Research Article

C^* -Algebra-Valued G -Metric Spaces and Related Fixed-Point Theorems

Congcong Shen ^{1,2}, Lining Jiang,² and Zhenhua Ma ³

¹School of Information, Beijing Wuzi University, Beijing, China

²School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China

³Department of Mathematics and Physics, Hebei University of Architecture, Zhangjiakou 075024, China

Correspondence should be addressed to Congcong Shen; shcc881111@163.com

Received 17 April 2018; Accepted 27 May 2018; Published 12 July 2018

Academic Editor: Liguang Wang

Copyright © 2018 Congcong Shen et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We introduce the notion of the C^* -algebra-valued G -metric space. The existence and uniqueness of some fixed-point theorems for self-mappings with contractive or expansive conditions on complete C^* -algebra-valued G -metric spaces are proved. As an application, we prove the existence and uniqueness of the solution of a type of differential equations.

1. Introduction

As is known to all, the proverbial fixed-point theorem of Banach has been widely used in many branches of mathematics and physics. There are a large number of generalizations for such a theorem. In general, the theorem has been extended in two directions. On the one hand, the usual contractive condition is replaced with weakly contractive conditions [1–7]. On the other hand, the action spaces are replaced with different types of metric spaces. Particularly, in 2006, Mustafa and Sims [8] introduced the concept of generalized metric spaces (G -metric spaces). Since then, many scholars studied fixed-point theory in G -metric spaces and many meaningful results are obtained.

Let us recall the basic definitions and conclusions on G -metric spaces. Details can be seen in [8–20].

Definition 1 (see [8]). Let X be a nonempty set. Suppose that $G : X \times X \times X \rightarrow \mathbb{R}^+$ is a mapping satisfying

- (1) $G(x, y, z) = 0$ if $x = y = z$;
- (2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
- (3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
- (4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables);

- (5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized metric, or a G -metric on X . The pair (X, G) is called a G -metric space.

Definition 2 (see [8]). Let (X, G) be a G -metric space, $\{x_n\} \subseteq X$. If there exists $x \in X$, such that $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$, we say that the sequence $\{x_n\}$ is called G -convergent to x . If for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that, for all $m, n, l > N$,

$$G(x_n, x_m, x_l) < \varepsilon, \quad (1)$$

$\{x_n\}$ is called a G -Cauchy sequence.

Notice that, in a G -metric space (X, G) , the following statements are equivalent:

- (1) $\{x_n\}$ is G -convergent to x ;
- (2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$.

As we have known that C^* -algebra, which was first proposed for its use in quantum mechanics to model algebras of physical observable, is an important research field of modern mathematics [21–29]. In 1947, I. Segal [28] introduced the term “ C^* -algebra” to describe a “uniformly

closed, self-adjoint algebra of bounded operators on a Hilbert space”.

Throughout this paper, $\mathcal{A}$ will denote a unital C^* -algebra with a unit I ; namely, $\mathcal{A}$ is a unital Banach algebra with an involution $*$ such that $\|A^*A\| = \|A\|^2$ ($A \in \mathcal{A}$). Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$, then $B(\mathcal{H})$ is a C^* -algebra with the operator norm. Let $\mathcal{A}_{sa}$ be the set of all self-adjoint elements in $\mathcal{A}$, and define the spectrum of $A \in \mathcal{A}$ to be the set $\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is not invertible}\}$. An element $A \in \mathcal{A}$ is positive (denoted by $A \geq \theta$) if $A \in \mathcal{A}_{sa}$ and $\sigma(A) \subseteq \mathbb{R}_+$, set $\mathcal{A}_+ = \{A \in \mathcal{A} \mid A \geq \theta\}$, then $\mathcal{A}_+ = \{A^*A \mid A \in \mathcal{A}\}$ [27]. Using the positive element, one can define a partial ordering “ $\leq$ ” on $\mathcal{A}_{sa}$ as follows: $A \leq B$ if and only if $B - A \geq \theta$. It is clear that if $A, B \in \mathcal{A}_{sa}$ and $C \in \mathcal{A}$, then $A \leq B \implies C^*AC \leq C^*BC$, and that if $A, B \in \mathcal{A}_+$ are invertible, then $A \leq B \implies \theta \leq B^{-1} \leq A^{-1}$.

Using the partial ordering “ $\leq$ ” on $\mathcal{A}_{sa}$, Ma introduced the notion of C^* -algebra-valued metric spaces and gave the fixed-point theory for contractive or expansion mapping on such a space [30, 31]. Let us recall the definition first.

Definition 3 (see [30]). Let X be a nonempty set. If $d : X \times X \longrightarrow \mathcal{A}_+$ is a mapping satisfying

- (1) $d(x, y) = \theta \iff x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$,

then d is called a C^* -algebra-valued metric on X . $(X, \mathcal{A}, d)$ is called a C^* -algebra-valued metric space.

Definition 4 (see [30]). Let $(X, \mathcal{A}, d)$ be a C^* -algebra-valued metric space, $\{x_n\} \subset X$, $x \in X$. If for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that for all $n > N$, $\|d(x_n, x)\| < \varepsilon$, then we say this $\{x_n\}$ converges to x . We denote it by $\lim_{n \rightarrow \infty} x_n = x$.

If for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that, for all $m, n > N$, $\|d(x_m, x_n)\| < \varepsilon$, then we say $\{x_n\}$ is a Cauchy sequence with respect to $\mathcal{A}$. We say that $(X, \mathcal{A}, d)$ is a complete C^* -algebra-valued metric space if every Cauchy sequence with respect to $\mathcal{A}$ is convergent.

Definition 5 (see [30]). Let $(X, \mathcal{A}, d)$ be a C^* -algebra-valued metric space. We call a mapping $T : X \longrightarrow X$ is a contractive mapping on $(X, \mathcal{A}, d)$ if there exists $A \in \mathcal{A}$ with $\|A\| < 1$ such that $d(Tx, Ty) \leq A^*d(x, y)A$, $\forall x, y \in X$.

