Research Article

Positive Solutions to n -Order Fractional Differential Equation with Parameter

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By employing the properties of the sum operators, we investigate the solutions of the n -order fractional differential equations multiple point boundary value problem (in short BVP) with the boundary conditions contains a parameter. We not only obtain the existence and uniqueness of solutions about this BVP but also construct an iterative scheme to approximate the solution which is important for practical application. An example is given to demonstrate the validity of our main results.

1. Introduction

In this paper, we focus on the n -order fractional differential equations of multiple point BVP:

$$\begin{aligned} -D_{0+}^{\alpha} x(t) &= f_1(t, x(t), x(t)) + f_2(t, x(t)), \\ 0 < t < 1, \quad n-1 < \alpha \leq n, \quad n \geq 2, \\ x(0) &= x'(0) = \cdots = x^{(k)}(0) = 0, \\ 0 \leq k &\leq n-2, \end{aligned} \quad (1)$$

$$x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i) + \lambda, \quad m \geq 3,$$

where D_{0+}^{α} is the α -order fractional derivative, $f_1 : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ and $f_2 : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ are continuous, $\alpha_i > 0$ ($i = 1, 2, \dots, m-2$), $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1} < 1$, λ is a parameter.

Boundary value problem is an important branch of differential equation. In recent years, as a mathematical model, BVP can be processed in many engineering and scientific models. Fractional differential equations BVP has become a hot issue; see the monographs of Lakshmikantham and Vatsala [1], Rudin [2], Samko et al. [3], Agarwal et al. [4, 5], and Webb and Zima [6]. Many excellent results have been

reported; see [7–23]. In [24], El-Sayed and Bin-Taher study the following m -point BVP:

$$\begin{aligned} -x''(t) &= f(t, D_{0+}^{\alpha} x(t)), \quad 0 < t < 1, \quad 0 < \alpha < 1, \\ x(0) &= 0, \\ x(1) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \\ 0 < a < \xi_i < b < 1. \end{aligned} \quad (2)$$

By using the technical of Kolmogorov compactness criterion, they obtained the results of existence at least of one positive solution. In [25], Zhang studied the following n -order BVP:

$$\begin{aligned} -D_{0+}^{\alpha} x(t) &= q(t) f(t, x(t), \dots, x^{(n-2)}(t)), \\ 0 < t < 1, \quad \alpha &\in (n-1, n], \quad n \geq 2, \\ x(0) &= x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ x^{(n-2)}(0) &= 1. \end{aligned} \quad (3)$$

The authors obtained the existence of positive solutions. However, many aspects of the theory to the solutions of the

fractional differential equations BVP need to be explored, such as the solutions of high-order multipoint BVP.

Motivated by the work mentioned above, we focus on the existence and uniqueness of positive solutions for the nonlocal BVP (1) by the technical of a fixed point theorem of a sum operator. In [12], we study the fractional equations with two and three point boundary conditions, where $\alpha \in (1, 2)$. In [20], we study the fractional equations of with integral boundary conditions. The new features of this paper are as follows. Firstly, the order of the fractional differential equation is high order with $\alpha \in (n-1, n]$, and the boundary condition is multipoint boundary condition. Compared with [12], the BVP is more general; secondly, the boundary conditions are dependent on a parameter λ which is different from paper [20], so the properties of Green function and the construct of the operator are differential. The Green function and the operator are the key to the two BVP.

The paper is organized as follows. In Section 2, we recall some definitions and lemmas. In Section 3, the existence of positive solutions to BVP (1) is obtained. Finally, in Section 4, an illustrative example is also presented.

2. Preliminaries

We first recall four classes operator.

The basic space used in this paper is the space $C[0, 1]$. Set

$$P = \{x \in C[0, 1] \mid x(t) \geq 0, t \in [0, 1]\}. \quad (4)$$

Let $D = P$ or $D = \frac{r}{p}P$ and r be a real number with $0 \leq r < 1$. An operator $T : P \rightarrow P$ is said to be

(1) positive homogeneous if it satisfies

$$T(tx) = tTx, \quad \text{for any } t \in (0, 1), x \in P; \quad (5)$$

(2) subhomogeneous if it satisfies

$$T(tx) \geq tTx, \quad \text{for any } t \in (0, 1), x \in P; \quad (6)$$

(3) r -concave if it satisfies

$$T(tx) \geq t^r Tx, \quad \text{for any } t \in (0, 1), x \in D; \quad (7)$$

(4) $T : P \times P \rightarrow P$ is said to be a mixed monotone operator if it satisfies $T(x_1, y_1) \leq T(x_2, y_2)$ for any x_i, y_i ($i = 1, 2$) $\in P, x_1 \leq x_2, y_1 \geq y_2$.

