Hindawi
Journal of Function Spaces
Volume 2018, Article ID 3048502, 7 pages
https://doi.org/10.1155/2018/3048502

Research Article

Hausdorff Operators on Modulation Spaces $M^s_{p,p}$

Guoping Zhao [1] and Qiong Lou [2]

[1] *School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China*
[2] *Department of Mathematics, Zhejiang University of Science and Technology, Hangzhou 310023, China*

Correspondence should be addressed to Guoping Zhao; guopingzhaomath@gmail.com

Received 10 October 2017; Accepted 28 December 2017; Published 22 February 2018

Academic Editor: Alberto Fiorenza

The sharp conditions are given for the boundedness of Hausdorff operators on modulation spaces with potential. By this, we extend some previous results in this topic.

1. Introduction and Preliminary

As we know, Hausdorff operator has a deep root in the study of the Fourier analysis and it has a long history in the study of real and complex analysis. The reader is referred to [1, 2] for a survey of some historic background and recent developments on Hausdorff operator.

The Hausdorff operator H_Φ can be defined for a suitable function Φ by

$$H_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(y) f\left(\frac{x}{|y|}\right) dy, \tag{1}$$

where the above integral makes sense for f belongs to some classes of nice functions. There are researches concerning the boundedness of Hausdorff operators on function spaces. Among them, the sharp conditions for the boundedness of Hausdorff operator can be characterized in only few cases. One can see [3, 4] for the characterization for the boundedness of Hausdorff operators on Lebesgue spaces, and see [5, 6] for that on Hardy spaces H^1 and h^1. There are some other results about the boundedness of Hausdorff operators on function spaces (see [7–10]).

The modulation spaces $M^s_{p,q}$, first introduced by Feichtinger [11] in 1983, are closely related to the topic of time-frequency analysis (see [12]). As function spaces associated with the uniform decomposition on frequency (see [13]), they have been regarded as appropriate function spaces for the study of partial differential equations (see [14]). We refer the reader to [15] for some motivations and historical remarks. One is also referred to our recent papers [16–18] for the properties of modulation spaces.

As a frequency decomposition space, the norm of f in the modulation space cannot be completely determined by the absolute value of the function. On the other hand, the scaling property of modulation spaces is not as simple as that of L^p (see [19]). Based on the above two observations, the characterization of bounded Hausdorff operator on modulation space is quite different from that on Lebesgue space. By introducing new technique, we established the sharp conditions for the boundedness of H_Φ on $M_{p,q}$ (see [20]).

Note that in [20], only the modulation spaces without potential were taken into consideration. Since the Hausdorff operator is not an operator of convolution type, and the dilation properties of modulation spaces are more complicated with potential, it is quite interesting to establish the sharp conditions for the boundedness of Hausdorff operator on modulation spaces with potential. As in the simple case in [20], the classical method used for the Lebesgue space is also not applicable here. As we will see, the potential index s plays important roles not only in the dilation property of $M^s_{p,q}$, but also in the boundedness of Hausdorff operator on $M^s_{p,q}$. Therefore, the method in [20] must be modified to cope with this new situation.

Remark 1 (basic assumptions on Φ). In order to establish the sharp conditions for the boundedness of H_Φ, we need to add some suitable assumptions on Φ.

Firstly, we assume $\Phi \geqslant 0$ in this article. To make the lower bound for H_Φ, this assumption appears in most of the previous papers which consider the characterizations for the boundedness of Hausdorff operator (see [4–6]).

Secondly, we make another assumption for Φ as follows:

$$\int_{B(0,1)} |y|^n \Phi(y)\, dy < \infty, \qquad \int_{B(0,1)^c} \Phi(y)\, dy < \infty. \tag{2}$$

In fact, assumption (2) is the weakest conditions to ensure that the Schwartz function can be mapped into tempered distribution by Hausdorff operator H_Φ; one can see [20] for details. We also remark that (2) can be deduced by the boundedness of H_Φ on modulation spaces, since $\mathcal{S} \subset M^s_{p,p} \subset \mathcal{S}'$.

We turn to give the definition of modulation space.

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. For $f \in \mathcal{S}(\mathbb{R}^n)$, we define the Fourier transform $\mathcal{F} f$ and the inverse Fourier transform $\mathcal{F}^{-1} f$ by

$$\mathcal{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x\cdot\xi} dx, \qquad \mathcal{F}^{-1} f(x) = \check{f}(x) = \widehat{f}(-x) = \int_{\mathbb{R}^n} f(\xi)\, e^{2\pi i x\cdot\xi} d\xi. \tag{3}$$

Definition 2. Let $s \in \mathbb{R}$, $0 < p,q \leqslant \infty$. The weighted Lebesgue space $L^s_{x,p}$ consists of all measurable functions f such that

$$\|f\|_{L^s_{x,p}} = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p \langle x\rangle^{ps} dx\right)^{1/p}, & p < \infty \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)\langle x\rangle^s|, & p = \infty \end{cases} \tag{4}$$

is finite, where $\langle x\rangle := (1+|x|^2)^{1/2}$. We write L^s_p for short if there is no confusion. If $s = 0$, we denote $L_p := L^0_p$ and $\|\cdot\|_p := \|\cdot\|_{L_p}$ for brevity.

