aligned} F_1 &= \frac{2}{2\sigma_{p,q} - p} J - \frac{p}{2\sigma_{p,q} - p} I, \\ F_2 &= -\frac{2}{2\sigma_{p,q} - p} J + \frac{p}{2\sigma_{p,q} - p} I. \end{aligned} \tag{4}$$

Conversely, any almost product structure F on $\overline{M}$ induces two metallic structures on $\overline{M}$ ([21]):

$$\begin{aligned} (i)\ J_1 &= \frac{2\sigma_{p,q} - p}{2} F + \frac{p}{2} I, \\ (ii)\ J_2 &= -\frac{2\sigma_{p,q} - p}{2} F + \frac{p}{2} I. \end{aligned} \tag{5}$$

If the almost product structure F is a Riemannian one, then J_1 and J_2 are also metallic Riemannian structures. Also, on a metallic manifold $(\overline{M}, J)$ there are two complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ corresponding to the projection operators P and Q ([21]), given by

$$\begin{aligned} P &= -\frac{1}{2\sigma_{p,q} - p} J + \frac{\sigma_{p,q}}{2\sigma_{p,q} - p} I, \\ Q &= \frac{1}{2\sigma_{p,q} - p} J + \frac{\sigma_{p,q} - p}{2\sigma_{p,q} - p} I \end{aligned} \tag{6}$$

and the operators P and Q verify the following relations:

$$\begin{aligned} P + Q &= I, \\ P^2 &= P, \\ Q^2 &= Q, \\ PQ &= QP = 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} JP &= PJ = \left(p - \sigma_{p,q}\right) P, \\ JQ &= QJ = \sigma_{p,q} Q. \end{aligned} \tag{8}$$

In particular, if $p = q = 1$, we obtain that every Golden structure J on $\overline{M}$ induces two almost product structures on this manifold and conversely, an almost product structure F on $\overline{M}$ induces two Golden structures on $\overline{M}$ ([17, 19]).

3. Submanifolds of Metallic Riemannian Manifolds

In the next issues we assume that M is an m'-dimensional submanifold, isometrically immersed in the m-dimensional metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ with $m, m' \in \mathbb{N}^*$ and $m > m'$. We denote by T_xM the tangent space of M in a point $x \in M$ and by $T_x^{\perp}M$ the normal space of M in x. The tangent space $T_x\overline{M}$ of $\overline{M}$ can be decomposed into the direct sum: $T_x\overline{M} = T_xM \oplus T_x^{\perp}M$, for any $x \in M$. Let i_* be the differential of the immersion $i : M \longrightarrow \overline{M}$. The induced Riemannian metric g on M is given by $g(X,Y) = \overline{g}(i_*X, i_*Y)$, for any $X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the set of all vector fields of M. For the simplification of the notations, in the rest of the paper we shall note by X the vector field i_*X, for any $X \in \Gamma(TM)$.

We consider the decomposition into the tangential and normal parts of JX and JV, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, are given by

$$\begin{aligned} &(i)\, JX = TX + NX,\\ &(ii)\, JV = tV + nV, \end{aligned} \tag{9}$$

where $T : \Gamma(TM) \longrightarrow \Gamma(TM), N : \Gamma(TM) \longrightarrow \Gamma(T^{\perp}M), t : \Gamma(T^{\perp}M) \longrightarrow \Gamma(TM)$ and $n : \Gamma(T^{\perp}M) \longrightarrow \Gamma(T^{\perp}M)$, with

$$\begin{aligned} TX &:= (JX)^T,\\ NX &:= (JX)^{\perp},\\ tV &:= (JV)^T,\\ nV &:= (JV)^{\perp}. \end{aligned} \tag{10}$$

We remark that the maps T and n are $\overline{g}$-symmetric ([29]):

$$\begin{aligned} &(i)\, \overline{g}(TX,Y) = \overline{g}(X,TY),\\ &(ii)\, \overline{g}(nU,V) = \overline{g}(U,nV) \end{aligned} \tag{11}$$

and

$$\overline{g}(NX,U) = \overline{g}(X,tU), \tag{12}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.

For an almost product structure F, the decompositions into tangential and normal parts of FX and FV, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, are given by ([7])

$$\begin{aligned} &(i)\, FX = fX + \omega X,\\ &(ii)\, FV = BV + CV, \end{aligned} \tag{13}$$

where $f : \Gamma(TM) \longrightarrow \Gamma(TM)$, $\omega : \Gamma(TM) \longrightarrow \Gamma(T^{\perp}M)$, $B : \Gamma(T^{\perp}M) \longrightarrow \Gamma(TM), C : \Gamma(T^{\perp}M) \longrightarrow \Gamma(T^{\perp}M)$, with

$$\begin{aligned} fX &:= (FX)^T,\\ \omega X &:= (FX)^{\perp},\\ BV &:= (FV)^T,\\ CV &:= (FV)^{\perp}. \end{aligned} \tag{14}$$

The maps f and C are $\overline{g}$-symmetric ([16]):

$$\begin{aligned} \overline{g}(fX,Y) &= \overline{g}(X,fY),\\ \overline{g}(CU,V) &= \overline{g}(U,CV) \end{aligned} \tag{15}$$

$$\overline{g}(\omega X,V) = \overline{g}(X,BV), \tag{16}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.

Remark 1. Let $(\overline{M}, \overline{g})$ be a Riemannian manifold endowed with an almost product structure F and let J be the metallic structure induced by F on $\overline{M}$. If M is a submanifold in the almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$, then

$$\begin{aligned} &(i)\, TX = \frac{p}{2}X \pm \frac{2\sigma_{p,q} - p}{2} fX,\\ &(ii)\, NX = \pm\frac{2\sigma_{p,q} - p}{2}\omega X \end{aligned} \tag{17}$$

$$\begin{aligned} &(i)\, tV = \pm\frac{2\sigma_{p,q} - p}{2}BV,\\ &(ii)\, nV = \frac{p}{2}V \pm \frac{2\sigma_{p,q} - p}{2}CV, \end{aligned} \tag{18}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Remark 2. Let $(\overline{M}, \overline{g})$ be a Riemannian manifold endowed with an almost product structure F and let J be the Golden structure induced by F on $\overline{M}$. If M is a submanifold in the almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$, then

$$\begin{aligned} &(i)\, TX = \frac{1}{2}X \pm \frac{2\phi - 1}{2} fX,\\ &(ii)\, NX = \pm\frac{2\phi - 1}{2}\omega X \end{aligned} \tag{19}$$

$$\begin{aligned} &(i)\, tV = \pm\frac{2\phi - 1}{2}BV,\\ &(ii)\, nV = \frac{1}{2}V \pm \frac{2\phi - 1}{2}CV, \end{aligned} \tag{20}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Let $r = m - m'$ be the codimension of M in $\overline{M}$ (where $r, m, m' \in \mathbb{N}^*$). We fix a local orthonormal basis $\{N_1, \ldots, N_r\}$ of the normal space $T_x^{\perp}M$. Hereafter we assume that the indices α, β, γ run over the range $\{1, \ldots, r\}$.

For any $x \in M$ and $X \in T_xM$, the vector fields $J(i_*X)$ and JN_α can be decomposed into tangential and normal components ([21]):

$$\begin{aligned} &(i)\, JX = TX + \sum_{\alpha=1}^{r} u_\alpha(X) N_\alpha,\\ &(ii)\, JN_\alpha = \xi_\alpha + \sum_{\beta=1}^{r} a_{\alpha\beta}N_\beta, \end{aligned} \tag{21}$$

where $(\alpha \in \{1, \ldots, r\})$, T is an $(1,1)$-tensor field on M, ξ_α are vector fields on M, u_α are 1-forms on M, and $(a_{\alpha\beta})_r$ is an $r \times r$ matrix of smooth real functions on M.

Using (9) and (21), we remark that

$$\begin{aligned} &(i)\, NX = \sum_{\alpha=1}^{r} u_\alpha(X) N_\alpha,\\ &(ii)\, tN_\alpha = \xi_\alpha,\\ &(iii)\, nN_\alpha = \sum_{\beta=1}^{r} a_{\alpha\beta}N_\beta. \end{aligned} \tag{22}$$

Theorem 3. *The structure* $\Sigma = (T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ *induced on the submanifold* M *by the metallic Riemannian structure* $(\overline{g}, J)$ *on* $\overline{M}$ *satisfies the following equalities ([30]):*

$$T^2 X = pTX + qX - \sum_{\alpha=1}^{r} u_\alpha (X) \xi_\alpha, \tag{23}$$

$$\begin{aligned}(i)\ & u_\alpha (TX) = pu_\alpha (X) - \sum_{\beta=1}^{r} a_{\alpha\beta} u_\beta (X), \\ (ii)\ & a_{\alpha\beta} = a_{\beta\alpha},\end{aligned} \tag{24}$$

$$\begin{aligned}(i)\ & u_\beta (\xi_\alpha) = q\delta_{\alpha\beta} + pa_{\alpha\beta} - \sum_{\gamma=1}^{r} a_{\alpha\gamma} a_{\gamma\beta}, \\ (ii)\ & T\xi_\alpha = p\xi_\alpha - \sum_{\beta=1}^{r} a_{\alpha\beta}\xi_\beta,\end{aligned} \tag{25}$$

$$u_\alpha (X) = g (X, \xi_\alpha) \tag{26}$$

for any $X \in \Gamma(TM)$*, where* $\delta_{\alpha\beta}$ *is the Kronecker delta and* p, q *are positive integers ([21]).*

A structure $\Sigma = (T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on the submanifold M by the metallic Riemannian structure $(\overline{g}, J)$ defined on $\overline{M}$ (determined by the $(1, 1)$-tensor field T on M, the vector fields ξ_α on M, the 1-forms u_α on M, and the $r \times r$ matrix $(a_{\alpha\beta})_r$ of smooth real functions on M) which verifies the relations (23), (24), (25), and (26) is called Σ- *metallic Riemannian structure* ([30]).