Lemma 1 (see [26]). Let $h > \theta$ and $\beta \in (0, 1)$. $T : P \times P \rightarrow P$ is a mixed monotone operator; $S : P \rightarrow P$ is an increasing subhomogeneous operator for any $x, y \in P$, and there exists a constant $\delta_0 > 0$ such that $T(x, y) \geq \delta_0 Sx$. Assume that

(i) for any $t \in (0, 1), x, y \in P$,

$$T(tx, t^{-1}y) \geq t^\beta T(x, y), \quad (8)$$

(ii) there exists a constant $h_0 \in P_h$ such that $T(h_0, h_0) \in P_h$ and $Sh_0 \in P_h$;

Then, the operator equation $T(x, x) + Sx = x$ has a unique solution x^* in P_h and for any initial values $x_0, y_0 \in P_h$, constructing successively sequences $x_n = T(x_{n-1}, y_{n-1}) + Sx_{n-1}$, $y_n = T(y_{n-1}, x_{n-1}) + Sy_{n-1}$, $n = 1, 2, \dots$, we have $x_n \rightarrow x^*$ and $y_n \rightarrow x^*$ as $n \rightarrow \infty$.

Lemma 2. Given $\in C(0, 1) \cap L(0, 1)$, the unique solution of

$$\begin{aligned} -D_{0+}^\alpha x(t) &= f(t), \quad 0 < t < 1, \\ x(0) &= x'(0) = \dots = x^{(k)}(0) = 0, \\ x(1) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i) + \lambda, \end{aligned} \quad (9)$$

is

$$x(t) = \int_0^1 G(t, s) f(s) ds + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}, \quad (10)$$

where $n-1 < \alpha \leq n$, $n \geq 2$, $m \geq 3$, $0 \leq k \leq n-2$,

$$\begin{aligned} G(t, s) &= H(t, s) + \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \\ H(t, s) &= \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} \quad (11)$$

Proof. By using the properties of the Riemann-Liouville fractional derivative, we can easily get the conclusion, so we omit. $\square$

In the follow, we verify the properties of $G(t, s)$.

Lemma 3. For all $t, s \in (0, 1)$, $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$ and

$$\begin{aligned} \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) &\leq G(t, s) \\ &\leq t^{\alpha-1} \left[1 + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \right]. \end{aligned} \quad (12)$$

Proof. Form the definition of $G(t, s)$, we can get $G(t, s)$ is continuous. Now, we need only to prove that (12) is satisfied. For all $t, s \in (0, 1)$, we have

$$\begin{aligned} G(t, s) &= H(t, s) + \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \\ &\leq [t(1-s)]^{\alpha-1} \end{aligned}$$

$$\begin{aligned}
& + \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \\
& = t^{\alpha-1} \left[1 + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \right], \\
G(t, s) & = H(t, s) + \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \\
& \geq \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \\
& = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} t^{\alpha-1} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s).
\end{aligned} \tag{13}$$

Hence (3) is held; the proof is complete. $\square$

3. Main Results

In this section, we establish the existence and uniqueness of positive solutions results for the problems (1).

Theorem 4. Assume that $f_1 \in C[[0, 1] \times [0, \infty) \times [0, \infty), [0, \infty)]$ and $f_2 \in C[[0, 1] \times [0, \infty), [0, \infty)]$ are continuous. A function $x \in P$ is a solution of the BVP (1) if and only if it is a solution of the integral equation

$$\begin{aligned}
x(t) & = \int_0^1 G(t, s) [f_1(s, x(s), x(s)) + f_2(s, x(s))] ds \\
& + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}.
\end{aligned} \tag{14}$$

Proof. The proof of this result is quite similar to that given earlier in Lemma 3 and omitted. $\square$

Define operators T and S as follows:

$$\begin{aligned}
T(x, y)(t) & = \int_0^1 G(t, s) f_1(s, x(s), y(s)) ds, \\
(Sx)(t) & = \int_0^1 G(t, s) f_2(s, x(s)) ds \\
& + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}.
\end{aligned} \tag{15}$$

Theorem 5. Assume that

- (H₁) $f_1(t, x, y) \in C[[0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty)]$, $f_2(t, x) \in C[[0, 1] \times [0, +\infty) \rightarrow [0, +\infty)]$, f_1 and f_2 increasing in $x \in [0, +\infty)$, f decreasing in $y \in [0, +\infty)$, and there exists a constant $\delta_0 > 0$ such that $f_1(t, x, y) \geq \delta_0 f_2(t, x) + \delta_0^2(1/(1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}))$;
- (H₂) for all $\mu \in (0, 1)$, $t \in [0, 1]$, $x, y \in [0, +\infty)$, there exists a constant $\xi \in (0, 1)$ such that $f_2(t, \mu x) \geq \mu f_2(t, x)$ and $f_1(t, \mu x, \mu^{-1} y) \geq \mu^\xi f_1(t, x, y)$.