The translation operator is defined as $T_{x_0} f(x) := f(x - x_0)$ and the modulation operator is defined as $M_\xi f(x) := e^{2\pi i \xi\cdot x} f(x)$, for $x, x_0, \xi \in \mathbb{R}^n$. Fixing a nonzero function $\phi \in \mathcal{S}$, the short-time Fourier transform of $f \in \mathcal{S}'$ with respect to the window ϕ is given by

$$V_\phi f(x,\xi) := \left\langle f, M_\xi T_x \phi \right\rangle, \tag{5}$$

and that can be written as

$$V_\phi f(x,\xi) = \int_{\mathbb{R}^n} f(y)\, \overline{\phi(y-x)} e^{-2\pi i y\cdot\xi} dy \tag{6}$$

if $f \in \mathcal{S}$. We give the definitions of weighted modulation space $M^{s,t}_{p,q}$.

Definition 3. Let $s,t \in \mathbb{R}$, $0 < p,q \leqslant \infty$. The weighted modulation space $\mathcal{M}^{s,t}_{p,q}$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the (quasi-)norm

$$\|f\|_{\mathcal{M}^{s,t}_{p,q}} := \left\| \left\| V_\phi f(x,\xi) \right\|_{L^t_{x,p}} \right\|_{L^s_{\xi,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| V_\phi f(x,\xi) \right|^p \langle x\rangle^{tp} dx \right)^{q/p} \langle \xi\rangle^{sq} d\xi \right)^{1/q} \tag{7}$$

is finite, with the usual modifications when $p = \infty$ or $q = \infty$. This definition is independent of the choice of the window ϕ.

In particular, for $t = 0$, the modulation space (continuous version) $\mathcal{M}^s_{p,q}$ is defined as follows.

Definition 4. Let $s \in \mathbb{R}$, $0 < p,q \leqslant \infty$. The modulation space $\mathcal{M}^s_{p,q}$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the (quasi-) norm

$$\|f\|_{\mathcal{M}^s_{p,q}} := \left\| \left\| V_\phi f(x,\xi) \right\|_{L_{x,p}} \right\|_{L^s_{\xi,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| V_\phi f(x,\xi) \right|^p dx \right)^{q/p} \langle \xi\rangle^{sq} d\xi \right)^{1/q} \tag{8}$$

is finite, with the usual modifications when $p = \infty$ or $q = \infty$. This definition is independent of the choice of the window $\phi \in \mathcal{S}$.

Applying the frequency-uniform localization techniques, one can give an alternative definition of modulation spaces (see [13, 21] for details).

Let Q_k be the unit cube with the center at k. Then the family $\{Q_k\}_{k\in\mathbb{Z}^n}$ constitutes a decomposition of $\mathbb{R}^n$. Let $\eta : \mathbb{R}^n \to [0,1]$ be a smooth function satisfying that $\eta(\xi) = 1$ for $\xi \in Q_0$ and $\eta(\xi) = 0$ for $\xi \notin (3/2)Q_0$. Denote the translation of η by

$$\eta_k(\xi) = \eta(\xi - k), \quad k \in \mathbb{Z}^n. \tag{9}$$

Since $\eta_k(\xi) = 1$ in Q_k, we have that $\sum_{k\in\mathbb{Z}^n} \eta_k(\xi) \geqslant 1$ for all $\xi \in \mathbb{R}^n$. Denote

$$\sigma_k(\xi) = \eta_k(\xi) \left(\sum_{l\in\mathbb{Z}^n} \eta_l(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n. \tag{10}$$

It is easy to know that $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ constitutes a smooth partition of the unity, and $\sigma_k(\xi) = \sigma(\xi - k)$. The frequency-uniform decomposition operators can be defined by

$$\Box_k := \mathcal{F}^{-1} \sigma_k \mathcal{F} \tag{11}$$

for $k \in \mathbb{Z}^n$. Now, we give the (discrete) definition of modulation space $M^s_{p,q}$.

Definition 5. Let $s \in \mathbb{R}$, $0 < p,q \leqslant \infty$. The modulation space $M^s_{p,q}$ consists of all $f \in \mathcal{S}'$ such that the (quasi-) norm

$$\|f\|_{M^s_{p,q}} := \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \left\| \Box_k f \right\|_p^q \right)^{1/q} \tag{12}$$

is finite. We write $M_{p,q} := M^0_{p,q}$ for short. We also recall that this definition is independent of the choice of $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ and the definitions of $\mathscr{M}^s_{p,q}$ and $M^s_{p,q}$ are equivalent [21] when $0 < p < \infty$ and $0 < q \leqslant \infty$.