Theorem 6 (see [30]). *If $(X, \mathcal{A}, d)$ is a complete C^* -algebra-valued metric space and T is a contractive mapping, there exists a unique fixed point in X .*

In this paper, we will define C^* -algebra-valued G -metric spaces and prove some fixed point theorems on such spaces. We also provide an application of the theory for a type of differential equations.

2. Main Results

In this section, we first give the definition of a C^* -algebra-valued G -metric spaces.

Definition 7. Let X be a nonempty set and S_3 be the permutation group on $\{1, 2, 3\}$. If $G : X \times X \times X \longrightarrow \mathcal{A}_+$ is a mapping satisfying

- (1) $G(x_1, x_2, x_3) = \theta \iff x_1 = x_2 = x_3$;
- (2) $G(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = G(x_1, x_2, x_3)$ for all $x_1, x_2, x_3 \in X, \sigma \in S_3$;
- (3) $G(x_1, x_1, x_2) \leq G(x_1, x_2, x_3)$ for all $x_1, x_2, x_3 \in X$ with $x_2 \neq x_3$;
- (4) $G(x_1, x_2, x_3) \leq G(x_1, a, a) + G(a, x_2, x_3)$ for all $x_1, x_2, x_3, a \in X$,

G is called a C^* -algebra-valued G -metric on X , and $(X, \mathcal{A}, G)$ is called a C^* -algebra-valued G -metric space.

Example 8. Let $X = \{x, y\}$. Notice that $M_2(\mathbb{C})$ of 2×2 -matrices with entries in $\mathbb{C}$ is identified with $B(\mathbb{C}^2)$. This is a C^* -algebra. Let

$$\begin{aligned} G(x, x, x) &= G(y, y, y) = \theta, \\ G(x, x, y) &= I, \\ G(x, y, y) &= 2I \end{aligned} \quad (2)$$

and extend G to all of $X \times X \times X$ by symmetry in the variables. Then G is a C^* -algebra-valued G -metric on X , and $(X, M_2(\mathbb{C}), G)$ is a C^* -algebra-valued G -metric space.

Definition 9. Let $(X, \mathcal{A}, G)$ be a C^* -algebra-valued G -metric space, $\{x_n\} \subset X$, $x \in X$. If for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that for all $m, n > N$, $\|G(x_m, x_n, x)\| < \varepsilon$, then we say $\{x_n\}$ is G -convergent to x , and denote by $x_n \xrightarrow{G} x$, ($n \rightarrow \infty$).

Proposition 10. *Let $(X, \mathcal{A}, G)$ be a C^* -algebra-valued G -metric space, $\{x_n\} \subset X$, $x \in X$. The following statements are equivalent:*

- (1) $x_n \xrightarrow{G} x$ ($n \rightarrow \infty$).
- (2) $G(x_n, x_n, x) \longrightarrow \theta$, as $n \rightarrow \infty$.
- (3) $G(x_n, x, x) \longrightarrow \theta$, as $n \rightarrow \infty$.

Proof. (1) $\implies$ (2) If $x_n \xrightarrow{G} x$, that is, for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$, such that, for all $m, n > N$, $\|G(x_m, x_n, x)\| < \varepsilon$, especially, $\|G(x_n, x_n, x)\| < \varepsilon$. Hence $G(x_n, x_n, x) \longrightarrow \theta$, as $n \rightarrow \infty$.

(2) $\implies$ (3) If $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that, for all $n > N$, $\|G(x_n, x_n, x)\| < \varepsilon/2$, then when $n > N$,

$$\begin{aligned} \|G(x_n, x, x)\| &= \|G(x, x_n, x)\| \\ &\leq \|G(x, x_n, x_n)\| + \|G(x_n, x_n, x)\| < \varepsilon, \end{aligned} \quad (3)$$

that is, $G(x_n, x, x) \longrightarrow \theta$, as $n \rightarrow \infty$.

(3) $\implies$ (1) If $G(x_n, x, x) \longrightarrow \theta$, as $n \rightarrow \infty$, then, for any $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$, such that for all $n > N_1$, $\|G(x_n, x, x)\| < \varepsilon/2$; $\exists N_2 \in \mathbb{N}$, such that, for all $m > N_2$, $\|G(x_m, x, x)\| < \varepsilon/2$. Let $N = N_1 + N_2$, for $m, n > N$,

$$\|G(x_m, x_n, x)\| \leq \|G(x_m, x, x)\| + \|G(x, x_n, x)\| < \varepsilon, \quad (4)$$

that is, $x_n \xrightarrow{G} x$ ($n \rightarrow \infty$). $\square$

Definition 11. Let $(X, \mathcal{A}, G)$ be a C^* -algebra-valued G -metric space, $\{x_n\} \subseteq X$. If for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $m, n, l > N$, $\|G(x_m, x_n, x_l)\| < \varepsilon$, then the sequence $\{x_n\}$ is called a G -Cauchy sequence. If any G -Cauchy sequence in $(X, \mathcal{A}, G)$ is G -convergent, then $(X, \mathcal{A}, G)$ is called a complete C^* -algebra-valued G -metric space.

Proposition 12. Let $(X, \mathcal{A}, G)$ be a C^* -algebra-valued G -metric space, $\{x_n\} \subseteq X$. Then $\{x_n\}$ is a G -Cauchy sequence if and only if for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that, for all $m, n > N$, $\|G(x_m, x_n, x_n)\| < \varepsilon$.