Then, for any $0 < \lambda \leq \delta_0$, BVP (1) has a unique positive solution x^* in P_h , where $h(t) = t^{\alpha-1}$, and for any $x_0, y_0 \in P_h$, construct successively the sequences

$$\begin{aligned}
x_{n+1}(t) & = \int_0^1 G(t, s) [f_1(s, x_n(s), y_n(s)) + f_2(s, x_n(s))] ds \\
& + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}, \quad n = 0, 1, 2, \dots, \\
y_{n+1}(t) & = \int_0^1 G(t, s) [f_1(s, y_n(s), x_n(s)) + f_2(s, y_n(s))] ds \\
& + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}, \quad n = 0, 1, 2, \dots,
\end{aligned} \tag{16}$$

we have $\|x_n - x^*\| \rightarrow 0$ and $\|y_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. According to Theorem 4 and the definition of operators T and S , it is straightforward to show that x is the solution of BVP (1) if and only if x solves the operator equation $x = T(x, x) + Sx$. We only need to investigate the solution of the above operator equation.

Now, we will show that $T : P \times P \rightarrow P$ and $S : P \rightarrow P$ satisfy all the assumptions of Lemma 1. To begin with, we prove that A is a mixed monotone operator. In fact, for $x_i, y_i \in P$, $i = 1, 2$ with $x_1 \geq x_2$, $y_1 \leq y_2$, we know that $x_1(t) \geq x_2(t)$ and $y_1(t) \leq y_2(t)$ for all $t \in [0, 1]$. According to (H₁) and Lemma 3, we get

$$f_1(s, x_1(s), y_1(s)) \geq f_1(s, x_2(s), y_2(s)), \tag{17}$$

since $G(t, s) > 0$, that is, $T(x_1, y_1) \geq T(x_2, y_2)$. Furthermore, an argument similar to the one used in T shows that S is increasing. Next we prove that T satisfies (8). For any $\mu, \xi \in (0, 1)$ and $x, y \in P$, from

$$f_1(s, \mu x(s), \mu^{-1} y(s)) \geq \mu^\xi f_1(s, x(s), y(s)) \tag{18}$$

we have $T(\mu x, \mu^{-1} y) \geq \mu^\xi T(x, y)$ for $\mu, \xi \in (0, 1)$, $x, y \in P$. Hence (8) holds. Also, for any $\mu \in (0, 1)$, $x \in P$, taking (H₂) into consideration, we have

$$\begin{aligned}
S(\mu x)(t) & = \int_0^1 G(t, s) f_2(s, \mu x(s)) ds \\
& + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \\
& \geq \mu \int_0^1 G(t, s) f_2(s, x(s)) ds \\
& + \mu \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} = \mu (Sx)(t).
\end{aligned} \tag{19}$$

Thus, $S(\mu x) \geq \mu S(x)$ for any $\mu \in (0, 1)$, $x \in P$. Hence the operator S is subhomogeneous. Now, we are in the position

to show that $T(h, h) \in P_h$ and $Sh \in P_h$. It follows from (H_1) , (H_2) and Lemma 3 that, for any $t \in [0, 1]$,

$$\begin{aligned} T(h, h)(t) &= \int_0^1 G(t, s) f_1(s, h(s), h(s)) ds \leq h(t) \\ &\cdot \int_0^1 \left[1 + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \right] \\ &\cdot f_1(s, 0, 1) ds = a_2 h(t), \end{aligned} \quad (20)$$

where

$$\begin{aligned} a_2 &:= \int_0^1 \left[1 + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \right] \\ &\cdot f_1(s, 0, 1) ds, \\ T(h, h)(t) &= \int_0^1 G(t, s) f_1(s, h(s), h(s)) ds \\ &\geq \frac{h(t)}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \int_0^1 \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) f_1(s, 1, 0) ds \\ &= a_1 h(t), \end{aligned} \quad (21)$$

where

$$a_1 := \int_0^1 \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) f_1(s, 1, 0) ds, \quad (22)$$