For $p = q = 1$, the structure $\Sigma = (T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is called Σ-*Golden Riemannian structure*.

Remark 4. If $\Sigma = (T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ is the induced structure on the submanifold M by the metallic (or Golden) Riemannian structure $(\overline{g}, J)$ on $\overline{M}$, then M is an invariant submanifold with respect to J if and only if (M, T, g) is a metallic (or Golden) Riemannian manifold, whenever T is nontrivial ([21]).

Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections on $(\overline{M}, \overline{g})$ and (M, g), respectively. The Gauss and Weingarten formulas are given by

$$\begin{aligned}(i)\ & \overline{\nabla}_X Y = \nabla_X Y + h (X, Y), \\ (ii)\ & \overline{\nabla}_X V = -A_V X + \nabla_X^\perp V,\end{aligned} \tag{27}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where h is the second fundamental form, A_V is the shape operator of M. The second fundamental form h and the shape operator A_V are related by

$$\overline{g} (h (X, Y), V) = \overline{g} (A_V X, Y). \tag{28}$$

Remark 5. Using a local orthonormal basis $\{N_1, \ldots, N_r\}$ of the normal space $T_x^\perp M$, where r is the codimension of M in $\overline{M}$ and $A_\alpha := A_{N_\alpha}$, for any $\alpha \in \{1, \ldots, r\}$, we obtain

$$\begin{aligned}(i)\ & \overline{\nabla}_X N_\alpha = -A_\alpha X + \nabla_X^\perp N_\alpha, \\ (ii)\ & h_\alpha (X, Y) = g (A_\alpha X, Y),\end{aligned} \tag{29}$$

for any $X, Y \in \Gamma(TM)$.

Remark 6. For $\alpha \in \{1, \ldots, r\}$, the normal connection $\nabla_X^\perp N_\alpha$ has the decomposition $\nabla_X^\perp N_\alpha = \sum_{\beta=1}^{r} l_{\alpha\beta}(X) N_\beta$, for any $X \in \Gamma(TM)$, where $(l_{\alpha\beta})_r$ is an $r \times r$ matrix of 1-forms on M. Moreover, from $\overline{g}(N_\alpha, N_\beta) = \delta_{\alpha\beta}$, we obtain ([30]): $\overline{g}(\nabla_X^\perp N_\alpha, N_\beta) + \overline{g}(N_\alpha, \nabla_X^\perp N_\beta) = 0$, which is equivalent to $l_{\alpha\beta} = -l_{\beta\alpha}$, for any $\alpha, \beta \in \{1, \ldots, r\}$ and $X \in \Gamma(TM)$.

The covariant derivatives of the tangential and normal parts of JX and JV are given by

$$\begin{aligned}(i)\ & (\nabla_X T) Y = \nabla_X TY - T (\nabla_X Y), \\ (ii)\ & (\overline{\nabla}_X N) Y = \nabla_X^\perp NY - N (\nabla_X Y),\end{aligned} \tag{30}$$

and

$$\begin{aligned}(i)\ & (\nabla_X t) V = \nabla_X tV - t (\nabla_X^\perp V), \\ (ii)\ & (\overline{\nabla}_X n) V = \nabla_X^\perp nV - n (\nabla_X^\perp V),\end{aligned} \tag{31}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$. From $\overline{g}(JX, Y) = \overline{g}(X, JY)$, it follows that

$$\overline{g} ((\overline{\nabla}_X J) Y, Z) = \overline{g} (Y, (\overline{\nabla}_X J) Z), \tag{32}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. Moreover, if M is an isometrically immersed submanifold of the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then ([23])

$$\overline{g} ((\nabla_X T) Y, Z) = \overline{g} (Y, (\nabla_X T) Z), \tag{33}$$

for any $X, Y, Z \in \Gamma(TM)$.

Using an analogy of a locally product manifold ([31]), we can define *locally metallic (or locally Golden) Riemannian manifold* as follows ([30]).

Definition 7. If $(\overline{M}, \overline{g}, J)$ is a metallic (or Golden) Riemannian manifold and J is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ (i.e., $\overline{\nabla} J = 0$), we say that $(\overline{M}, \overline{g}, J)$ is a *locally metallic (or locally Golden) Riemannian manifold.*

Proposition 8. *If* M *is a submanifold of a locally metallic (or locally Golden) Riemannian manifold* $(\overline{M}, \overline{g}, J)$*, then*

$$T ([X, Y]) = \nabla_X TY - \nabla_Y TX - A_{NY} X + A_{NX} Y \tag{34}$$

and

$$\begin{aligned}N ([X, Y]) = {} & h (X, TY) - h (TX, Y) + \nabla_X^\perp NY \\ & - \nabla_Y^\perp NX,\end{aligned} \tag{35}$$

for any $X, Y \in \Gamma(TM)$*, where* ∇ *is the Levi-Civita connection on* M.

Proof. From $(\overline{M}, \overline{g}, J)$ locally metallic (or locally Golden) Riemannian manifold, we have $(\overline{\nabla}_X J)Y = 0$, for any $X, Y \in \Gamma(TM)$.

Thus, $\overline{\nabla}_X(TY + NY) = J(\nabla_X Y + h(X, Y))$, which is equivalent to

$$\begin{aligned}& \nabla_X TY + h (X, TY) - A_{NY} X + \nabla_X^\perp NY \\ & \quad = T (\nabla_X Y) + N (\nabla_X Y) + th (X, Y) + nh (X, Y).\end{aligned} \tag{36}$$

Taking the normal and the tangential components of this equality, we get

$$N(\nabla_X Y) - \nabla_X^{\perp} NY = h(X, TY) - nh(X, Y) \tag{37}$$

and

$$\nabla_X TY - T(\nabla_X Y) = A_{NY}X + th(X, Y). \tag{38}$$

Interchanging X and Y and subtracting these equalities, we obtain the tangential and normal components of $[X, Y] = \nabla_X Y - \nabla_Y X$, which give us (34) and (35). □

From (30), (31), (37), and (38) we obtain the following.

Proposition 9. *If M is a submanifold of a locally metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then the covariant derivatives of T and N verify*

$$\begin{aligned} &(i)\ (\nabla_X T) Y = A_{NY}X + th(X, Y), \\ &(ii)\ (\overline{\nabla}_X N) Y = nh(X, Y) - h(X, TY), \end{aligned} \tag{39}$$

and

$$\begin{aligned} &(i)\ (\nabla_X t) V = A_{nV}X - TA_V X, \\ &(ii)\ (\overline{\nabla}_X n) V = -h(X, tV) - NA_V X, \end{aligned} \tag{40}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proposition 10. *If M is an n-dimensional submanifold of codimension r in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then the structure $\Sigma = (T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on M by the metallic (or Golden) Riemannian structure $(\overline{g}, J)$ has the following properties ([30]):*

$$(\nabla_X T) Y = \sum_{\alpha=1}^{r} h_\alpha(X, Y)\xi_\alpha + \sum_{\alpha=1}^{r} u_\alpha(Y) A_\alpha X, \tag{41}$$

$$\begin{aligned} (\nabla_X u_\alpha) Y = &-h_\alpha(X, TY) \\ &+ \sum_{\beta=1}^{r} \left[u_\beta(Y) l_{\alpha\beta}(X) + h_\beta(X, Y) a_{\beta\alpha}\right], \end{aligned} \tag{42}$$

for any $X, Y \in \Gamma(TM)$.