Now, we state our main result as follows.

Theorem 6. *Let $1 \leqslant p \leqslant 2$, $s \geqslant 0$ and Φ be a nonnegative function. Then H_Φ is bounded on $M^s_{p,p}$ if and only if*

$$\int_{\mathbb{R}^n} \left(|y|^{n/p} + |y|^{n/p'}\right) \cdot \left(1 + |y|^{-s}\right) \Phi(y)\, dy < \infty. \tag{13}$$

In Section 2, we will collect some basic properties of modulation spaces. The proof of Theorem 6 will be also given in Section 2.

Throughout this paper, we will adopt the following notations. We use $X \lesssim Y$ to denote the statement that $X \leqslant CY$, with a positive constant C that may depend on n, p, but it might be different from line to line. The notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$. We use $X \lesssim_\lambda Y$ to denote $X \leqslant C_\lambda Y$, meaning that the implied constant C_λ depends on the parameter λ. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we denote $\langle x\rangle := (1 + |x|^2)^{1/2}$.

2. Proof of the Main Theorem

Lemma 7 ([20] the Fourier transform of H_Φ). *Let Φ be a nonnegative function satisfying the basic assumption (2). For $f, g \in \mathcal{S}(\mathbb{R}^n)$, define $\widetilde{H_\Phi}$*

$$\widetilde{H_\Phi} f(x) = \int_{\mathbb{R}^n} \Phi(y)\, |y|^n f(|y|\, x)\, dy. \tag{14}$$

Then

(1) *$\widetilde{H_\Phi} f$ is a tempered distribution and that the map $\widetilde{H_\Phi} : \mathcal{S} \to \mathcal{S}'$ is continuous.*

(2) *$\widehat{H_\Phi f} = \widetilde{H_\Phi} \widehat{f}$ in the distribution sense.*

Lemma 8 ([22] weight lifting). *For any $s, t, t_0 \in \mathbb{R}$, $p, q \in [1, \infty]$, the map*

$$f \longmapsto \langle\cdot\rangle^{t_0} f \tag{15}$$

is a bijection from $\mathscr{M}^{s,t+t_0}_{p,q}$ to $\mathscr{M}^{s,t}_{p,q}$.

Lemma 9 ([22] potential lifting). *For any $t, s, s_0 \in \mathbb{R}$, $p, q \in [1, \infty]$, the map*

$$f \longmapsto \mathscr{F}^{-1}\left(\langle\cdot\rangle^{s_0} \mathscr{F}(f)\right) \tag{16}$$

is a bijection from $\mathscr{M}^{s+s_0,t}_{p,q}$ to $\mathscr{M}^{s,t}_{p,q}$.

Lemma 10 (symmetry of time and frequency). $\|\mathscr{F}^{-1} f\|_{M_{p,p}} \sim \|\mathscr{F} f\|_{M_{p,p}}$.

Proof. By the fact that

$$\left|V_\phi f(x, \xi)\right| = \left|V_{\widehat{\phi}} \widehat{f}(\xi, -x)\right|, \tag{17}$$

the conclusion follows by the definition of modulation space. □

Lemma 11 ([19] dilation property of weighted modulation space). *Let $1 \leqslant p \leqslant \infty$. Set $f_\lambda(x) = f(\lambda x)$. Then*

$$\begin{aligned}\|f_\lambda\|_{M^s_{p,p}} &\lesssim \max\left\{\lambda^{-n/p}, \lambda^{-n/p'}\right\} \cdot \max\{1, \lambda^s\} \\ &\quad \cdot \|f\|_{M^s_{p,p}}.\end{aligned} \tag{18}$$

Lemma 12 (embedding relations between modulation and Lebesgue spaces). *The following embedding relations are right:*

(1) *$M^s_{p,p} \hookrightarrow L^p$ for $1 \leqslant p \leqslant 2$ and $s \geqslant 0$;*

(2) *$\mathscr{M}^{0,s}_{p,p} \hookrightarrow L^s_p$ for $1 \leqslant p \leqslant 2$.*

Proof. The first relationship can be obtained by $M^s_{p,p} \hookrightarrow M_{p,p} \hookrightarrow L^p$, where the $M_{p,p} \hookrightarrow L^p$ is a known inclusion relation (see [23]).

For the second one, recalling the definition of L^s_p and the fact $M_{p,p} \hookrightarrow L^p$ with $1 \leqslant p \leqslant 2$, we use Lemma 8 to deduce that

$$\|f\|_{L^s_p} = \|\langle\cdot\rangle^s f(\cdot)\|_{L^p} \lesssim \|\langle\cdot\rangle^s f(\cdot)\|_{M_{p,p}} \sim \|f\|_{\mathscr{M}^{0,s}_{p,p}}. \tag{19}$$

□

In order to make the proof more clear, we give the following two technical propositions.