Proof. It suffices to show the necessity. For any $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$, such that for all $m, n > N_1$, $\|G(x_m, x_n, x_n)\| < \varepsilon/2$; $\exists N_2 \in \mathbb{N}$, such that for all $n, l > N_2$, $\|G(x_l, x_n, x_n)\| < \varepsilon/2$. So for the above ε , let $N = N_1 + N_2$, when $m, n, l > N$,

$$\|G(x_m, x_n, x_l)\| \leq \|G(x_m, x_n, x_n)\| + \|G(x_n, x_n, x_l)\| < \varepsilon, \quad (5)$$

that is, $\{x_n\}$ is a G -Cauchy sequence. $\square$

Example 13. Set $X = \mathbb{C}$, $\mathcal{A} = M_3(\mathbb{C})$. Let $\alpha, \beta > 0$; set

$$G(x, y, z) = \begin{pmatrix} g(x, y, z) & 0 & 0 \\ 0 & \alpha g(x, y, z) & 0 \\ 0 & 0 & \beta g(x, y, z) \end{pmatrix}, \quad (6)$$

where $g(x, y, z) = |x - y| + |y - z| + |z - x|$, then G is a C^* -algebra-valued G -metric on X , and $(X, \mathcal{A}, G)$ is a complete C^* -algebra-valued G -metric space.

It is easy to see that $(X, \mathcal{A}, G)$ is a C^* -algebra-valued G -metric space. We only need to prove the completeness. Let $\{x_n\} \subseteq (X, \mathcal{A}, G)$ be a G -Cauchy sequence. Then for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$, such that for all $m, n > N$,

$$\|G(x_m, x_n, x_n)\| = \max\{g(x_m, x_n, x_n), \alpha g(x_m, x_n, x_n), \beta g(x_m, x_n, x_n)\} < \varepsilon. \quad (7)$$

So $g(x_m, x_n, x_n) = 2|x_m - x_n| < \varepsilon$. Since X is complete, there exists $x \in X$, such that $x_n \rightarrow x$. Hence there is $N_0 \in \mathbb{N}$ such that, for any $n > N_0$, $|x_n - x| < \varepsilon/2$. It follows that

$$\begin{aligned} \|G(x_n, x, x)\| &= \max\{2|x - x_n|, 2\alpha|x - x_n|, 2\beta|x - x_n|\} \\ &= 2 \max\{1, \alpha, \beta\} |x_n - x| < \max\{1, \alpha, \beta\} \varepsilon. \end{aligned} \quad (8)$$

Therefore, $x_n \xrightarrow{G} x$, and $(X, \mathcal{A}, G)$ is complete.

Example 14. Suppose Ω is a compact Hausdorff space and μ is a positive regular Borel measure on Ω . Let $X = L^\infty(\Omega, \mu)$, the set of all essentially bounded complex-valued measurable functions on Ω , then X is a Banach space with the essential supremum norm $\|f\|_\infty$. Set $\mathcal{H} = L^2(\Omega, \mu) = \{f : \Omega \rightarrow \mathbb{C} \mid$

$\int_\Omega |f(\omega)|^2 d\mu(\omega) < \infty\}$; $\mathcal{H}$ is a Hilbert space with the inner-product

$$\langle f, g \rangle = \int_\Omega f(\omega) \overline{g(\omega)} d\mu(\omega). \quad (9)$$

For $f \in L^\infty(\Omega, \mu)$, define

$$\begin{aligned} M_f : \mathcal{H} &\longrightarrow \mathcal{H} \\ \varphi &\longmapsto f\varphi. \end{aligned} \quad (10)$$

Then M_f is bounded and moreover $\|M_f\| = \|f\|_\infty$. Let $G : X \times X \times X \rightarrow B(\mathcal{H})$ by

$$G(f, g, h) = M_{|f-g|+|g-h|+|h-f|}. \quad (11)$$

Then G is a C^* -algebra-valued G -metric and $(X, B(\mathcal{H}), G)$ is a complete C^* -algebra-valued G -metric space. We omit its proof and leave it to readers.

Next, we define the contractive mapping on C^* -algebra-valued G -metric space and prove the fixed point theorem for contractive mappings.

Definition 15. Let $(X, \mathcal{A}, G)$ be a C^* -algebra-valued G -metric space and $T : X \rightarrow X$ is a mapping. If there exists $A \in \mathcal{A}$ with $\|A\| < 1$ such that

$$G(Tx, Ty, Tz) \leq A^* G(x, y, z) A, \quad \forall x, y, z \in X, \quad (12)$$

then T is called a contractive mapping on $(X, \mathcal{A}, G)$.

Theorem 16. Let $(X, \mathcal{A}, G)$ be a complete C^* -algebra-valued G -metric space. If $T : X \rightarrow X$ is a contractive mapping on $(X, \mathcal{A}, G)$, then there is a unique fixed point of T on X .

Proof. Let $d(x, y) = G(x, x, y) + G(x, y, y)$.

We first show that d is a C^* -algebra-valued metric on X . It suffices to show that

$$d(x, y) \leq d(x, z) + d(y, z), \quad \forall x, y, z \in X. \quad (13)$$

That is,

$$\begin{aligned} G(x, x, y) + G(x, y, y) &\leq G(x, x, z) + G(x, z, z) + G(y, y, z) \\ &\quad + G(y, z, z). \end{aligned} \quad (14)$$

Since

$$\begin{aligned} G(x, x, y) + G(x, y, y) &= G(y, x, x) + G(x, y, y) \\ &\leq G(y, z, z) + G(z, x, x) + G(x, z, z) \\ &\quad + G(z, y, y) \\ &= G(x, x, z) + G(x, z, z) + G(y, z, z) \\ &\quad + G(y, y, z), \end{aligned} \quad (15)$$

we have $d(x, y) \leq d(x, z) + d(y, z)$, $\forall x, y, z \in X$.