Proof. From $\overline{\nabla} J = 0$ we obtain $\overline{\nabla}_X JY = J(\overline{\nabla}_X Y)$, for any $X, Y \in \Gamma(T\overline{M})$. Using (27)(i), (29), and (21)(ii), we get

$$\begin{aligned} \overline{\nabla}_X JY = {} & \nabla_X TY - \sum_{\alpha=1}^{r} u_\alpha(Y) A_\alpha X + \sum_{\alpha=1}^{r} \left[h_\alpha(X, TY) \right. \\ & \left. + X(u_\alpha(Y)) + \sum_{\beta=1}^{r} u_\beta(Y) l_{\beta\alpha}(X) \right] N_\alpha \\ J(\overline{\nabla}_X Y) = {} & T(\nabla_X Y) + \sum_{\alpha=1}^{r} h_\alpha(X, Y)\xi_\alpha \\ & + \sum_{\alpha=1}^{r} \left[u_\alpha(\nabla_X Y) + \sum_{\beta=1}^{r} h_\beta(X, Y) a_{\beta\alpha} \right] N_\alpha, \end{aligned} \tag{43}$$

for any $X, Y \in \Gamma(TM)$. Identifying the tangential and normal components, respectively, of the last two equalities, we get (41) and (42). □

Using (34), (35), (41), and (42), we obtain the following.

Proposition 11. *If M is a submanifold of a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then*

$$\begin{aligned} T([X, Y]) = {} & \nabla_X TY - \nabla_Y TX - \sum_{i=1}^{r} \left[u_\alpha(Y) A_\alpha X \right. \\ & \left. - u_\alpha(X) A_\alpha Y\right] \end{aligned} \tag{44}$$

$$\begin{aligned} N([X, Y]) = {} & \sum_{\alpha=1}^{n} \left[((\nabla_Y u_\alpha) X - (\nabla_X u_\alpha) Y) \right. \\ & \left. + \left(u_\alpha(X) l_{\alpha\beta}(Y) - u_\alpha(Y) l_{\alpha\beta}(X)\right) \right] N_\alpha, \end{aligned} \tag{45}$$

for any $X, Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on M.

4. Slant Submanifolds in Metallic or Golden Riemannian Manifolds

Let M be an m'-dimensional submanifold, isometrically immersed in an m-dimensional metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, where $m, m' \in \mathbb{N}^*$ and $m > m'$. Using the Cauchy-Schwartz inequality ([6]), we have

$$\left|\overline{g}(JX, TX)\right| \leq \|JX\| \cdot \|TX\|, \tag{46}$$

for any $X \in \Gamma(TM)$. Thus, there exists a function $\theta : \Gamma(TM) \longrightarrow [0, \pi]$, such that

$$\overline{g}(JX_x, TX_x) = \cos\theta(X_x) \|TX_x\| \cdot \|JX_x\|, \tag{47}$$

for any $x \in M$ and any nonzero tangent vector $X_x \in T_x M$. The angle $\theta(X_x)$ between JX_x and $T_x M$ is called the *Wirtinger angle* of X and it verifies

$$\cos\theta(X_x) = \frac{\overline{g}(JX_x, TX_x)}{\|TX_x\| \cdot \|JX_x\|}. \tag{48}$$

Definition 12 (see [29]). A submanifold M in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ is called slant submanifold if the angle $\theta(X_x)$ between JX_x and $T_x M$ is constant, for any $x \in M$ and $X_x \in T_x M$. In such a case, $\theta =: \theta(X_x)$ is called the *slant angle* of M in $\overline{M}$, and it verifies

$$\cos\theta = \frac{\overline{g}(JX, TX)}{\|JX\| \cdot \|TX\|} = \frac{\|TX\|}{\|JX\|}. \tag{49}$$

The immersion $i : M \longrightarrow \overline{M}$ is named slant immersion of M in $\overline{M}$.

Remark 13. The invariant and anti-invariant submanifolds in the metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ are particular cases of slant submanifolds with the slant angle

$\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold M in $\overline{M}$, which is neither invariant nor anti-invariant, is called proper slant submanifold and the immersion $i : M \longrightarrow \overline{M}$ is called proper slant immersion.

Proposition 14. *([29]) Let M be an isometrically immersed submanifold of the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. If M is a slant submanifold with the slant angle θ, then, for any $X, Y \in \Gamma(TM)$ we get*

$$\overline{g}(TX, TY) = \cos^2\theta\left[p\overline{g}(X, TY) + q\overline{g}(X, Y)\right] \quad (50)$$

$$\overline{g}(NX, NY) = \sin^2\theta\left[p\overline{g}(X, TY) + q\overline{g}(X, Y)\right]. \quad (51)$$

Moreover, we have

$$T^2 = \cos^2\theta\,(pT + qI), \quad (52)$$

where I is the identity on $\Gamma(TM)$ and

$$\nabla\left(T^2\right) = p\cos^2\theta\,(\nabla T). \quad (53)$$

Remark 15. Let I be the identity on $\Gamma(TM)$. From (23) and (52), we have

$$\sin^2\theta\,(pT + qI) = \sum_{\alpha=1}^{r} u_\alpha \otimes \xi_\alpha. \quad (54)$$

Proposition 16. *If M is an isometrically immersed slant submanifold of the Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the slant angle θ, then*

$$\overline{g}(TX, TY) = \cos^2\theta\left[\overline{g}(X, TY) + \overline{g}(X, Y)\right], \quad (55)$$

$$\overline{g}(NX, NY) = \sin^2\theta\left[\overline{g}(X, TY) + \overline{g}(X, Y)\right], \quad (56)$$

for any $X, Y \in \Gamma(TM)$. If I is the identity on $\Gamma(TM)$, we have

$$\begin{aligned} T^2 &= \cos^2\theta\,(T + I), \\ \nabla\left(T^2\right) &= \cos^2\theta\,(\nabla T), \end{aligned} \quad (57)$$

$$\sin^2\theta\,(T + I) = \sum_{\alpha=1}^{r} u_\alpha \otimes \xi_\alpha. \quad (58)$$

Definition 17 (see [8]). A submanifold M in an almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$ is a slant submanifold if the angle $\vartheta(X_x)$ between JX_x and T_xM is constant, for any $x \in M$ and $X_x \in T_xM$. In such a case, $\vartheta =: \vartheta(X_x)$ is called the slant angle of the submanifold M in $\overline{M}$ and it verifies

$$\cos\vartheta = \frac{\overline{g}(FX, fX)}{\|FX\| \cdot \|fX\|} = \frac{\|fX\|}{\|FX\|}. \quad (59)$$

Proposition 18 (see [16]). *If M is a slant submanifold isometrically immersed in an almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$ with the slant angle ϑ then, for any $X, Y \in \Gamma(TM)$, we get*

$$\begin{aligned} (i)\ \overline{g}(fX, fY) &= \cos^2\vartheta\,\overline{g}(X, Y), \\ (ii)\ \overline{g}(\omega X, \omega Y) &= \sin^2\vartheta\,\overline{g}(X, Y). \end{aligned} \quad (60)$$

In the next proposition we find a relation between the slant angles θ of the submanifold M in the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ and the slant angle ϑ of the submanifold M in the almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$.

Theorem 19. *Let M be a submanifold in the Riemannian manifold $(\overline{M}, \overline{g})$ endowed with an almost product structure F on $\overline{M}$ and let J be the induced metallic structure by F on $(\overline{M}, \overline{g})$. If M is a slant submanifold in the almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$ with the slant angle ϑ and $F \neq -I$ (I is the identity on $\Gamma(TM)$) and $J = ((2\sigma_{p,q} - p)/2)F + (p/2)I$, then M is a slant submanifold in the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with slant angle θ given by*

$$\sin\theta = \frac{2\sigma_{p,q} - p}{2\sigma_{p,q}}\sin\vartheta. \quad (61)$$

Proof. From (17)(ii), we obtain $\overline{g}(NX, NY) = ((2\sigma_{p,q} - p)^2/4)\overline{g}(\omega X, \omega Y)$, for any $X, Y \in \Gamma(TM)$. From (51) and (60)(ii) and $\overline{g}(X, JY) = \overline{g}(X, TY)$, we get

$$\begin{aligned} &\frac{\left(2\sigma_{p,q} - p\right)^2}{4}\overline{g}(X, Y)\sin^2\vartheta \\ &\quad = \left[p\overline{g}(X, JY) + q\overline{g}(X, Y)\right]\sin^2\theta, \end{aligned} \quad (62)$$

for any $X, Y \in \Gamma(TM)$. Using $J = (p/2)I + ((2\sigma_{p,q} - p)/2)F$, we have

$$\begin{aligned} &\left(2\sigma_{p,q} - p\right)^2\overline{g}(X, Y)\sin^2\vartheta \\ &\quad = \left[\left(2p^2 + 4q\right)\overline{g}(X, Y) + 2p\sqrt{p^2 + 4q}\,\overline{g}(X, FY)\right] \\ &\quad \cdot \sin^2\theta, \end{aligned} \quad (63)$$

for any $X, Y \in \Gamma(TM)$. Replacing Y by FY and using $F^2Y = Y$, for any $Y \in \Gamma(TM)$, we obtain

$$\begin{aligned} &\left(2\sigma_{p,q} - p\right)^2\overline{g}(X, FY)\sin^2\vartheta \\ &\quad = \left[\left(2p^2 + 4q\right)\overline{g}(X, FY) + 2p\sqrt{p^2 + 4q}\,\overline{g}(X, Y)\right] \\ &\quad \cdot \sin^2\theta. \end{aligned} \quad (64)$$

for any $X, Y \in \Gamma(TM)$. Summing equalities (63) and (64), we obtain

$$\begin{aligned} &\overline{g}(X, FY + Y) \\ &\quad \cdot \left[\left(2\sigma_{p,q} - p\right)^2\sin^2\vartheta - 4\left(q + p\sigma_{p,q}\right)\sin^2\theta\right] = 0, \end{aligned} \quad (65)$$

for any $X, Y \in \Gamma(TM)$. Using $q + p\sigma_{p,q} = \sigma_{p,q}^2$, $FY \neq -Y$, and $\theta, \vartheta \in [0, \pi)$ in (65), we get (61). □

In particular, for $p = q = 1$, we obtain the relation between slant angle θ of the immersed submanifold M in a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ and the slant angle ϑ of M immersed in the almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$.