Proposition 13 (for technique). *Let $1 \leqslant p \leqslant 2$, $s \geqslant 0$ and Φ be a nonnegative function. Then*

(1) *if $H_\Phi : M^s_{p,p} \to L^p$ is bounded, one has*

$$\int_{\mathbb{R}^n} |y|^{n/p} \Phi(y)\, dy < \infty; \tag{20}$$

(2) *if $\widetilde{H_\Phi} : \mathscr{M}^{0,s}_{p,p} \to L^s_p$ is bounded, one has*

$$\int_{\mathbb{R}^n} |y|^{n/p'-s} \Phi(y)\, dy < \infty. \tag{21}$$

Proof. We first give the proof of statement (1). Suppose $H_\Phi : M^s_{p,p} \to L^p$ is bounded. Let $\psi : \mathbb{R}^n \to [0,1]$ be a smooth bump function supported in the ball $\{\xi : |\xi| < 3/2\}$ and be equal to 1 on the ball $\{\xi : |\xi| \leqslant 4/3\}$. Let $\rho(\xi) = \psi(\xi) - \psi(2\xi)$. Then ρ is a positive smooth function supported in the annulus $\{\xi : 2/3 < |\xi| < 3/2\}$, satisfying $\rho(\xi) = 1$ on a smaller annulus $\{\xi : 3/4 \leqslant |\xi| \leqslant 4/3\}$. Denote $\rho_j(\xi) := \rho(\xi/2^j)$. We have $\operatorname{supp} \rho_j \subset \{\xi : 2/3 \cdot 2^j \leqslant |\xi| \leqslant 3/2 \cdot 2^j\}$ and $\rho_j(\xi) = 1$ on $\{\xi : 3/4 \cdot 2^j \leqslant |\xi| \leqslant 4/3 \cdot 2^j\}$. Thus, we have $\operatorname{supp} \sum_{j=1}^N \rho_j(\xi) \subset \{\xi : 4/3 \leqslant |\xi| \leqslant 3/2 \cdot 2^N\}$ and $\sum_{j=1}^N \rho_j(\xi) = 1$ on $\{\xi : 3/2 \leqslant |\xi| \leqslant 4/3 \cdot 2^N\}$. Take φ to be a nonnegative smooth function satisfying that $\operatorname{supp} \widehat{\varphi} \subset B(0, 1/2)$, $\varphi(0) = 1$. Choose $f_N(x) = (\sum_{j=0}^{N+1} \rho_j(x) \cdot |x|^{-n/p}) * \varphi$. Therefore, we have

$$\begin{aligned}&\operatorname{supp} \widehat{f_N} \subset B\left(0, \frac{1}{2}\right), \\ &f_N(x) \gtrsim \sum_{j=1}^{N} \rho_j(x) \cdot |x|^{-n/p}.\end{aligned} \tag{22}$$

Indeed, the previous inclusion relation of (22) follows from the support condition of $\widehat{\varphi}$. To verify the latter inequality, we only need to prove it when the right hand is nonzero; that is, $x \in \{4/3 \leqslant |x| \leqslant 3/2 \cdot 2^N\}$. For the nonnegative function φ satisfying $\varphi(0) = 1$, there exists a positive constant $\delta < 1/12$, such that $\varphi(x) > 1/2$ when $|x| < \delta$. By the triangle inequality and the properties of φ we have that

$$
\begin{aligned}
f_N(x) &= \left(\sum_{j=0}^{N+1} \rho_j(x) \cdot |x|^{-n/p}\right) * \varphi \\
&= \int_{\mathbb{R}^n} \left(\sum_{j=0}^{N+1} \rho_j(x-y) \cdot |x-y|^{-n/p}\right) \varphi(y)\, dy \\
&\gtrsim \int_{\substack{4/3 \leq |x-y| \leq 3/2 \cdot 2^N \\ |y| < \delta}} \sum_{j=0}^{N+1} \rho_j(x-y) \cdot |x|^{-n/p}\, dy \\
&\gtrsim \sum_{j=1}^{N} \rho_j(x) \cdot |x|^{-n/p},
\end{aligned} \tag{23}
$$

so we prove (22). Recalling that Φ is nonnegative, for $0 < M \ll N$, we have the following estimate:

$$
\begin{aligned}
\|H_\Phi f_N\|_{L^p} &= \left\| \int_{\mathbb{R}^n} \Phi(y) f_N\left(\frac{x}{|y|}\right) dy \right\|_{L^p} \\
&\gtrsim \left\| \int_{\mathbb{R}^n} \Phi(y) |y|^{n/p} \cdot \sum_{j=1}^{N} \rho_j\left(\frac{x}{|y|}\right) \cdot |x|^{-n/p}\, dy \right\|_{L^p} \\
&\geqslant \left\| \int_{B(0,2/3\cdot 2^M)\setminus B(0,3/4\cdot 2^{-M})} \Phi(y) |y|^{n/p} \right. \\
&\quad \left. \cdot \sum_{j=1}^{N} \rho_j\left(\frac{x}{|y|}\right) \cdot |x|^{-n/p}\, dy \right\|_{L^p} \\
&\geqslant \left\| \int_{B(0,2/3\cdot 2^M)\setminus B(0,3/4\cdot 2^{-M})} \Phi(y) |y|^{n/p} \right. \\
&\quad \left. \cdot \chi_{\{2^M < |x| < 2^{N-M}\}}(x) \cdot |x|^{-n/p}\, dy \right\|_{L^p} \\
&= \int_{B(0,2/3\cdot 2^M)\setminus B(0,3/4\cdot 2^{-M})} \Phi(y) |y|^{n/p}\, dy \\
&\quad \cdot \left\| |x|^{-n/p} \chi_{\{2^M < |x| < 2^{N-M}\}}(x) \right\|_{L^p} \\
&\gtrsim \int_{B(0,2/3\cdot 2^M)\setminus B(0,3/4\cdot 2^{-M})} \Phi(y) |y|^{n/p}\, dy \\
&\quad \cdot \left(\lg 2^{N-2M}\right)^{1/p},
\end{aligned} \tag{24}
$$

where we use the fact that $\sum_{j=1}^{N} \rho_j(x/|y|) = 1$ for $y \in B(0, 2/3 \cdot 2^M) \setminus B(0, 3/4 \cdot 2^{-M})$ and $x \in B(0, 2^{N-M}) \setminus B(0, 2^M)$. On the other hand, observing that $\operatorname{supp}\widehat{f_N} \subset B(0, 1/2)$, so $\#\{k : \sigma_k \widehat{f_N} \neq 0\} \lesssim 1$ for all N and $|l| \lesssim 1$ if $l \in \{k : \sigma_k \widehat{f_N} \neq 0\}$. And then we have that

$$
\begin{aligned}
\|f_N\|_{M^s_{p,p}} &= \left(\sum_{\substack{\sigma_k \widehat{f_N} \neq 0 \\ k \in \mathbb{Z}^n}} \langle k \rangle^{sp} \left\| \mathscr{F}^{-1}\left(\sigma_k \widehat{f_N}\right) \right\|_{L^p}^p \right)^{1/p} \\
&\lesssim \left(\sum_{\substack{\sigma_k \widehat{f_N} \neq 0 \\ k \in \mathbb{Z}^n}} \langle k \rangle^{sp} \|f_N\|_{L^p}^p \right)^{1/p} \lesssim \|f_N\|_{L^p} \\
&\lesssim \left\| \sum_{j=0}^{N+1} \rho_j(x) \cdot |x|^{-n/p} \right\|_{L^p} \sim \left(\ln 2^N\right)^{1/p}.
\end{aligned} \tag{25}
$$

Using the boundedness of H_Φ and the above estimates for $H_\Phi f_N$ and f_N, we have that

$$
\begin{aligned}
\|H_\Phi\|_{M^s_{p,p} \to L^p} &\geqslant \frac{\|H_\Phi f_N\|_{L^p}}{\|f_N\|_{M^s_{p,p}}} \\
&\gtrsim \int_{B(0,2/3\cdot 2^M)\setminus B(0,3/4\cdot 2^{-M})} \Phi(y) |y|^{n/p}\, dy \\
&\quad \cdot \left(\frac{\lg 2^{N-2M}}{\lg 2^N}\right)^{1/p}.
\end{aligned} \tag{26}
$$

Letting $N \to \infty$, we have

$$
\int_{B(0,2/3\cdot 2^M)\setminus B(0,3/4\cdot 2^{-M})} \Phi(y) |y|^{n/p}\, dy \lesssim \|H_\Phi\|_{M^s_{p,p} \to L^p}. \tag{27}
$$

By the arbitrariness of M, we let $M \to \infty$ and obtain that $\int_{\mathbb{R}^n} \Phi(y)|y|^{n/p} dy \lesssim \|H_\Phi\|_{M^s_{p,p} \to L^p}$.

Now we turn to give the proof of (2). Suppose $\widetilde{H_\Phi} : \mathscr{M}^{0,s}_{p,p} \to L^s_p$ is bounded. As in the proof of conclusion (1), we take $g_N(x) = \sum_{j=1}^{N} \rho_j(x) \cdot |x|^{-n/p-s}$. A direction calculation yields that