Next, we show $(X, \mathcal{A}, d)$ is a complete C^* -algebra-valued metric space. Let $\{x_n\} \subseteq X$ be a Cauchy sequence with respect to $\mathcal{A}$. Then for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $m, n > N$, $\|d(x_n, x_m)\| < \varepsilon$, that is,

$$\|G(x_n, x_m, x_m) + G(x_n, x_n, x_m)\| < \varepsilon. \quad (16)$$

Since $\theta \leq G(x_n, x_m, x_m) \leq G(x_n, x_m, x_m) + G(x_n, x_n, x_m)$,

$$\begin{aligned} \|G(x_n, x_m, x_m)\| &\leq \|G(x_n, x_m, x_m) + G(x_n, x_n, x_m)\| \\ &< \varepsilon. \end{aligned} \quad (17)$$

So $\{x_n\} \subseteq X$ is a G -Cauchy sequence. By the completeness of $(X, \mathcal{A}, G)$, there exists an $x \in X$, such that $x_n \xrightarrow{G} x (n \rightarrow \infty)$. That is, for any $\varepsilon > 0$, there is an $N_1 \in \mathbb{N}$ such that, for all $n > N_1$, $\|G(x_n, x, x)\| < \varepsilon/2$; there is an $N_2 \in \mathbb{N}$ such that for all $m > N_2$, $\|G(x_m, x_m, x)\| < \varepsilon/2$. Let $N = N_1 + N_2$, then for all $n > N$, we have

$$\begin{aligned} \|G(x_n, x, x)\| &< \frac{\varepsilon}{2}, \\ \|G(x_n, x_n, x)\| &< \frac{\varepsilon}{2}. \end{aligned} \quad (18)$$

Therefore,

$$\begin{aligned} \|d(x_n, x)\| &= \|G(x_n, x, x) + G(x_n, x_n, x)\| \\ &\leq \|G(x_n, x, x)\| + \|G(x_n, x_n, x)\| < \varepsilon. \end{aligned} \quad (19)$$

Hence $\lim_{n \rightarrow \infty} x_n = x$, and $(X, \mathcal{A}, d)$ is complete.

Moreover, T is a contractive mapping on $(X, \mathcal{A}, d)$. In fact,

$$\begin{aligned} d(Tx, Ty) &= G(Tx, Tx, Ty) + G(Tx, Ty, Ty) \\ &\leq A^* G(x, x, y) A + A^* G(x, y, y) A \\ &= A^* (G(x, x, y) + G(x, y, y)) A \\ &= A^* d(x, y) A. \end{aligned} \quad (20)$$

It follows from Theorem 6 that there is an $x \in X$ such that $Tx = x$.

Finally, we show the uniqueness of this fixed point. Let y is another fixed point of T . If $x \neq y$, then

$$\theta \not\leq G(x, y, y) = G(Tx, Ty, Ty) \leq A^* G(x, y, y) A. \quad (21)$$

Since $\|G(x, y, y)\| \neq 0$,

$$\begin{aligned} \|G(x, y, y)\| &\leq \|A^* G(x, y, y) A\| \\ &\leq \|A\|^2 \|G(x, y, y)\| < \|G(x, y, y)\|. \end{aligned} \quad (22)$$

This is a contradiction. So $x = y$. $\square$

Remark 17. (1) In the theorem above, the completeness of X is essential. For example, let $X = (0, 1) \subseteq \mathbb{R}$ and $G : X \times X \times X \rightarrow B(\mathcal{H})$ satisfy $G(x, y, z) = (|x - y| + |y - z| + |z - x|)I$,

then $(X, B(\mathcal{H}), G)$ is a C^* -algebra-valued G -metric space, but $(X, B(\mathcal{H}), G)$ is not complete. Considering the mapping

$$\begin{aligned} T : X &\longrightarrow X \\ x &\longrightarrow \frac{1}{2}x \end{aligned} \quad (23)$$

T is a contractive mapping, but T has no fixed point.

(2) In the definition of contractive mapping, the element $A \in \mathcal{A}$ does not depend on the choice of x, y, z . If A depends on x, y, z , then T may not have a fixed point.

Indeed, let $X = \mathbb{R}$ and $G : X \times X \times X \rightarrow B(\mathcal{H})$ be defined by $G(x, y, z) = (|x - y| + |y - z| + |z - x|)I$. Then $(X, B(\mathcal{H}), G)$ is a complete C^* -algebra-valued G -metric space. Let $f(x) = \pi/(2 + x - \arctan x)$. If $x < y < z$, then

$$\begin{aligned} G(f(x), f(y), f(z)) &= (|x - y - \arctan x + \arctan y| \\ &\quad + |y - z - \arctan y + \arctan z| \\ &\quad + |z - x - \arctan z + \arctan x|) I \\ &= \left(\left| x - y + \frac{y - x}{1 + \xi^2} \right| + \left| y - z + \frac{z - y}{1 + \eta^2} \right| \right. \\ &\quad \left. + \left| z - x + \frac{x - z}{1 + \zeta^2} \right| \right) I = \left(\frac{\xi^2}{1 + \xi^2} |x - y| \right. \\ &\quad \left. + \frac{\eta^2}{1 + \eta^2} |y - z| + \frac{\zeta^2}{1 + \zeta^2} |z - x| \right) I \leq \alpha (|x - y| \\ &\quad + |y - z| + |z - x|) I = \sqrt{\alpha} IG(x, y, z) \sqrt{\alpha} I, \end{aligned} \quad (24)$$

where $\xi \in (x, y)$, $\eta \in (y, z)$, $\zeta \in (x, z)$, and $\alpha = \max\{\xi^2/(1 + \xi^2), \eta^2/(1 + \eta^2), \zeta^2/(1 + \zeta^2)\} < 1$ depends on x, y, z , but f has no fixed point.

(3) When $\|A\| = 1$, T may not have a unique fixed point.

Consider $X = l^\infty(\mathbb{N}) = \{(x_1, x_2, \dots) \mid x_n \in \mathbb{C}, n \in \mathbb{N} \text{ and } \sup_n |x_n| < +\infty\}$, for $x = (x_1, x_2, \dots) \in l^\infty(\mathbb{N})$, and let $\|x\|_\infty = \sup_n |x_n|$. Define $G : X \times X \times X \rightarrow M_2(\mathbb{C})$ by

$$G(x, y, z) = (\|x - y\|_\infty + \|y - z\|_\infty + \|z - x\|_\infty) I. \quad (25)$$

Then $(X, M_2(\mathbb{C}), G)$ is a complete C^* -algebra-valued G -metric space. Let

$$T(x_1, x_2, x_3, \dots) = \left(1 + x_2, \frac{1}{2} + x_3, \frac{1}{2^2} + x_4, \dots\right). \quad (26)$$

Then

$$G(Tx, Ty, Tz) \leq G(x, y, z). \quad (27)$$

But for each $\alpha \in \mathbb{C}$, $x = (\alpha, \alpha - 1, \alpha - 3/2, \dots, \alpha - \sum_{i=0}^{n-1} (1/2^i), \dots)$ is a fixed point, which means the fixed point is not unique.