Thanks to (H_1) , (H_2) and (H_3) , we get

$$\begin{aligned} f_1(s, 1, 0) &\geq f_1(s, 0, 1) \\ &\geq \delta_0 f_2(s, 0) + \delta_0^2 \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} > 0. \end{aligned} \quad (23)$$

So

$$\begin{aligned} \int_0^1 f_1(s, 1, 0) ds &\geq \delta_0 \int_0^1 f_2(s, 0) ds \\ &+ \delta_0^2 \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} > 0, \end{aligned} \quad (24)$$

and, in consequence, $a_1 > 0$ and $a_2 > 0$. Thus, for $t \in [0, 1]$, inequality $a_1 h(t) \leq T(h, h)(t) \leq a_2 h(t)$ holds, and then we get $T(h, h) \in P_h$. An argument similar to the one used in T shows that

$$\begin{aligned} Sh(t) &= \int_0^1 G(t, s) f_2(s, h(s)) ds + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \\ &\leq h(t) \left\{ \int_0^1 \left[1 + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \right] \right. \\ &\quad \cdot f_2(s, 0) ds + \frac{\delta_0}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \left. \right\} = b_2 h(t), \end{aligned} \quad (25)$$

where

$$\begin{aligned} b_2 &:= \int_0^1 \left[1 + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) \right] \\ &\cdot f_2(s, 0) ds + \frac{\delta_0}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}, \\ Sh(t) &= \int_0^1 G(t, s) f_2(s, h(s)) ds + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \\ &\geq h(t) \left\{ \int_0^1 \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \right. \\ &\quad \cdot \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) f_2(s, 1) ds \left. \right\} = b_1 h(t), \end{aligned} \quad (26)$$

where

$$b_1 := \int_0^1 \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \sum_{i=1}^{m-2} \alpha_i H(\xi_i, s) f_2(s, 1) ds. \quad (27)$$

Obviously, $b_1 > 0$ and $b_2 > 0$; thus we are led to the conclusion that $b_1 h(t) \leq Sh(t) \leq b_2 h(t)$ with $t \in [0, 1]$, i.e., $Sh \in P_h$. Hence condition (i) of Lemma 1 is proved. It remains to show that condition (ii) of Lemma 1 is satisfied. For any $x, y \in P$, $t \in [0, 1]$, from (H_3) we have

$$\begin{aligned} T(x, y)(t) &= \int_0^1 G(t, s) f_1(s, x(s), y(s)) ds \\ &\geq \delta_0 \int_0^1 G(t, s) \left[f_2(s, x(s)) + \frac{\delta_0}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \right] ds \\ &\geq \delta_0 (Sx)(t). \end{aligned} \quad (28)$$

That is, for any $x, y \in P$, inequality $T(x, y) \geq \delta_0 Sx$ holds. Applying Lemma 2, we can get the conclusion of Theorem 5. $\square$

As a special case of Theorem 5, we obtain the following results.

Corollary 6. Assume that

- $(H_1)'$ $f(t, x, y) \in C[[0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty)]$ is increasing in $x \in [0, +\infty)$, decreasing in $y \in [0, +\infty)$ and $f(t, 0, 1) \neq 0$;
- $(H_2)'$ for all $t \in [0, 1]$, $\mu \in (0, 1)$, $x, y \in [0, +\infty)$, there exists a constant $\xi \in (0, 1)$ such that $f(t, \mu x, \mu^{-1} y) \geq \mu^\xi f(t, x, y)$.

Then, the problem

$$\begin{aligned} -D_{0+}^\alpha x(t) &= f(t, x(t), x(t)), \\ 0 < t < 1, \quad n-1 < \alpha \leq n, \quad n \geq 2, \end{aligned}$$

$$\begin{aligned}
x(0) &= x^{(l)}(0) = \cdots = x^{(k)}(0) = 0, \\
0 &\leq k \leq n-2, \\
x(1) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i) + \lambda, \quad m \geq 3
\end{aligned} \tag{29}$$

has a unique positive solution x^* in P_h , where $h(t) = t^{\alpha-1}$ and, for any $x_0, y_0 \in P_h$, constructing successively the sequences

$$\begin{aligned}
x_{n+1}(t) &= \int_0^1 G(t, s) f(s, x_n(s), y_n(s)) ds \\
&\quad + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}, \quad n = 0, 1, 2, \dots, \\
y_{n+1}(t) &= \int_0^1 G(t, s) f(s, y_n(s), x_n(s)) ds \\
&\quad + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}, \quad n = 0, 1, 2, \dots,
\end{aligned} \tag{30}$$

we have $\|x_n - x^*\| \rightarrow 0$ and $\|y_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Letting $g \equiv 0$ in Theorem 5, by suitable modification to the proof of Theorem 5, we can prove the desired conclusion. $\square$

If $T : P \times P \rightarrow P$ is a mixed monotone operator and satisfies

$$T(tx, t^{-1}y) \geq tT(x, y), \quad \forall t \in (0, 1), \quad x, y \in P. \tag{31}$$

$S : P \rightarrow P$ is an increasing α -concave operator. We can also obtain the following result.