$$
\begin{aligned}
\left\|\widetilde{H_\Phi} g_N\right\|_{L^s_p} &= \left\| \int_{\mathbb{R}^n} \Phi(y) |y|^n g_N(|y| x)\, dy \right\|_{L^s_p} \\
&= \left\| \int_{\mathbb{R}^n} \Phi(y) |y|^{n/p'-s} \cdot \sum_{j=1}^{N} \rho_j(|y| x) \right. \\
&\quad \left. \cdot |x|^{-n/p-s}\, dy \right\|_{L^s_p} \\
&\geqslant \left\| \int_{B(0,4/3\cdot 2^M)\setminus B(0,3/2\cdot 2^{-M})} \Phi(y) |y|^{n/p'-s} \right.
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \sum_{j=1}^{N} \rho_j\left(|y| x\right) \cdot |x|^{-n/p-s}\, dy \Bigg\|_{L_p^s} \\
&\geqslant \int_{B(0,4/3\cdot 2^M)\setminus B(0,3/2\cdot 2^{-M})} \Phi(y)\, |y|^{n/p'-s}\, dy \\
&\cdot \left\| |x|^{-n/p-s} \chi_{\{2^M<|x|<2^{N-M}\}}(x) \right\|_{L_p^s} \\
&\gtrsim \int_{B(0,4/3\cdot 2^M)\setminus B(0,3/2\cdot 2^{-M})} \Phi(y)\, |y|^{n/p'-s}\, dy \\
&\cdot \left(\lg 2^{N-2M}\right)^{1/p}.
\end{aligned} \tag{28}
$$

On the other hand,

$$
\begin{aligned}
\left\|\mathscr{F}^{-1}(\sigma_k g_N)\right\|_{L^p} &= \left\|\mathscr{F}^{-1}\left(\sigma_k \sum_{j=1}^{N} \rho_j(x) \cdot |x|^{-n/p-s}\right)\right\|_{L^p} \\
&\lesssim \langle k\rangle^{-n/p-s} \left\|\mathscr{F}^{-1}\left(\sigma_k \sum_{j=1}^{N} \rho_j(x)\right)\right\|_{L^p} \\
&\leqslant \langle k\rangle^{-n/p-s} \left\|\mathscr{F}^{-1}\sigma_k\right\|_{L^p} \\
&\quad \cdot \sum_{1\leqslant j\leqslant N:\sigma_k\rho_j\neq 0} \left\|\mathscr{F}^{-1}\left(\rho_j(x)\right)\right\|_{L^1} \\
&\lesssim \langle k\rangle^{-n/p-s}.
\end{aligned} \tag{29}
$$

Using Lemma 10, we obtain that

$$
\begin{aligned}
\|g_N\|_{\mathscr{M}_{p,p}^{0,s}} &= \left\|\langle\cdot\rangle^s g_N(\cdot)\right\|_{M_{p,p}} \\
&= \left\|\mathscr{F}^{-1}\left(\langle\xi\rangle^s g_N(\xi)\right)(\cdot)\right\|_{M_{p,p}} = \left\|\mathscr{F}^{-1} g_N\right\|_{M_{p,p}^s} \\
&= \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sp} \left\|\mathscr{F}^{-1}(\sigma_k g_N)\right\|_{L^p}^p\right)^{1/p} \\
&\lesssim \left(\sum_{\substack{|k|<2^{N+1}\\ k\in\mathbb{Z}^n}} \langle k\rangle^{-n}\right)^{1/p} \sim \left(\lg 2^N\right)^{1/p}.
\end{aligned} \tag{30}
$$

We deduce that

$$
\begin{aligned}
\left\|\widetilde{H_\Phi}\right\|_{\mathscr{M}_{p,p}^{0,s}\to L_p^s} &\geqslant \frac{\left\|\widetilde{H_\Phi} g_N\right\|_{L_p^s}}{\|g_N\|_{M_{p,p}^{0,s}}} \\
&\gtrsim \int_{B(0,4/3\cdot 2^M)\setminus B(0,3/2\cdot 2^{-M})} \Phi(y)\, |y|^{n/p'-s}\, dy \\
&\cdot \left(\frac{\lg 2^{N-2M}}{\lg 2^N}\right)^{1/p}.
\end{aligned} \tag{31}
$$

Letting $N \to \infty$, we have

$$
\begin{aligned}
&\int_{B(0,4/3\cdot 2^M)\setminus B(0,3/2\cdot 2^{-M})} \Phi(y)\, |y|^{n/p'-s}\, dy \\
&\quad \lesssim \left\|\widetilde{H_\Phi}\right\|_{\mathscr{M}_{p,p}^{0,s}\to L_p^s}.
\end{aligned} \tag{32}
$$

By the arbitrariness of M, we let $M \to \infty$ and obtain that $\int_{\mathbb{R}^n} \Phi(y)|y|^{n/p'-s} dy \lesssim \|\widetilde{H_\Phi}\|_{M_{p,p}^{0,s}\to L_p^s}$. □

Next, we establish the following two propositions for reduction.