What follows is the definition of the expansion mapping on C^* -algebra-valued G -metric space and the fixed-point theorem for expansion mappings.

Definition 18. Let $(X, \mathcal{A}, G)$ be a C^* -algebra-valued G -metric space. If $T : X \rightarrow X$ satisfies the condition

$$G(Tx, Ty, Tz) \geq A^* G(x, y, z) A, \quad (28)$$

where $A \in \mathcal{A}$ is invertible and $\|A^{-1}\| < 1$, we call T an expansion mapping on $(X, \mathcal{A}, G)$.

Theorem 19. Let $(X, \mathcal{A}, G)$ be a complete C^* -algebra-valued G -metric space and $T : X \rightarrow X$ a expansion mapping on $(X, \mathcal{A}, G)$. If T is surjective, then there is a unique fixed point for T .

Proof. First we show T is injective. Indeed, if $Tx = Ty$, then

$$\theta = G(Tx, Ty, Ty) \geq A^* G(x, y, y) A. \quad (29)$$

Since $A^* G(x, y, y) A \in \mathcal{A}_+$, $A^* G(x, y, y) A = \theta$, and since A is invertible, $G(x, y, y) = \theta$. Therefore $x = y$.

Next we show that T has a unique fixed point in X . In fact, T is bijective and so T is invertible. Since $\forall x, y, z \in X$,

$$G(Tx, Ty, Tz) \geq A^* G(x, y, z) A. \quad (30)$$

Replace x, y, z by $T^{-1}x, T^{-1}y, T^{-1}z$, respectively, we get

$$G(x, y, z) \geq A^* G(T^{-1}x, T^{-1}y, T^{-1}z) A. \quad (31)$$

Hence

$$G(T^{-1}x, T^{-1}y, T^{-1}z) \leq (A^*)^{-1} G(x, y, z) A^{-1}. \quad (32)$$

Thus

$$G(T^{-1}x, T^{-1}y, T^{-1}z) \leq (A^{-1})^* G(x, y, z) A^{-1}. \quad (33)$$

By Theorem 16, there is a unique $x \in X$, such that $T^{-1}x = x$, and therefore there is a unique $x \in X$, such that $Tx = x$. $\square$

The following lemma is necessary for another fixed-point theorem, for detail, see [27].

Lemma 20. Set $\mathcal{A}' = \{A \in \mathcal{A} : AB = BA, \forall B \in \mathcal{A}\}$.

(1) If $A, B \in \mathcal{A}'_+$ and $AB = BA$, then $AB \in \mathcal{A}'_+$.

(2) If $A \in \mathcal{A}'$, $B, C \in \mathcal{A}$ with $B \geq C \geq \theta$ and $I - A \in \mathcal{A}'$ is invertible, then $(I - A)^{-1}B \geq (I - A)^{-1}C$.

Theorem 21. Let $(X, \mathcal{A}, G)$ be a complete C^* -algebra-valued G -metric space, $T : X \rightarrow X$. If there exists an $A \in (\mathcal{A}')_+$, $\|A\| < 1/6$ such that for any $x, y \in X$,

$$\begin{aligned} G(Tx, Ty, Tz) \\ \leq A(G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)), \end{aligned} \quad (34)$$

or

$$\begin{aligned} G(Tx, Ty, Tz) \\ \leq A(G(Tx, Ty, z) + G(x, Ty, Tz) + G(Tx, y, Tz)), \end{aligned} \quad (35)$$

then T has a unique fixed point in X .

Proof. Without loss of generality, we can assume $A \neq \theta$.

(1) Suppose that T satisfies

$$\begin{aligned} G(Tx, Ty, Tz) \\ \leq A(G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)), \end{aligned} \quad (36)$$

$$\forall x, y, z \in X.$$

Since $A \in (\mathcal{A}')_+$,

$$A(G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)) \geq \theta. \quad (37)$$

For $x_0 \in X$, set $x_{n+1} = Tx_n = T^{n+1}x_0$, $n = 1, 2, \dots$, and $B = G(x_1, x_0, x_0)$. Then

$$\begin{aligned} G(x_{n+1}, x_n, x_n) &= G(Tx_n, Tx_{n-1}, Tx_{n-1}) \\ &\leq A(G(Tx_n, x_{n-1}, x_{n-1}) + G(x_n, Tx_{n-1}, x_{n-1}) \\ &\quad + G(x_n, x_{n-1}, Tx_{n-1})) = AG(x_{n+1}, x_{n-1}, x_{n-1}) \\ &\quad + 2AG(x_n, x_n, x_{n-1}) \leq A(G(x_{n+1}, x_n, x_n) \\ &\quad + G(x_n, x_{n-1}, x_{n-1})) + 2A(G(x_n, x_{n-1}, x_{n-1}) \\ &\quad + G(x_{n-1}, x_n, x_{n-1})) = AG(x_{n+1}, x_n, x_n) \\ &\quad + 5AG(x_n, x_{n-1}, x_{n-1}), \end{aligned} \quad (38)$$