Proposition 20. *Let M be a submanifold in the Riemannian manifold $(\overline{M},\overline{g})$ endowed with an almost product structure F on $\overline{M}$ and let J be the induced Golden structure by F on $(\overline{M},\overline{g})$. If M is a slant submanifold in the almost product Riemannian manifold $(\overline{M},\overline{g},F)$ with the slant angle ϑ and $F \neq -I$ (I is the identity on $\Gamma(TM)$) and $J = ((2\phi-1)/2)F + (1/2)I$, then M is a slant submanifold in the Golden Riemannian manifold $(\overline{M},\overline{g},J)$ with slant angle θ given by*

$$\sin\theta = \frac{2\phi-1}{2\phi}\sin\vartheta, \tag{66}$$

where $\phi = (1+\sqrt{5})/2$ is the Golden number.

5. Semi-Slant Submanifolds in Metallic or Golden Riemannian Manifolds

We define the slant distribution of a metallic (or Golden) Riemannian manifold, using a similar definition as for Riemannian product manifold ([7, 16]).

Definition 21. Let M be an immersed submanifold of a metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$. A differentiable distribution D on M is called a slant distribution if the angle θ_D between JX_x and the vector subspace D_x is constant, for any $x \in M$ and any nonzero vector field $X_x \in \Gamma(D_x)$. The constant angle θ_D is called the slant angle of the distribution D.

Proposition 22. *Let D be a differentiable distribution on a submanifold M of a metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$. The distribution D is a slant distribution if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$(P_DT)^2X = \lambda\left(pP_DTX + qX\right), \tag{67}$$

for any $X \in \Gamma(D)$, where P_D is the orthogonal projection on D. Moreover, if θ_D is the slant angle of D, then it satisfies $\lambda = \cos^2\theta_D$.

Proof. If the distribution D is a slant distribution on M, by using

$$\cos\theta_D = \frac{\overline{g}(JX, P_DTX)}{\|JX\|\cdot\|P_DTX\|} = \frac{\|P_DTX\|}{\|JX\|}, \tag{68}$$

we get $\overline{g}(P_DTX, P_DTX) = \cos^2\theta_D\overline{g}(pP_DTX + qX, X)$, for any $X \in \Gamma(D)$ and we obtain (67).

Conversely, if there exists a constant $\lambda \in [0,1]$ such that (67) holds for any $X \in \Gamma(D)$, we obtain $\overline{g}(JX, P_DTX) = \overline{g}(X, JP_DTX) = \overline{g}(X,(P_DT)^2X)$ and $\overline{g}(JX,P_DTX) = \lambda\overline{g}(X, pP_DTX + qX) = \lambda\overline{g}(X, pJTX + qX) = \lambda\overline{g}(X, J^2X) = \lambda\overline{g}(JX, JX)$. Thus, $\cos\theta_D = \lambda(\|JX\|/\|P_DTX\|)$, and using $\cos\theta = \|P_DTX\|/\|JX\|$ we get $\cos^2\theta_D = \lambda$. Thus, $\cos^2\theta_D$ is constant and D is a slant distribution on M. □

Definition 23. Let M be an immersed submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$. We say that M is a *bi-slant submanifold* of $\overline{M}$ if there exist two orthogonal differentiable distributions D_1 and D_2 on M such that $TM = D_1 \oplus D_2$ and D_1, D_2 are slant distributions with the slant angles θ_1 and θ_2, respectively.

For a differentiable distribution D_1 on M, we denote by $D_2 := D_1^{\perp}$ the orthogonal distribution of D_1 in M (i.e., $TM = D_1 \oplus D_2$). Let P_1 and P_2 be the orthogonal projections on D_1 and D_2. Thus, for any $X \in \Gamma(TM)$, we can consider the decomposition of $X = P_1X + P_2X$, where $P_1X \in \Gamma(D_1)$ and $P_2X \in \Gamma(D_2)$.

If M is a bi-slant submanifold of a metallic Riemannian manifold $(\overline{M},\overline{g},J)$ with the orthogonal distribution D_1 and D_2 and the slant angles θ_1 and θ_2, respectively, then $JX = P_1TX + P_2TX + NX = TP_1X + TP_2X + NP_1X + NP_2X$, for any $X \in \Gamma(TM)$. In a similar manner as in ([16]), we can prove the following.

Proposition 24. *If M is a bi-slant submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$, with the slant angles $\theta_1 = \theta_2 = \theta$ and $g(JX,Y) = 0$, for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$, then M is a slant submanifold in the metallic Riemannian manifold $(\overline{M},\overline{g},J)$ with the slant angle θ.*

Proof. From $\overline{g}(JX,Y) = \overline{g}(TX,Y) = 0$, for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$, it follows that $\overline{g}(X,JY) = \overline{g}(X,TY) = 0$. Thus, we obtain $TX \in \Gamma(D_1)$, for any $X \in \Gamma(D_1)$ and $TY \in \Gamma(D_2)$, for any $Y \in \Gamma(D_2)$. Moreover, using the projections of any $X \in \Gamma(TM)$ on $\Gamma(D_1)$ and $\Gamma(D_2)$, respectively, we obtain the decomposition $X = P_1X + P_2X$, where $P_1X \in \Gamma(D_1)$ and $P_2X \in \Gamma(D_2)$.

From $\overline{g}(TP_iX, TP_iX) = \cos^2\theta_i g(JP_iX, JP_iX)$ (for $i \in \{1, 2\}$) and using $\theta_1 = \theta_2 = \theta$, we obtain $\overline{g}(TX,TX)/\overline{g}(JX,JX) = \cos^2\theta$, for any $X \in \Gamma(TM)$. Thus, M is a slant submanifold in the metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$ with the slant angle θ. □

If M is a *bi-slant submanifold* of a manifold $\overline{M}$, for particular values of the angles $\theta_1 = 0$ and $\theta_2 \neq 0$, we obtain the following.

Definition 25. An immersed submanifold M in a metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$ is a semi-slant submanifold if there exist two orthogonal distributions D_1 and D_2 on M such that

(1) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2$;

(2) The distribution D_1 is invariant distribution (i.e., $J(D_1) = D_1$);

(3) The distribution D_2 is slant with angle $\theta \neq 0$.

Moreover, if $\dim(D_1)\cdot\dim(D_2) \neq 0$, then M is a proper semi-slant submanifold.

Remark 26. If M is a semi-slant submanifold of a metallic Riemannian manifold $(\overline{M},\overline{g},J)$ with the slant angle θ of the distributions D_2, then we get that

(1) M is an invariant submanifold if $\dim(D_2) = 0$;

(2) M is an anti-invariant submanifold if $\dim(D_1) = 0$ and $\theta = \pi/2$;

(3) M is a semi-invariant submanifold if D_2 is anti-invariant (i.e., $\theta = \pi/2$).

If M is a semi-slant submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ then, for any $X \in \Gamma(TM)$,

$$\begin{aligned} JX &= TP_1X + TP_2X + NP_2X \\ &= P_1TX + P_2TX + NP_2X, \end{aligned} \tag{69}$$

$$\begin{aligned} &(i)\, JP_1X = TP_1X, \\ &(ii)\, NP_1X = 0, \\ &(iii)\, TP_2X \in \Gamma(D_2). \end{aligned} \tag{70}$$

Moreover, we have $\overline{g}(JP_2X, TP_2X) = \cos\theta(X)\|TP_2X\| \cdot \|JP_2X\|$ and the cosine of the slant angle $\theta(X)$ of the distribution D_2 is constant, for any nonzero $X \in \Gamma(TM)$. If $\theta(X) =: \theta$, for any nonzero $X \in \Gamma(TM)$ we get

$$\cos\theta = \frac{\overline{g}(JP_2X, TP_2X)}{\|TP_2X\| \cdot \|JP_2X\|} = \frac{\|TP_2X\|}{\|JP_2X\|}. \tag{71}$$