Theorem 7. Assume that

- (H₃) $f_1(t, x, y) \in C[[0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty)]$, $f_2(t, x) \in C[[0, 1] \times [0, +\infty) \rightarrow [0, +\infty)]$, f_1 and f_2 increasing in $x \in [0, +\infty)$, f decreasing in $y \in [0, +\infty)$, and there exists a constant $\delta_0 > 0$ such that $f_1(t, x, y) \leq \delta_0 f_2(t, x) + \delta_0^2(1/(1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}))$;
- (H₄) for all $t \in [0, 1]$, $\mu \in (0, 1)$, $x, y \in [0, +\infty)$, there exists a constant $\xi \in (0, 1)$ such that $f_2(t, \mu x) \geq \mu^\xi f_2(t, x)$ and $f_1(t, \mu x, \mu^{-1}y) \geq \mu f_1(t, x, y)$.

Then, BVP (1) has a unique positive solution x^* in P_h , where $h(t) = t^{\alpha-1}$ and, for any $x_0, y_0 \in P_h$, constructing successively the sequences

$$\begin{aligned}
x_{n+1}(t) &= \int_0^1 G(t, s) [f_1(s, x_n(s), y_n(s)) + f_2(s, x_n(s))] ds \\
&\quad + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}, \quad n = 0, 1, 2, \dots,
\end{aligned}$$

$$\begin{aligned}
y_{n+1}(t) &= \int_0^1 G(t, s) [f_1(s, y_n(s), x_n(s)) + f_2(s, y_n(s))] ds \\
&\quad + \lambda \frac{t^{\alpha-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}}, \quad n = 0, 1, 2, \dots,
\end{aligned} \tag{32}$$

we have $\|x_n - x^*\| \rightarrow 0$ and $\|y_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. In the light of [26, Theorem 2.4], a similar argument used in Theorem 5. shows that the desired result. We omit it. $\square$

4. Example

In this section, we give one example to illustrate our results.

Example 1. Consider the BVP (1). Letting $\alpha = 5/2$, $\alpha_1 = 1/2$, $\alpha_2 = 1/4$, $\xi_1 = 1/16$, $\xi_2 = 1/4$, $f_1(t, x, y) = x^{1/2} + y^{-1/2} + t^2 + 8/123$ and $f_2(t, x) = x^{1/2} + t^3$. A simple calculation gives $\sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1} = 5/128 < 1$. It is not difficult to verify that (H₁) and (H₂) are satisfied. Taking $\delta_0 = 1/4 > 0$, we have

$$\begin{aligned}
f_1(t, x, y) &= x^{1/2} + y^{-1/2} + t^2 + \frac{8}{123} \\
&\geq x^{1/2} + t^3 + \frac{8}{123} \\
&\geq \delta_0 \left(x^{1/2} + t^3 + \delta_0 \frac{128}{123} \right) \\
&= \delta_0 \left[f_2(t, x) + \delta_0 \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-1}} \right].
\end{aligned} \tag{33}$$

Thus, (H₃) is satisfied. Moreover, for any $\mu \in (0, 1)$, $t \in [0, 1]$, $x \in [0, \infty)$, $y \in [0, \infty)$, we have

$$\begin{aligned}
f_2(t, \mu x) &= \mu^{1/2} x^{1/2} + t^3 \geq \mu^{1/2} (x^{1/2} + t^3) \\
&\geq \mu f_2(t, x), \\
f_1(t, \mu x, \mu^{-1}y) &= \mu^{1/2} x^{1/2} + \mu^{1/2} y^{-1/2} + t^2 + \frac{8}{123} \\
&\geq \mu^{1/2} \left(x^{1/2} + y^{-1/2} + t^2 + \frac{8}{123} \right) \\
&= \mu^\gamma f_1(t, x, y),
\end{aligned} \tag{34}$$

where $\gamma = 1/2$. We conclude that condition (H₄) is satisfied. Therefore, Theorem 5 ensures that the BVP (1) has a unique positive solution in P_h with $h(t) = t^{3/2}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors typed, read, and approved the final manuscript.

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