That is,

$$(I - A)G(x_{n+1}, x_n, x_n) \leq 5AG(x_n, x_{n-1}, x_{n-1}). \quad (39)$$

Since $A \in (\mathcal{A}')_+$ with $\|A\| < 1/6$, we have $(I - A)^{-1} \in (\mathcal{A}')_+$ and furthermore $A(I - A)^{-1} \in (\mathcal{A}')_+$ with $\|5A(I - A)^{-1}\| < 1$. Therefore,

$$G(x_{n+1}, x_n, x_n) \leq 5A(I - A)^{-1}G(x_n, x_{n-1}, x_{n-1}). \quad (40)$$

Let $t = 5A(I - A)^{-1}$, for $n + 1 > m$,

$$\begin{aligned} G(x_{n+1}, x_m, x_m) \\ \leq G(x_{n+1}, x_n, x_n) + G(x_n, x_{n-1}, x_{n-1}) \cdots \\ + G(x_{m+1}, x_m, x_m) \\ \leq (t^n + t^{n-1} + \cdots + t^m)G(x_1, x_0, x_0) = \sum_{k=m}^n t^k B \\ = \sum_{k=m}^n |t^{k/2} B^{1/2}|^2 \leq \left\| \sum_{k=m}^n |t^{k/2} B^{1/2}|^2 \right\| I \\ \leq \sum_{k=m}^n \|t^{k/2}\|^2 \|B^{1/2}\|^2 I \leq \|B^{1/2}\|^2 \sum_{k=m}^n \|t\|^k I \\ \leq \|B^{1/2}\|^2 \frac{\|t\|^m}{1 - \|t\|} I \rightarrow \theta \quad (m \rightarrow \infty). \end{aligned} \quad (41)$$

This implies that $\{x_n\}$ is a G -Cauchy sequence in X . Since $(X, \mathcal{A}, G)$ is complete, there exists an $x \in X$ such that $x_n \xrightarrow{G} x (n \rightarrow \infty)$, i.e., $Tx_{n-1} \xrightarrow{G} x (n \rightarrow \infty)$. Since

$$\begin{aligned} G(Tx, x, x) &\leq G(Tx, Tx_n, Tx_n) + G(Tx_n, x, x) \\ &\leq A(G(Tx, x_n, x_n) + G(x, Tx_n, x_n) \\ &\quad + G(x, x_n, Tx_n)) + G(x_{n+1}, x, x) \leq AG(Tx, x, x) \quad (42) \\ &\quad + AG(x, x_n, x_n) + 2AG(x, x_n, x_{n+1}) \\ &\quad + G(x_{n+1}, x, x). \end{aligned}$$

That is

$$\begin{aligned} (I - A)G(Tx, x, x) &\leq AG(x, x_n, x_n) \\ &\quad + 2AG(x, x_n, x_{n+1}) \quad (43) \\ &\quad + G(x_{n+1}, x, x). \end{aligned}$$

Then

$$\begin{aligned} \|G(Tx, x, x)\| &\leq \|A(I - A)^{-1}\| \\ &\quad \cdot (\|G(x, x_n, x_n)\| + 2\|G(x, x_n, x_{n+1})\|) \quad (44) \\ &\quad + \|(I - A)^{-1}\| \|G(x_{n+1}, x, x)\| \rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

and hence $Tx = x$ and x is the fixed point of T on X .

Next, we show the uniqueness of x . If there exists another fixed point $y \in X$, then

$$\begin{aligned} \theta &\leq G(x, x, y) = G(Tx, Tx, Ty) \\ &\leq A(G(Tx, x, y) + G(x, Tx, y) + G(x, x, Ty)) \quad (45) \\ &= AG(x, x, y) + 2AG(x, x, y), \end{aligned}$$

i.e.,

$$G(x, x, y) \leq 2A(I - A)^{-1}G(x, x, y). \quad (46)$$

Since $\|5A(I - A)^{-1}\| < 1$, $\|2A(I - A)^{-1}\| < 1$. Hence $G(x, x, y) = \theta$, and $x = y$.

(2) The case when

$$\begin{aligned} G(Tx, Ty, Tz) \\ \leq A(G(Tx, Ty, z) + G(x, Ty, Tz) + G(Tx, y, Tz)), \quad (47) \\ \forall x, y, z \in X \end{aligned}$$

can be proved similarly and we omitted it. $\square$

Definition 22. Let $(X, \mathcal{A}, G)$ be a C^* -algebra-valued G -metric space. We say G is symmetric if $G(x, x, y) = G(x, y, y)$ for all $x, y \in X$.

It is easy to show that, in Example 8, G is not symmetric and, in Example 13, G is symmetric.

Theorem 23. Let $(X, \mathcal{A}, G)$ be a complete C^* -algebra-valued G -metric space and G symmetric. If $T : X \rightarrow X$ is a mapping satisfying that for $x, y \in X$

$$\begin{aligned} G(Tx, Ty, Tz) \\ \leq A(G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)), \quad (48) \end{aligned}$$

or

$$\begin{aligned} G(Tx, Ty, Tz) \\ \leq A(G(Tx, Ty, z) + G(x, Ty, Tz) + G(Tx, y, Tz)), \quad (49) \end{aligned}$$

where $A \in (\mathcal{A}')_+$ and $\|A\| < 1/4$, then T has a unique fixed point in X .

Proof. Without loss of generality, one can assume $A \neq \theta$.

We only consider the case when T satisfies

$$\begin{aligned} G(Tx, Ty, Tz) \\ \leq A(G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)). \quad (50) \end{aligned}$$

Since $A \in (\mathcal{A}')_+$, $A(G(Tx, y, z) + G(x, Ty, z) + G(x, y, Tz)) \geq \theta$.

For $x_0 \in X$, set $x_{n+1} = Tx_n = T^{n+1}x_0$, $n = 1, 2, \dots$, and $B = G(x_1, x_0, x_0)$. If G is symmetric,

$$\begin{aligned} G(x_{n+1}, x_n, x_n) &= G(Tx_n, Tx_{n-1}, Tx_{n-1}) \\ &\leq A(G(Tx_n, x_{n-1}, x_{n-1}) + G(x_n, Tx_{n-1}, x_{n-1}) \\ &\quad + G(x_n, x_{n-1}, Tx_{n-1})) = AG(x_{n+1}, x_{n-1}, x_{n-1}) \quad (51) \\ &\quad + 2AG(x_n, x_n, x_{n-1}) \leq A(G(x_{n+1}, x_n, x_n) \\ &\quad + G(x_n, x_{n-1}, x_{n-1})) + 2AG(x_n, x_{n-1}, x_{n-1}) \\ &= AG(x_{n+1}, x_n, x_n) + 3AG(x_n, x_{n-1}, x_{n-1}). \end{aligned}$$