Proposition 27. *If M is a semi-slant submanifold of the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the slant angle θ of the distribution D_2 then, for any $X, Y \in \Gamma(TM)$, we get*

$$\begin{aligned} &\overline{g}(TP_2X, TP_2Y) \\ &\quad = \cos^2\theta\left[p\overline{g}(TP_2X, P_2Y) + q\overline{g}(P_2X, P_2Y)\right], \end{aligned} \tag{72}$$

$$\begin{aligned} &\overline{g}(NX, NY) \\ &\quad = \sin^2\theta\left[p\overline{g}(TP_2X, P_2Y) + q\overline{g}(P_2X, P_2Y)\right]. \end{aligned} \tag{73}$$

Proof. Taking $X + Y$ in (71) we have $\overline{g}(TP_2X, TP_2Y) = \cos^2\theta\overline{g}(JP_2X, JP_2Y) = \cos^2\theta[p\overline{g}(JP_2X, P_2Y) + q\overline{g}(P_2X, P_2Y)]$, for any $X, Y \in \Gamma(TM)$ and using (70)(iii) we get (72). From (70)(ii) we get $TP_2X = JP_2X - NX$, for any $X \in \Gamma(TM)$. Thus, we obtain $\overline{g}(TP_2X, TP_2Y) = \overline{g}(JP_2X, JP_2Y) - \overline{g}(NX, NY)$, for any $X, Y \in \Gamma(TM)$ and it implies (73). ☐

Remark 28. A semi-slant submanifold M of a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the slant angle θ of the distribution D_2 verifies

$$\begin{aligned} &\overline{g}(TP_2X, TP_2Y) \\ &\quad = \cos^2\theta\left[\overline{g}(TP_2X, P_2Y) + \overline{g}(P_2X, P_2Y)\right], \end{aligned} \tag{74}$$

$$\begin{aligned} &\overline{g}(NX, NY) \\ &\quad = \sin^2\theta\left[\overline{g}(TP_2X, P_2Y) + \overline{g}(P_2X, P_2Y)\right], \end{aligned} \tag{75}$$

for any $X, Y \in \Gamma(TM)$.

Proposition 29. *Let M be a semi-slant submanifold of a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the slant angle θ of the distribution D_2. Then*

$$(TP_2)^2 = \cos^2\theta(pTP_2 + qI), \tag{76}$$

where I is the identity on $\Gamma(D_2)$ and

$$\nabla\left((TP_2)^2\right) = p\cos^2\theta\nabla(TP_2). \tag{77}$$

Proof. Using $\overline{g}(TP_2X, TP_2Y) = \overline{g}((TP_2)^2X, P_2Y)$, for any $X, Y \in \Gamma(TM)$ and (72), we obtain (76). Moreover, we have $(\nabla_X(TP_2)^2)Y = \cos^2\theta(p(\nabla_XTP_2)Y + q(\nabla_XI)Y) = p\cos^2\theta\nabla_X(P_2T)Y$, for any $X \in \Gamma(D_2)$ and $Y \in \Gamma(TM)$. For the identity I on $\Gamma(D_2)$ we have $(\nabla_XI)P_2Y = 0$; thus, we get (77). ☐

Remark 30. A semi-slant submanifold M of a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the slant angle θ of the distribution D_2 verifies

$$(TP_2)^2 = \cos^2\theta(TP_2 + I), \tag{78}$$

where I is the identity on $\Gamma(D_2)$ and

$$\nabla\left((TP_2)^2\right) = \cos^2\theta\nabla(TP_2). \tag{79}$$

Proposition 31. *Let M be an immersed submanifold of a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then M is a semi-slant submanifold in $\overline{M}$ if and only if exists a constant $\lambda \in [0, 1)$ such that $D = \{X \in \Gamma(TM) | T^2X = \lambda(pTX + qX)\}$ is a distribution and $NX = 0$, for any $X \in \Gamma(TM)$ orthogonal to D, where p and q are given in (1).*

Proof. If we consider M a semi-slant submanifold of the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ then, in (72) we put $\lambda = \cos^2\theta \in [0, 1)$. Thus, we obtain $T^2X = \lambda(pTX + qX)$ and we get $D_2 \subseteq D$. For a nonzero vector field $X \in \Gamma(D)$, let $X = X_1 + X_2$, where $X_1 = P_1X \in \Gamma(D_1)$ and $X_2 = P_2X \in \Gamma(D_2)$. Because D_1 is invariant, then $JX_1 = TX_1$ and using the property of the metallic structure (1), we obtain $pTX_1 + qX_1 = pJX_1 + qX_1 = J^2X_1 = T^2X_1 = \lambda(pTX_1 + qX_1)$, which implies $(pTX_1 + qX_1)(\lambda - 1) = 0$. Because $\lambda \in [0, 1)$, we obtain $TX_1 = -(q/p)X_1$ and we get $X_1 = 0$ ($q^2/p^2 \neq 0$ because p and q are nonzero natural numbers). Thus, we obtain $X \in \Gamma(D_2)$ and $D \subseteq D_2$, which implies $D = D_2$. Therefore, $D_1 = D^\perp$.

Conversely, if there exists a real number $\lambda \in [0, 1)$ such that we have $T^2X = \lambda(pTX + qX)$, for any $X \in \Gamma(D)$, it follows that $\cos^2(\theta(X)) = \lambda$ which implies that $\theta(X) = \arccos(\sqrt{\lambda})$ does not depend on X. We can consider the orthogonal direct sum $TM = D \oplus D^\perp$. For $Y \in \Gamma(D^\perp) := \Gamma(D_1)$ and $X \in \Gamma(D)$ (with $D := D_2$), we have $\overline{g}(X, J^2Y) = \overline{g}(X, T(JY)) = \overline{g}(TX, JY) = \overline{g}(TX, TY) = \overline{g}(T^2X, Y) = \lambda[p\overline{g}(TX, Y) + q\overline{g}(X, Y)]$. From $\overline{g}(X, J^2Y) = p\overline{g}(X, JY) + q\overline{g}(X, Y)$ and $\overline{g}(X, Y) = 0$, we obtain $\overline{g}(X, JY) = \lambda\overline{g}(X, TY)$ and this implies $(1 - \lambda)TY \in \Gamma(D^\perp)$ and $TY \in \Gamma(D^\perp)$. Thus, $JY \in \Gamma(D^\perp)$, for any $X \in \Gamma(D^\perp)$ and we obtain that $D^\perp$ is an invariant distribution. ☐

Remark 32. An immersed submanifold M of the Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ is a semi-slant submanifold in $\overline{M}$ if and only if there exists a constant $\lambda \in [0, 1)$ such that

$$D = \left\{X \in \Gamma(TM) \mid T^2X = \lambda(TX + X)\right\} \tag{80}$$

is a distribution and $NX = 0$, for any $X \in \Gamma(TM)$ orthogonal to D.

Examples 1. Let $\mathbb{R}^7$ be the Euclidean space endowed with the usual Euclidean metric $\langle\cdot,\cdot\rangle$. Let $f : M \longrightarrow \mathbb{R}^7$ be the immersion given by

$$f(u,t_1,t_2) = (u\cos t_1, u\sin t_1, u\cos t_2, u\sin t_2, u, t_1, t_2), \tag{81}$$

where $M \coloneqq \{(u,t_1,t_2)/u > 0, t_1,t_2 \in [0,\pi/2]\}$.

We can find a local orthonormal frame on TM given by

$$\begin{aligned} Z_1 &= \cos t_1 \frac{\partial}{\partial x_1} + \sin t_1 \frac{\partial}{\partial x_2} + \cos t_2 \frac{\partial}{\partial x_3} + \sin t_2 \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_1},\\ Z_2 &= -u\sin t_1 \frac{\partial}{\partial x_1} + u\cos t_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2},\\ Z_3 &= -u\sin t_2 \frac{\partial}{\partial x_3} + u\cos t_2 \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_3}. \end{aligned} \tag{82}$$

We define the metallic structure $J : \mathbb{R}^7 \longrightarrow \mathbb{R}^7$ given by

$$J\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = \left(\sigma\frac{\partial}{\partial x_1}, \sigma\frac{\partial}{\partial x_2}, \overline{\sigma}\frac{\partial}{\partial x_3}, \overline{\sigma}\frac{\partial}{\partial x_4}, \overline{\sigma}\frac{\partial}{\partial y_1}, \sigma\frac{\partial}{\partial y_2}, \overline{\sigma}\frac{\partial}{\partial y_3}\right), \tag{83}$$

for $i \in \{1,2,3,4\}$ and $j \in \{1,2,3\}$ where $\sigma \coloneqq \sigma_{p,q} = (p + \sqrt{p^2+4q})/2$ is the metallic number $(p,q \in N^*)$ and $\overline{\sigma} = p - \sigma$. We can verify that $\overline{\nabla}J = 0$ and we obtain that $(\mathbb{R}^7, \langle\cdot,\cdot\rangle, J)$ is a locally metallic Riemannian manifold.