Proposition 14 (for reduction). *Let $1 \leqslant p \leqslant 2$, $s \geqslant 0$ and Φ be a nonnegative function. If the Hausdorff operator H_Φ is bounded on $M_{p,p}^s$, one has the following:*

(1) *$H_\Phi : M_{p,p}^s \to L^p$ is bounded,*

(2) *$\widetilde{H_\Phi} : \mathscr{M}_{p,p}^{0,s} \to L_p^s$ is bounded.*

Proof. The first conclusion can be deduced by the embedding relation $M_{p,p}^s \hookrightarrow L^p$ (see Lemma 12) directly. We turn to prove the second conclusion. For any Schwartz function f, by the property of H_Φ and Lemmas 8, 9, and 10, we have

$$
\begin{aligned}
\|f\|_{M_{p,p}^s} &= \left\|\mathscr{F}^{-1}\left(\langle\xi\rangle^s \mathscr{F} f\right)\right\|_{M_{p,p}} = \left\|\langle\xi\rangle^s \mathscr{F} f\right\|_{M_{p,p}} \\
&= \|\mathscr{F} f\|_{\mathscr{M}_{p,p}^{0,s}}
\end{aligned} \tag{33}
$$

and similarly we obtain

$$
\|H_\Phi f\|_{M_{p,p}^s} = \|\mathscr{F} H_\Phi f\|_{\mathscr{M}_{p,p}^{0,s}} = \left\|\widetilde{H_\Phi} \widehat{f}\right\|_{\mathscr{M}_{p,p}^{0,s}}, \tag{34}
$$

where we use Lemma 7 in the last equality. Thus, if H_Φ is bounded on $M_{p,p}^s$, we have

$$
\left\|\widetilde{H_\Phi} \widehat{f}\right\|_{\mathscr{M}_{p,p}^{0,s}} \lesssim \left\|\widehat{f}\right\|_{\mathscr{M}_{p,p}^{0,s}}. \tag{35}
$$

Lemma 12, that is, the embedding relation $\mathscr{M}_{p,p}^{0,s} \hookrightarrow L_p^s$, then yields that

$$
\left\|\widetilde{H_\Phi} f\right\|_{L_p^s} \lesssim \|f\|_{\mathscr{M}_{p,p}^{0,s}} \tag{36}
$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. □

Now, we are ready to give the proof of Theorem 6.

Proof of Theorem 6. We divide this proof into two parts.

IF Part. Using the Minkowski inequality, we deduce that

$$
\begin{aligned}
\|H_\Phi f\|_{M_{p,p}^s} &\lesssim \left\|\int_{\mathbb{R}^n} \Phi(y)\, f\left(\frac{x}{|y|}\right) dy\right\|_{M_{p,p}^s} \\
&\lesssim \int_{\mathbb{R}^n} \Phi(y) \left\|f\left(\frac{x}{|y|}\right)\right\|_{M_{p,p}^s} dy.
\end{aligned} \tag{37}
$$

Recalling the dilation properties of modulation space (see Lemma 11), we obtain that

$$\begin{aligned}\|H_\Phi f\|_{M^s_{p,p}} &\lesssim \int_{\mathbb{R}^n}\Phi(y)\cdot\max\left\{|y|^{n/p},|y|^{n/p'}\right\}\\&\quad\cdot\max\left\{1,|y|^{-s}\right\}dy\,\|f\|_{M^s_{p,p}}\\&\lesssim \int_{\mathbb{R}^n}\left(|y|^{n/p}+|y|^{n/p'}\right)\left(1+|y|^{-s}\right)\Phi(y)\,dy\\&\quad\cdot\|f\|_{M^s_{p,p}}.\end{aligned}\tag{38}$$

This implies the boundedness of H_Φ on $M^s_{p,p}$.

ONLY IF Part. Using Proposition 14, we obtain $H_\Phi : M^s_{p,p} \to L^p$ and $\widetilde{H_\Phi} : \mathscr{M}^{0,s}_{p,p} \to L^s_p$. Next, Proposition 13 yields that

$$\begin{aligned}\int_{\mathbb{R}^n}|y|^{n/p}\Phi(y)\,dy<\infty,\\\int_{\mathbb{R}^n}|y|^{n/p'-s}\Phi(y)\,dy<\infty.\end{aligned}\tag{39}$$

Hence, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(|y|^{n/p}+|y|^{n/p'}\right)\cdot\left(1+|y|^{-s}\right)\Phi(y)\,dy\\&\quad=\left(\int_{B(0,1)}+\int_{B^c(0,1)}\right)\left(|y|^{n/p}+|y|^{n/p'}\right)\\&\qquad\cdot\left(1+|y|^{-s}\right)\Phi(y)\,dy\\&\quad\leqslant 4\int_{B(0,1)}|y|^{n/p'-s}\Phi(y)\,dy\\&\qquad+4\int_{B^c(0,1)}|y|^{n/p}\Phi(y)\,dy\\&\quad\lesssim\int_{\mathbb{R}^n}|y|^{n/p'-s}\Phi(y)\,dy+\int_{\mathbb{R}^n}|y|^{n/p}\Phi(y)\,dy\\&\quad<\infty,\end{aligned}\tag{40}$$

which is the desired conclusion. □

Remark 15. For some technical reasons, our main theorems only characterize the boundedness of Hausdorff operator on $M^s_{p,p}$ for $1 \leqslant p \leqslant 2$, $s \geqslant 0$. By dual argument, one can easily get the similar conclusion for the range $2 \leqslant p \leqslant \infty$, $s \leqslant 0$. Moreover, similar results for Wiener amalgam spaces can be also established as in [20]. Our theorem remains an open problem for the characterization of Hausdorff operator on the full range $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Foundation of China (nos. 11601456, 11371316, 11771388).