That is,

$$(I - A)G(x_{n+1}, x_n, x_n) \leq 3AG(x_n, x_{n-1}, x_{n-1}). \quad (52)$$

Since $A \in (\mathcal{A}')_+$ with $\|A\| < 1/4$, then $I - A$ is invertible and $(I - A)^{-1} \in (\mathcal{A}')_+$ and $\|3A(I - A)^{-1}\| < 1$. Therefore

$$G(x_{n+1}, x_n, x_n) \leq 3A(I - A)^{-1}G(x_n, x_{n-1}, x_{n-1}). \quad (53)$$

Just like the proof of Theorem 21, we can prove that, for $n+1 > m$,

$$G(x_{n+1}, x_m, x_m) \rightarrow \theta \quad (m \rightarrow \infty). \quad (54)$$

This implies that $\{x_n\}$ is a G -Cauchy sequence in X ; thus there exists an $x \in X$ such that $x_n \xrightarrow{G} x (n \rightarrow \infty)$ i.e., $Tx_{n-1} \xrightarrow{G} x (n \rightarrow \infty)$. Similarly, we can show that $Tx = x$, x is the fixed point of T on X .

The uniqueness of the fixed point can be proved similarly. $\square$

3. Applications

For fixed-point theorems, there are a number of applications in differential equations and integral equations.

Consider the second-order differential equation:

$$\begin{aligned} \frac{d^2 u}{dt^2} &= K(t, u(t)), \\ u(0) &= u(1) = 0, \end{aligned} \quad (*)$$

where $u : [0, 1] \rightarrow [0, \infty)$ and $K : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are continuous. Then the equation has a unique solution if and only if $u(t) = \int_0^1 g(t, s) K(s, u(s)) ds$, $t \in [0, 1]$ has a unique solution, where

$$g(t, s) = \begin{cases} t(1-s), & 0 \leq s < t \leq 1; \\ s(1-t), & 0 \leq t < s \leq 1. \end{cases} \quad (55)$$

Let $X = C([0, 1], \mathbb{R}^+) \subseteq L^\infty([0, 1], \mathbb{R}^+)$, $\mathcal{A} = B(L^2[0, 1])$, $G(u, v, w) = M_{|u-v|+|v-w|+|w-u|}$, then $(X, \mathcal{A}, G)$ is a complete C^* -algebra-valued G -metric space. Let $T : X \rightarrow X$ be a mapping defined by

$$Tu(t) = \int_0^1 g(t, s) K(s, u(s)) ds, \quad (56)$$

then $(*)$ has a unique solution if and only if T has a unique fixed point.

Theorem 24. If $|K(s, x) - K(s, y)| \leq |x - y|$, $\forall s \in [0, 1]$, $x, y \in \mathbb{R}^+$, equation $(*)$ has a unique solution.

Proof.

$$\begin{aligned} \|G(Tu, Tv, Tw)\| &= \|M_{|Tu-Tv|+|Tv-Tw|+|Tw-Tu|}\| = \|Tu \\ &\quad - Tv\|_\infty + \|Tv - Tw\|_\infty + \|Tw - Tu\|_\infty \\ &= \max_{t \in [0, 1]} \left| \int_0^1 g(t, s) (K(s, u(s)) - K(s, v(s))) ds \right| \\ &\quad + \max_{t \in [0, 1]} \left| \int_0^1 g(t, s) (K(s, v(s)) - K(s, w(s))) ds \right| \\ &\quad + \max_{t \in [0, 1]} \left| \int_0^1 g(t, s) (K(s, w(s)) - K(s, u(s))) ds \right| \\ &\leq \max_{t \in [0, 1]} \int_0^1 g(t, s) |K(s, u(s)) - K(s, v(s))| ds \\ &\quad + \max_{t \in [0, 1]} \int_0^1 g(t, s) |K(s, v(s)) - K(s, w(s))| ds \\ &\quad + \max_{t \in [0, 1]} \int_0^1 g(t, s) |K(s, w(s)) - K(s, u(s))| ds \\ &\leq \max_{t \in [0, 1]} \int_0^1 g(t, s) |u(s) - v(s)| ds + \max_{t \in [0, 1]} \int_0^1 g(t, s) \\ &\quad \cdot |v(s) - w(s)| ds + \max_{t \in [0, 1]} \int_0^1 g(t, s) |w(s) \end{aligned}$$

$$\begin{aligned} -u(s)| ds &= \max_{t \in [0, 1]} \int_0^1 g(t, s) (|u(s) - v(s)| \\ &\quad + |v(s) - w(s)| + |w(s) - u(s)|) ds \leq \|G(u, \\ &\quad v, w)\| \max_{t \in [0, 1]} \int_0^1 g(t, s) ds \end{aligned} \quad (57)$$

For a fixed $t \in [0, 1]$,

$$\begin{aligned} \int_0^1 g(t, s) ds &= \int_0^t t(1-s) ds + \int_t^1 s(1-t) ds \\ &= \frac{1}{2} + \frac{t^2}{2} - \frac{t}{2} \leq \frac{1}{2}. \end{aligned} \quad (58)$$

Therefore,

$$\|G(Tu, Tv, Tw)\| \leq \frac{1}{2} \|G(u, v, w)\| \quad (59)$$

and T has a unique fixed point. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The project is supported by the Scientific Research Foundation of Education Bureau of Hebei Province (QN2016191) and Natural Science Foundation of China (11371222).