Moreover, we have $JZ_2 = \sigma Z_2$, $JZ_3 = \overline{\sigma}Z_3$, and

$$JZ_1 = \sigma\cos t_1 \frac{\partial}{\partial x_1} + \sigma\sin t_1 \frac{\partial}{\partial x_2} + \overline{\sigma}\cos t_2 \frac{\partial}{\partial x_3} + \overline{\sigma}\sin t_2 \frac{\partial}{\partial x_4} + \overline{\sigma}\frac{\partial}{\partial y_1}. \tag{84}$$

We can verify that $\|JZ_2\|^2 = \sigma^2(u^2+1)$, $\|JZ_3\|^2 = \overline{\sigma}^2(u^2+1)$,

$$\begin{aligned} \|JZ_1\|^2 &= \sigma^2 + 2\overline{\sigma}^2,\\ \|JZ_2\|^2 &= \sigma^2\left(u^2+1\right),\\ \|JZ_3\|^2 &= \overline{\sigma}^2\left(u^2+1\right). \end{aligned} \tag{85}$$

On the other hand, we have $\langle JZ_1, Z_1\rangle = \sigma + 2\overline{\sigma}$ and $\langle JZ_i, Z_j\rangle = 0$, for any $i \neq j$, where $i,j \in \{1,2,3\}$. We remark that

$$\cos\theta = \frac{\langle JZ_1, Z_1\rangle}{\|Z_1\|\cdot\|JZ_1\|} = \frac{\sigma + 2\overline{\sigma}}{\sqrt{3\left(\sigma^2 + 2\overline{\sigma}^2\right)}}. \tag{86}$$

We define the distributions $D_1 = span\{Z_2, Z_3\}$ and $D_2 = span\{Z_1\}$. We have $J(D_1) \subset D_1$ (i.e., D_1 is an invariant distribution with respect to J). The Riemannian metric tensor of $D_1 \oplus D_2$ is given by $g = 3du^2 + (u^2+1)(dt_1^2 + dt_2^2)$. Thus, $D_1 \oplus D_2$ is a warped product semi-slant submanifold in the locally metallic Riemannian manifold $(\mathbb{R}^7, \langle\cdot,\cdot\rangle, J)$ with the slant angle $\arccos((\sigma + 2\overline{\sigma})/\sqrt{3(\sigma^2 + 2\overline{\sigma}^2)})$.

If J is the Golden structure $J : \mathbb{R}^7 \longrightarrow \mathbb{R}^7$ given by

$$J\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = \left(\phi\frac{\partial}{\partial x_1}, \phi\frac{\partial}{\partial x_2}, \overline{\phi}\frac{\partial}{\partial x_3}, \overline{\phi}\frac{\partial}{\partial x_4}, \overline{\phi}\frac{\partial}{\partial y_1}, \phi\frac{\partial}{\partial y_2}, \overline{\phi}\frac{\partial}{\partial y_3}\right), \tag{87}$$

for $i \in \{1,2,3,4\}$ and $j \in \{1,2,3\}$, where $\phi \coloneqq (1+\sqrt{5})/2$ is the Golden number and $\overline{\phi} = 1 - \phi$, in the same manner we obtain

$$\cos\theta = \frac{\langle JZ_1, Z_1\rangle}{\|Z_1\|\cdot\|JZ_1\|} = \frac{\phi + 2\overline{\phi}}{\sqrt{3\left(\phi^2 + 2\overline{\phi}^2\right)}}. \tag{88}$$

We define the distributions $D_1 = span\{Z_2, Z_3\}$ and $D_2 = span\{Z_1\}$. We obtain that $D_1 \oplus D_2$ is a warped product semi-slant submanifold in the locally Golden Riemannian manifold $(\mathbb{R}^7, \langle\cdot,\cdot\rangle, J)$, with the slant angle $\arccos((\phi + 2\overline{\phi})/\sqrt{3(\phi^2 + 2\overline{\phi}^2)})$.

Examples 2. Let $M \coloneqq \{(u,\alpha_1,\alpha_2,\dots,\alpha_n)/u > 0, \alpha_i \in [0,\pi/2], i \in \{1,\dots,n\}\}$ and $f : M \longrightarrow \mathbb{R}^{3n+1}$ is the immersion given by

$$f(u,\alpha_1,\dots,\alpha_n) = (u\cos\alpha_1,\dots,u\cos\alpha_n, u\sin\alpha_1,\dots,u\cdot\sin\alpha_n, \alpha_1,\dots,\alpha_n, u). \tag{89}$$

We can find a local orthonormal frame of the submanifold TM in $\mathbb{R}^{3n+1}$, spanned by the vectors:

$$\begin{aligned} Z_0 &= \sum_{j=1}^{n}\left(\cos\alpha_j \frac{\partial}{\partial x_j} + \sin\alpha_j \frac{\partial}{\partial x_{n+j}}\right) + \frac{\partial}{\partial x_{3n+1}},\\ Z_i &= -u\sin\alpha_i \frac{\partial}{\partial x_i} + u\cos\alpha_i \frac{\partial}{\partial x_{n+i}} + \frac{\partial}{\partial x_{2n+i}}, \end{aligned} \tag{90}$$

for any $i \in \{1,\dots,n\}$. We remark that $\|Z_0\|^2 = n+1$, $\|Z_i\|^2 = u^2+1$, for any $i \in \{1,\dots,n\}$, $Z_0 \perp Z_i$, for any $i \in \{1,\dots,n\}$, and $Z_i \perp Z_j$, for $i \neq j$, where $i,j \in \{1,\dots,n\}$.

Let $J : \mathbb{R}^{3n+1} \longrightarrow \mathbb{R}^{3n+1}$ be the $(1,1)$-tensor field defined by

$$J\left(X^1,\dots,X^{3n},X^{3n+1}\right) = \left(\sigma X^1,\dots,\sigma X^{3n},\overline{\sigma}X^{3n+1}\right), \tag{91}$$

where $\sigma \coloneqq \sigma_{p,q}$ is the metallic number and $\overline{\sigma} = p - \sigma$. It is easy to verify that J is a metallic structure on $\mathbb{R}^{3n+1}$ (i.e., $J^2 = pJ + qI$). The metric $\overline{g}$, given by the scalar product

$\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{3n+1}$, is J compatible and $(\mathbb{R}^{3n+1}, \overline{g}, J)$ is a metallic Riemannian manifold.

Also, $JZ_0 = \sigma\sum_{j=1}^{n}(\cos\alpha_j(\partial/\partial x_j) + \sin\alpha_j(\partial/\partial x_{n+j})) + \overline{\sigma}(\partial/\partial x_{3n+1})$ and, for any $i \in \{1,\ldots,n\}$, we get $JZ_i = \sigma(-u\sin\alpha_i(\partial/\partial x_i) + u\cos\alpha_i(\partial/\partial x_{n+i}) + \partial/\partial x_{2n+i}) = \sigma Z_i$. We can verify that JZ_0 is orthogonal to $span\{Z_1,\ldots,Z_n\}$ and $\cos(\widehat{JZ_0,Z_0}) = (n\sigma+\overline{\sigma})/\sqrt{(n+1)(n\sigma^2+\overline{\sigma}^2)}$.

If we consider the distributions $D_1 = span\{Z_i/i \in \{1,\ldots,n\}\}$ and $D_2 = span\{Z_0\}$, then $D_1 \oplus D_2$ is a semi-slant submanifold in the metallic Riemannian manifold $(\mathbb{R}^{3n+1}, \langle\cdot,\cdot\rangle, J)$, with the Riemannian metric tensor $g = (n+1)du^2 + (u^2+1)\sum_{j=1}^{n} d\alpha_j^2$.

6. On the Integrability of the Distributions of Semi-Slant Submanifolds

In this section we investigate the conditions for the integrability of the distributions of semi-slant submanifolds in metallic (or Golden) Riemannian manifolds.

Theorem 33. *If M is a semi-slant submanifold of a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then*

(i) the distribution D_1 is integrable if and only if

$$(\nabla_Y u_\alpha)(X) = (\nabla_X u_\alpha)(Y), \tag{92}$$

for any $X, Y \in \Gamma(D_1)$;

(ii) the distribution D_2 is integrable if and only if

$$\begin{aligned} &P_1(\nabla_X TY - \nabla_Y TX) \\ &\quad = \sum_{i=1}^{r}\left[u_\alpha(Y)P_1(A_\alpha X) - u_\alpha(X)P_1(A_\alpha Y)\right], \end{aligned} \tag{93}$$

for any $X, Y \in \Gamma(D_2)$.

Proof. (i) For $X, Y \in \Gamma(D_1)$, we have $X = P_1X$ and $Y = P_1Y$. The distribution D_1 is integrable if and only if $[X,Y] \in \Gamma(D_1)$, which is equivalent to $N([X,Y]) = 0$, for any $X, Y \in \Gamma(D_1)$. From $J(D_1) \subseteq D_1$ we obtain $NX = NY = 0$ and from (22)(i) we get $u_\alpha(X)l_{\alpha\beta}(Y) = u_\alpha(Y)l_{\alpha\beta}(X) = 0$. Thus, using (45) we have the distribution D_1 is integrable if and only if (92) holds.