References

[1] J. Chen, D. Fan, and S. Wang, "Hausdorff operators on Euclidean spaces," *Applied Mathematics-A Journal of Chinese Universities Series B*, vol. 28, no. 4, pp. 548–564, 2013.

[2] E. Liflyand, "Hausdorff operators on Hardy spaces," *Eurasian Mathematical Journal*, vol. 4, no. 4, pp. 101–141, 2013.

[3] G. Gao and F. Zhao, "Sharp weak bounds for Hausdorff operators," *Analysis Mathematica*, vol. 41, no. 3, pp. 163–173, 2015.

[4] X. Wu and J. Chen, "Best constants for Hausdorff operators on n-dimensional product spaces," *Science China Mathematics*, vol. 57, no. 3, pp. 569–578, 2014.

[5] D. Fan and X. Lin, "Hausdorff operator on real Hardy spaces," *Analysis*, vol. 34, no. 4, pp. 319–337, 2014.

[6] J. Ruan and D. Fan, "Hausdorff operators on the power weighted Hardy spaces," *Journal of Mathematical Analysis and Applications*, vol. 433, no. 1, pp. 31–48, 2016.

[7] G. Gao, X. Wu, and W. Guo, "Some results for hausdorff operators," *Journal of Inequalities and Applications*, vol. 18, no. 1, pp. 155–168, 2015.

[8] G. Gao and Y. Zhong, "Some inequalities for hausdorff operators," *Journal of Inequalities and Applications*, vol. 17, no. 3, pp. 1061–1078, 2014.

[9] E. Liflyand and A. Miyachi, "Boundedness of the Hausdorff operators in hp spaces, 0 < p < 1," *Studia Mathematica*, vol. 194, no. 3, pp. 279–292, 2009.

[10] E. Liflyand and F. Móricz, "The Hausdorff operator is bounded on the real Hardy space H1(R)," *Proceedings of the American Mathematical Society*, vol. 128, no. 5, pp. 1391–1396, 2000.

[11] H. G. Feichtinger, "Modulation spaces on locally compact Abelian group," in *Proceedings of the Proc. Internat. Conf. on Wavelet and Applications*, pp. 99–140, New Delhi Allied Publishers, New Delhi, India, 1983.

[12] K. Gröchenig, *Foundations of Time–Frequency Analysis*, Birkhauser, Boston, Mass, USA, 2000.

[13] H. Triebel, "Modulation spaces on the euclidean n-space," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 2, no. 5, pp. 443–457, 1983.

[14] M. Ruzhansky, M. Sugimoto, and B. Wang, "Modulation spaces and nonlinear evolution equations," *Progress in Mathematics*, vol. 301, pp. 267–283, 2012.

[15] H. G. Feichtinger, "Modulation spaces: looking back and ahead," *Sampling Theory in Signal and Image Processing. An International Journal*, vol. 5, no. 2, pp. 109–140, 2006.

[16] W. Guo, D. Fan, H. Wu, and G. Zhao, "Sharpness of some properties of weighted modulation spaces," *Science China Mathematics*, vol. 59, no. 1, pp. 169–190, 2016.

[17] W. Guo, H. Wu, and G. Zhao, "Inclusion relations between modulation and Triebel-Lizorkin spaces," *Proceedings of the American Mathematical Society*, vol. 145, no. 11, pp. 4807–4820, 2017.

[18] W. Guo, D. Fan, H. Wu, and G. Zhao, "Sharp weighted convolution inequalities and some applications," *Studia Math*, In press.

[19] E. Cordero and K. A. Okoudjou, "Dilation properties for weighted modulation spaces," *Journal of Function Spaces And Applications*, vol. 2012, Article ID 145491, 29 pages, 2012.

[20] G. Zhao, D. Fan, and W. Guo, "Hausdorff operators on modulation and Wiener amalgam spaces," *Annals of Functional Analysis*, In press.

[21] B. Wang and H. Hudzik, "The global Cauchy problem for the NLS and NLKG with small rough data," *Journal of Differential Equations*, vol. 232, no. 1, pp. 36–73, 2007.

[22] J. Toft, *Convolutions and Embeddings for Weighted Modulation Spaces, Advances in Pseudo-Differential Operators*, vol. 155, Birkhäuser Basel, 2004.

[23] M. Kobayashi and M. Sugimoto, "The inclusion relation between Sobolev and modulation spaces," *Journal of Functional Analysis*, vol. 260, no. 11, pp. 3189–3208, 2011.