References

- [1] X. Du and Z. Zhao, "On fixed point theorems of mixed monotone operators," *Fixed Point Theory and Applications*, 8 pages, 2011.
- [2] A. Jiménez-Vargas, L. Li, A. M. Peralta, L. Wang, and Y.-S. Wang, "2-local standard isometries on vector-valued Lipschitz function spaces," *Journal of Mathematical Analysis and Applications*, vol. 461, no. 2, pp. 1287–1298, 2018.
- [3] D. Kong, L. Liu, and Y. Wu, "Best approximation and fixed-point theorems for discontinuous increasing maps in Banach lattices," *Fixed Point Theory and Applications*, vol. 18, 10 pages, 2014.
- [4] L. G. Wang, B. Liu, and R. Bai, "Stability of a mixed type functional equation on multi-Banach spaces: a fixed point approach," *Fixed Point Theory and Applications*, vol. 2010, Article ID 283827, 9 pages, 2010.
- [5] L. Wang, "The fixed point method for intuitionistic fuzzy stability of a quadratic functional equation," *Fixed Point Theory and Applications*, vol. 1, 7 pages, 2010.
- [6] X. Zhang, L. Liu, and Y. Wu, "Fixed point theorems for the sum of three classes of mixed monotone operators and applications," *Fixed Point Theory and Applications*, vol. 49, 22 pages, 2016.

- [7] Z. Zhao, "Fixed points of τ - φ -convex operators and applications," *Applied Mathematics Letters*, vol. 23, no. 5, pp. 561–566, 2010.
- [8] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.
- [9] A. H. Ansari, M. A. Barakat, and H. Aydi, "New approach for common fixed point theorems via C -class functions in G_p -metric spaces," *Journal of Function Spaces*, vol. 2017, Article ID 2624569, 9 pages, 2017.
- [10] H. Aydi, A. Felhi, and S. Sahmim, "Related fixed point results for cyclic contractions on G -metric spaces and application," *Filomat*, vol. 31, no. 3, pp. 853–869, 2017.
- [11] B. C. Dhage, "Generalised metric spaces and mappings with fixed point," *Bulletin of the Calcutta Mathematical Society*, vol. 84, no. 4, pp. 329–336, 1992.
- [12] L. Gajić and Z. Lozanov Crvenković, "Some fixed point results for a sequence of mappings on G -metric space," *PAMM*, vol. 13, no. 1, pp. 375–376, 2013.
- [13] S. Manro, S. S. Bhatia, S. Kumar, and C. Vetro, "A common fixed point theorem for two weakly compatible pairs in G -metric spaces using the property E. A.," *Fixed Point Theory and Applications*, vol. 41, 9 pages, 2013.
- [14] B. Moeini, P. Kumar, and H. Aydi, "Zamfirescu type contractions on C^* -algebra-valued metric spaces and an application," *Journal of Mathematical Analysis*, vol. 9, no. 1, pp. 150–161, 2016.
- [15] F. Moradlou, P. Salimi, and P. Vetro, "Some new extensions of Edelstein-Suzuli-type fixed point theorem to G -metric and G -cone metric spaces," *Acta Mathematica Scientia B*, vol. 33, no. 4, pp. 1049–1058, 2013.
- [16] Z. Mustafa, H. Aydi, and E. Karapinar, "Generalized Meir-Keeler type contractions on G -metric spaces," *Applied Mathematics and Computation*, vol. 219, no. 21, pp. 10441–10447, 2013.
- [17] Z. Mustafa, H. Aydi, and E. Karapinar, "On common fixed points in G -metric spaces using (E.A) property," *Computers & Mathematics with Applications*, vol. 64, no. 6, pp. 1944–1956, 2012.
- [18] Z. Mustafa, F. Awawdeh, and W. Shatanawi, "Fixed point theorem for expansive mappings in G -metric spaces," *International Journal of Contemporary Mathematical Sciences*, vol. 5, no. 49–52, pp. 2463–2472, 2010.
- [19] H. K. Nashine, Z. Golubovi, and Z. Kadelburg, "Nonlinear cyclic weak contractions in G -metric spaces and applications to boundary value problems," *Fixed Point Theory and Applications*, vol. 227, 17 pages, 2012.
- [20] N. Tahat, H. Aydi, E. Karapinar, and W. Shatanawi, "Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G -metric spaces," *Fixed Point Theory and Applications*, vol. 48, 9 pages, 2012.
- [21] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, vol. 179 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, NY, USA, 2nd edition, 1998.
- [22] D. Hadwin, J. Li, J. Shen, and L. Wang, "Reduced free products of unital AH algebras and Blackadar and Kirchberg's MF algebras," *The Journal of Operator Theory*, vol. 68, no. 1, pp. 275–302, 2012.
- [23] S. Liu and X. Fang, " K -theory for certain extension algebras of purely infinite simple C^* -algebras," *The Proceedings of the Indian Academy of Sciences – Mathematical Sciences*, vol. 120, no. 4, pp. 409–415, 2010.
- [24] X. Ma, D. Hadwin, and L. Wang, "On representations of $M_2(C) * M_2(C)$," *Journal of Mathematical Analysis and Applications*, vol. 362, no. 1, pp. 107–113, 2010.
- [25] Q. Meng, "The Weak Haagerup property for C^* -algebras," *Annals of Functional Analysis*, vol. 8, no. 4, pp. 502–511, 2017.
- [26] Q. Meng, "Weak Haagerup property of W^* -crossed products," *Bulletin of the Australian Mathematical Society*, vol. 97, no. 1, pp. 119–126, 2018.
- [27] G. J. Murphy, *C^* -Algebras and Operator Theory*, Academic Press, Inc., Boston, Mass, USA, 1990.
- [28] I. E. Segal, "Irreducible representations of operator algebras," *Bulletin (New Series) of the American Mathematical Society*, vol. 53, pp. 73–88, 1947.
- [29] Q. Xu, N. B. Turdebek, and Z. Chen, *Introduction to Operator Algebras and Noncommutative L_p Spaces*, Science Press, Beijing, China, 2010.
- [30] Z. Ma, L. Jiang, and H. Sun, " C^* -algebra-valued metric spaces and related fixed point theorems," *Fixed Point Theory and Applications*, 11 pages, 2014.
- [31] Z. Ma and L. Jiang, " C^* -algebra-valued b -metric spaces and related fixed point theorems," *Fixed Point Theory and Applications*, vol. 222, 12 pages, 2015.

Submit your manuscripts at
www.hindawi.com