(ii) For $X, Y \in \Gamma(D_2)$, we have $X = P_2X$, $Y = P_2Y$. The distribution D_2 is integrable if and only if $[X,Y] \in \Gamma(D_2)$, which is equivalent to $P_1T([X,Y]) = 0$. Thus, from (44), we obtain that the distribution D_2 is integrable if and only if (93) holds. □

Remark 34. If M is a semi-slant submanifold of a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then

(i) the distribution D_1 is integrable if and only if

$$h(X, TY) = h(TX, Y), \tag{94}$$

for any $X, Y \in \Gamma(D_1)$;

(ii) the distribution D_1 is integrable if and only if the shape operator of M satisfies

$$JA_V X = A_V JX, \tag{95}$$

for any $X \in \Gamma(D_1)$ and $V \in \Gamma(T^\perp M)$;

(iii) the distribution D_2 is integrable if and only if

$$P_1(\nabla_X TY - \nabla_Y TX) = P_1(A_{NY}X - A_{NX}Y), \tag{96}$$

for any $X, Y \in \Gamma(D_2)$.

Proof. (i) For any $X, Y \in \Gamma(D_1)$, we have $[X,Y] \in \Gamma(D_1)$ if and only if $N([X,Y]) = 0$ and from (35), we obtain (94).

(ii) For any $X, Y \in \Gamma(D_1)$ and any $V \in \Gamma(T^\perp M)$, from (28) and (2) we have

$$\begin{aligned} &g(JA_V X - A_V JX, Y) \\ &\quad = g(h(X, JY) - h(JX, Y), V). \end{aligned} \tag{97}$$

From (35) and $NX = NY = 0$ (because $J(D_1) \subseteq D_1$) we have

$$g(JA_V X - A_V JX, Y) = g(N([X,Y]), V) = 0, \tag{98}$$

for any $X, Y \in \Gamma(D_1)$ and any $V \in \Gamma(T^\perp M)$. Thus, we have $[X,Y] \in \Gamma(D_1)$.

(iii) For any $X, Y \in \Gamma(D_2)$, we have $X = P_2X$ and $Y = P_2Y$. The distribution D_2 is integrable if and only if $[X,Y] \in \Gamma(D_2)$, which is equivalent to $T([X,Y]) \in \Gamma(D_2)$ or $P_1T([X,Y]) = 0$. Thus, from (34), we obtain that D_2 is integrable if and only if (96) holds. □

Theorem 35. *Let M be a semi-slant submanifold of a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. If $\nabla T = 0$, then the distributions D_1 and D_2 are integrable.*

Proof. First of all, we consider $X, Y \in \Gamma(D_1)$ and we prove $[X,Y] \in \Gamma(D_1)$. For any $Y \in \Gamma(D_1)$, we get $NY = 0$ and using $\nabla T = 0$ in (39)(i) we obtain $th(X,Y) = 0$, for any $X, Y \in \Gamma(D_1)$, which implies $Jh(X,Y) = nh(X,Y)$. From

$$\begin{aligned} \overline{g}(th(X,Y), Z) &= \overline{g}(Jh(X,Y), Z) \\ &= \overline{g}(h(X,Y), JZ) \end{aligned} \tag{99}$$

and (69) we get $\overline{g}(h(X,Y), NP_2Z) = 0$, for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(TM)$. Thus, from (1) and (2) we get

$$\begin{aligned} \overline{g}(Jh(X,Y), JZ) &= \overline{g}\left(J^2h(X,Y), Z\right) \\ &= p\overline{g}(Jh(X,Y), Z) \\ &\quad + q\overline{g}(h(X,Y), Z) = 0, \end{aligned} \tag{100}$$

for any $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(TM)$. Moreover, for $Z = \nabla_X Y$, we obtain

$$\begin{aligned} 0 &= \overline{g}(Jh(X,Y), NP_2\nabla_X Y) \\ &= \overline{g}\left(\overline{\nabla}_X JY, NP_2\nabla_X Y\right) - \overline{g}(J\nabla_X Y, NP_2\nabla_X Y), \end{aligned} \tag{101}$$

which implies

$$\overline{g}(h(X, JY), NP_2\nabla_X Y) = \overline{g}(NP_2\nabla_X Y, NP_2\nabla_X Y). \tag{102}$$

On the other hand, from (73) and (102) we have

$$\begin{aligned} &\overline{g}(h(X, JY), NP_2\nabla_X Y) = \sin^2\theta \\ &\quad \cdot\left[p\overline{g}(TP_2\nabla_X Y, P_2\nabla_X Y) + q\overline{g}(P_2\nabla_X Y, P_2\nabla_X Y)\right]. \end{aligned} \tag{103}$$

Using (102), (12) and $JY = TY$, for any $Y \in \Gamma(D_1)$, we obtain

$$\overline{g}\left(h(X,JY), NP_2\nabla_X Y\right) = \overline{g}\left(th(X,TY), P_2\nabla_X Y\right) = 0. \tag{104}$$

Thus, from (102) and (104) we have

$$\sin^2\theta \left[p\overline{g}\left(TP_2\nabla_X Y, P_2\nabla_X Y\right) + q\overline{g}\left(P_2\nabla_X Y, P_2\nabla_X Y\right)\right] = 0. \tag{105}$$

From $\theta \neq 0$, we obtain $p\overline{g}(TP_2\nabla_X Y, P_2\nabla_X Y) + q\overline{g}(P_2\nabla_X Y, P_2\nabla_X Y) = 0$.

By using $\overline{g}(TP_2\nabla_X Y, P_2\nabla_X Y) = \overline{g}(JP_2\nabla_X Y, P_2\nabla_X Y)$, we have

$$\begin{aligned}\overline{g}\left(J^2 P_2\nabla_X Y, P_2\nabla_X Y\right) &= p\overline{g}\left(JP_2\nabla_X Y, P_2\nabla_X Y\right) \\ &\quad + q\overline{g}\left(P_2\nabla_X Y, P_2\nabla_X Y\right) = 0,\end{aligned} \tag{106}$$

which implies $\overline{g}(J(P_2\nabla_X Y), J(P_2\nabla_X Y)) = 0$. Thus, we get $J(P_2\nabla_X Y) = 0$ and we obtain $P_2\nabla_X Y = 0$. In conclusion, $\nabla_X Y \in \Gamma(D_1)$ for any $X, Y \in \Gamma(D_1)$ and this implies $[X,Y] \in \Gamma(D_1)$. Thus, the distribution D_1 is integrable.

Moreover, because D_2 is orthogonal to D_1 and $(\overline{M}, \overline{g})$ is a Riemannian manifold, we obtain the integrability of the distribution D_2. □

In the next propositions, we consider semi-slant submanifolds in the locally metallic (or locally Golden) Riemannian manifolds and we find some conditions for these submanifolds to be $D_1 - D_2$ mixed totally geodesic (i.e., $h(X,Y) = 0$, for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$), in a similar manner as in the case of semi-slant submanifolds in locally product manifolds ([16]).

Proposition 36. *If M is a semi-slant submanifold of a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then M is a $D_1 - D_2$ mixed totally geodesic submanifold if and only if $A_V X \in \Gamma(D_1)$ and $A_V Y \in \Gamma(D_2)$, for any $X \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$ and $V \in \Gamma(T^\perp M)$.*

Proof. From (28) we remark that M is a $D_1 - D_2$ mixed totally geodesic submanifolds in the locally metallic (or locally Golden) Riemannian manifolds if and only if $g(A_V X, Y) = g(A_V Y, X) = 0$, for any $X \in \Gamma(D_1), Y \in \Gamma(D_2)$ and $V \in \Gamma(T^\perp M)$, which is equivalent to $A_V X \in \Gamma(D_1)$ and $A_V Y \in \Gamma(D_2)$. □

Proposition 37. *Let M be a proper semi-slant submanifold of a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. If M is a $D_1 - D_2$ mixed totally geodesic submanifold, then $(\overline{\nabla}_X N)Y = 0$, for any $X \in \Gamma(D_1)$, and $Y \in \Gamma(D_2)$.*

Proof. If M is a $D_1 - D_2$ mixed geodesic submanifold, then $nh(X,Y) = h(X,TY)$ and using (39)(ii), we obtain $(\overline{\nabla}_X N)Y = 0$, for any $X \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$. □

Theorem 38. *Let M be a proper semi-slant submanifold of a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. If $(\overline{\nabla}_X N)Y = 0$, for any $X \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$, and $h(X,Y)$ is not an eigenvector of the tensor field n with the eigenvalue $-q/p$, then M is a $D_1 - D_2$ mixed totally geodesic submanifold.*

Proof. If M is a semi-slant submanifold of a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ and $(\overline{\nabla}_X N)Y = 0$, then from (39)(ii), we get

$$n^2 h(X,Y) = nh(X,TY) = h\left(X, T^2 Y\right), \tag{107}$$

for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$, where $nV := (JV)^\perp$ for any $V \in \Gamma(T^\perp M)$. By using $TY = TP_2 Y$, for any $Y \in \Gamma(D_2)$ and (76), we obtain

$$n^2 h(X,Y) = \cos^2\theta\left(pnh(X,Y) + qh(X,Y)\right), \tag{108}$$

where θ is the slant angle of the distribution D_2. Using $TX = TP_1 X = JX$, for any $X \in \Gamma(D_1)$, we obtain

$$\begin{aligned}n^2 h(X,Y) &= h\left(T^2 X, Y\right) = h\left(J^2 X, Y\right) \\ &= h\left(pJX + qX, Y\right) \\ &= ph(TX,Y) + qh(X,Y).\end{aligned} \tag{109}$$

Thus, we get

$$n^2 h(X,Y) = pnh(X,Y) + qh(X,Y), \tag{110}$$

for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. From (108), (110) and $\cos^2\theta \neq 0$ (D_2 is a proper semi-slant distribution), we remark that $nh(X,Y) = -(q/p)h(X,Y)$ and this implies $h(X,Y) = 0$, for any $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$, because $h(X,Y)$ is not an eigenvector of n with the eigenvalue $-q/p$. Thus, M is a $D_1 - D_2$ mixed totally geodesic submanifold in the locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. □

In a similar manner as in ([8], Theorem 4.8), we get the following.

Proposition 39. *Let M be a semi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then N is parallel if and only if the shape operator A verifies*

$$A_{nV} X = TA_V X = A_V TX, \tag{111}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Proof. From (2), we get $\overline{g}(nh(X,Y),V) = \overline{g}(Jh(X,Y),V) = \overline{g}(h(X,Y),nV)$, for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^\perp M)$. Thus, by using (39)(ii), we obtain

$$\begin{aligned}\overline{g}\left(\left(\overline{\nabla}_X N\right)Y, V\right) &= \overline{g}\left(h(X,Y), nV\right) \\ &\quad - \overline{g}\left(h(X,TY), V\right) \\ &= \overline{g}\left(A_{nV} X, Y\right) - \overline{g}\left(A_V X, TY\right),\end{aligned} \tag{112}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$ and we have

$$\begin{aligned}\overline{g}\left(\left(\overline{\nabla}_X N\right) Y, V\right) &= \overline{g}\left(A_{nV}X - TA_V X, Y\right) \\ &= \overline{g}\left(A_{nV}Y - A_V TY, X\right),\end{aligned} \tag{113}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$. Thus, from (113) we obtain (111). □

Theorem 40. *Let M be a proper semi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. If the shape operator A verifies $A_{nV}X = TA_V X = A_V TX$, for any $X \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$ and $h(X,Y)$ is not an eigenvector of the tensor field n with the eigenvalue $-q/p$ then, M is a $D_1 - D_2$ mixed totally geodesic submanifold.*

Proof. If $A_{nV}X = TA_V X = A_V TX$, for any $X \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$ then, from (113) we obtain $(\overline{\nabla}_X N)Y = 0$ for any $X, Y \in \Gamma(TM)$ and using the Theorem 38, we obtain that M is a $D_1 - D_2$ mixed totally geodesic submanifold. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] B. Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit Leuven, Leuven, Belgium, 1990.

[2] B.-Y. Chen, "Slant immersions," *Bulletin of the Australian Mathematical Society*, vol. 41, no. 1, pp. 135–147, 1990.

[3] A. Lotta, "Slant submanifolds in contact geometry," *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 39, pp. 183–198, 1996.

[4] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, and M. Fernandez, "Slant submanifolds in Sasakian manifolds," *Glasgow Mathematical Journal*, vol. 42, no. 1, pp. 125–138, 2000.

[5] J. L. Cabrerizo, A. Carriazo, and L. M. Fernandez, "Semi-slant submanifolds of a Sasakian manifold," *Geometriae Dedicata*, vol. 78, no. 2, pp. 183–199, 1999.

[6] P. Alegre and A. Carriazo, "Slant submanifolds of para-Hermitian manifolds," *Mediterranean Journal of Mathematics*, vol. 14, no. 5, pp. 1–14, 2017.

[7] B. Sahin, "Slant submanifolds of an almost product Riemannian manifold," *Journal of the Korean Mathematical Society*, vol. 43, no. 4, pp. 717–732, 2006.

[8] M. Atçeken, "Slant submanifolds of a Riemannian product manifold," *Acta Mathematica Scientia B*, vol. 30, no. 1, pp. 215–224, 2010.

[9] M. Atçeken, "Warped product semi-invariant submanifolds in locally decomposable Riemannian manifolds," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 3, pp. 401–407, 2011.

[10] N. Papaghiuc, "Semi-slant submanifolds of a Kaehlerian manifold," *Scientific Annals of the Alexandru Ioan Cuza University of Iasi, s. I. a, Mathematics*, vol. 40, no. 1, pp. 55–61, 1994.

[11] A. Carriazo, *New Developments in Slant Submanifolds Theory*, Narasa Publishing House, New Delhi, India, 2002.

[12] M. Atçeken and S. Dirik, "Pseudo-slant submanifolds of a nearly Kenmotsu manifold," *Serdica Mathematical Journal*, vol. 41, no. 2-3, pp. 243–262, 2015.

[13] M. Atçeken and S. Dirik, "On the geometry of pseudo-slant submanifolds of a Kenmotsu manifold," *Gulf Journal of Mathematics*, vol. 2, no. 2, pp. 51–66, 2014.

[14] M. Atçeken and S. K. Hui, "Slant and pseudo-slant submanifolds in LCS-manifolds," *Czechoslovak Mathematical Journal*, vol. 63, no. 1, pp. 177–190, 2013.

[15] M. Atçeken and S. Dirik, "Pseudo-slant submanifolds of a locally decomposable Riemannian manifold," *Journal of Advances in Math*, vol. 11, no. 8, pp. 5587–5599, 2015.

[16] H. Li and X. Liu, "Semi-slant submanifolds of a locally product manifold," *Georgian Mathematical Journal*, vol. 12, no. 2, pp. 273–282, 2005.

[17] M. Crasmareanu and C.-E. Hreţcanu, "Golden differential geometry," *Chaos, Solitons & Fractals*, vol. 38, no. 5, pp. 1229–1238, 2008.

[18] C. E. Hretcanu and M. Crasmareanu, "On some invariant submanifolds in a Riemannian manifold with Golden structure," *Scientific Annals of the Alexandru Ioan Cuza University of Iasi (New Series). Mathematics*, vol. 53, no. 1, pp. 199–211, 2007.

[19] C. E. Hretcanu and M. C. Crasmareanu, "Applications of the Golden ratio on Riemannian manifolds," *Turkish Journal of Mathematics*, vol. 33, no. 2, pp. 179–191, 2009.

[20] M. Crasmareanu, C.-E. Hreţcanu, and M.-I. Munteanu, "Golden- and product-shaped hypersurfaces in real space forms," *International Journal of Geometric Methods in Modern Physics*, vol. 10, no. 4, Article ID 1320006, 2013.

[21] C. E. Hretcanu and M. Crasmareanu, "Metallic structures on Riemannian manifolds," *Revista de la Unión Matemática Argentina*, vol. 54, no. 2, pp. 15–27, 2013.

[22] A. M. Blaga, "The geometry of golden conjugate connections," *Sarajevo Journal of Mathematics*, vol. 10, 2, no. 23, pp. 237–245, 2014.

[23] A. M. Blaga and C. E. Hretcanu, "Metallic conjugate connections," *Revista de la Unión Matemática Argentina*, vol. 59, no. 1, pp. 179–192, 2018.

[24] F. Etayo, R. Santamaria, and A. Upadhyay, "On the geometry of almost Golden Riemannian manifolds," *Mediterranean Journal of Mathematics*, vol. 14, no. 5, pp. 1–14, 2017.

[25] A. Gezer, N. Cengiz, and A. Salimov, "On integrability of golden Riemannian structures," *Turkish Journal of Mathematics*, vol. 37, no. 4, pp. 693–703, 2013.

[26] S. I. Goldberg and K. Yano, "Polynomial structures on manifolds," *Kodai Mathematical Seminar Reports*, vol. 22, pp. 199–218, 1970.

[27] S. I. Goldberg and N. C. Petridis, "Differentiable solutions of algebraic equations on manifolds," *Kodai Mathematical Seminar Reports*, vol. 25, pp. 111–128, 1973.

[28] V. W. de Spinadel, "The metallic means family and forbidden symmetries," *International Mathematical Journal*, vol. 2, no. 3, pp. 279–288, 2002.

[29] A. M. Blaga and C. E. Hretcanu, *Invariant, Anti-Invariant and Slant Submanifolds of a Metallic Riemannian Manifold*, 2018.

[30] C. E. Hretcanu and A. M. Blaga, "Submanifolds in metallic Riemannian manifolds," *Differential Geometry - Dynamical Systems*, vol. 20, pp. 83–97, 2018.

[31] G. Pitis, "On some submanifolds of a locally product manifold," *Kodai Mathematical Journal*, vol. 9, no. 3, pp. 327–333, 1986.


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