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Research Article

Approximate Cubic Lie Derivations on ρ-Complete Convex Modular Algebras

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In this article, we present generalized Hyers–Ulam stability results of a cubic functional equation associated with an approximate cubic Lie derivations on convex modular algebras χ_ρ with Δ_2-condition on the convex modular functional ρ.

1. Introduction

In 1940, S. M. Ulam [1] raised the question concerning the stability of group homomorphisms. Let G be a group and let G' be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping $f : G \longrightarrow G'$ satisfies the inequality

$$d\left(f\left(xy\right), f\left(x\right) f\left(y\right)\right) < \delta \quad (1)$$

for all $x, y \in G$, then there exists a homomorphism $F : G \longrightarrow G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$? D. H. Hyers [2] has solved the problem of Ulam for the case of additive mappings in 1941. The result was generalized by T. Aoki [3] in 1950, by Th.M. Rassias [4] in 1978, by J. M. Rassias [5] in 1992, and by P. Găvruta [6] in 1994. Over the past few decades, many mathematicians have investigated the generalized Hyers–Ulam stability theorems of various functional equations [7–12].

Now, we recall some basic definitions and remarks of modular spaces with modular functions, which are primitive notions corresponding to norms or metrics, as in the following [13–15].

Definition 1. Let χ be a linear space.

(a) A function $\rho : \chi \longrightarrow [0, \infty]$ is called a modular if, for arbitrary $x, y \in \chi$,

(1) $\rho(x) = 0$ if and only if $x = 0$,

(2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,

(3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ for any scalars α, β, where $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$;

(b) alternatively, if (3) is replaced by

(3)' $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ for every scalars α, β, where $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$,

then we say that ρ is a convex modular.

It is well known that a modular ρ defines a corresponding modular space, i.e., the linear space χ_ρ given by

$$\chi_\rho = \left\{x \in \chi : \rho\left(\lambda x\right) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\right\}. \quad (2)$$

Let ρ be a convex modular. Then, we remark the modular space χ_ρ can be a Banach space equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \le 1\right\}. \quad (3)$$

If ρ is a modular on χ, we note that $\rho(tx)$ is an increasing function in $t \ge 0$ for each fixed $x \in \chi$; that is, $\rho(ax) \le \rho(bx)$, whenever $0 \le a < b$. In addition, if ρ is a convex modular on χ, then $\rho(\alpha x) \le \alpha\rho(x)$ for all $x \in \chi$ and $0 \le \alpha \le 1$. Moreover, we see that $\rho(\alpha x) \le |\alpha|\rho(x)$ for all $x \in \chi$ and $|\alpha| \le 1$.

Remark. (a) In general, we note that $\rho(\sum_{i=1}^{n} \alpha_i x_i) \le \sum_{i=1}^{n} \alpha_i \rho(x_i)$ for all $x_i \in \chi$ and $\alpha_i \ge 0$ $(i = 1, \ldots, n)$ whenever $0 < \sum_{i=1}^{n} \alpha_i := \alpha \le 1$ [14].

(b) Consequently, we lead to $\rho(\sum_{i=1}^{n}\alpha_i x_i) \le \sum_{i=1}^{n}|\alpha_i|\rho(x_i)$ for all $x_i \in \chi$ and $0 < \sum_{i=1}^{n}|\alpha_i| := \alpha \le 1$.

Definition 2. Let χ_ρ be a modular space and let $\{x_n\}$ be a sequence in χ_ρ. Then,

(1) $\{x_n\}$ is ρ-convergent to $x \in \chi_\rho$ and we write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \longrightarrow 0$ as $n \longrightarrow \infty$;

(2) $\{x_n\}$ is called ρ-Cauchy in χ_ρ if $\rho(x_n - x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$;

(3) a subset K of χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence in K is ρ-convergent to an element in K.

They say that the modular ρ has the Fatou property if and only if $\rho(x) \le \liminf_{n\longrightarrow\infty}\rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x. A modular function ρ is said to satisfy the Δ_2-condition if there exists $\kappa > 0$ such that $\rho(2x) \le \kappa\rho(x)$ for all $x \in \chi_\rho$.

In 2014, G. Sadeghi [16] has demonstrated generalized Hyers–Ulam stability via the fixed point method of a generalized Jensen functional equation $f(rx + sy) = rg(x) + sh(y)$ in convex modular spaces with the Fatou property satisfying the Δ_2-condition with $0 < \kappa \le 2$. In [15], the authors have proved the generalized Hyers–Ulam stability of quadratic functional equations via the extensive studies of fixed point theory in the framework of modular spaces whose modulars are convex and lower semicontinuous but do not satisfy any relatives of Δ_2-conditions (see also [17, 18]). Recently, the authors [14, 19, 20] have investigated stability theorems of functional equations in modular spaces without using the Fatou property and Δ_2-condition. In 2001, J. M. Rassias [21] has introduced to study Hyers–Ulam stability of the following cubic functional equation:

$$\begin{aligned} &f(2x+y)+f(x-y)+3f(y) \\ &\quad = 3f(x+y)+6f(x), \end{aligned} \tag{4}$$

which is equivalent to

$$\begin{aligned} &f(2x+y)+f(2x-y) \\ &\quad = 2f(x+y)+2f(x-y)+12f(x), \end{aligned} \tag{5}$$

whose general solution is characterized as $f(x) = B(x, x, x)$ where B is symmetric and additive for each fixed one variable [22]. For this reason, every solution of the cubic functional equation is said to be a cubic mapping.

Now, we say that χ_ρ is called a (convex) modular algebra if the fundamental space χ is an algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with (convex) modular ρ subject to $\rho(ab) \le \rho(a)\rho(b)$ for all $a, b \in \chi$. A subset K of a convex modular algebra χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence in K is ρ-convergent to an element in K. Throughout the paper, χ_ρ will be a ρ-complete convex modular algebra and the symbol $[a, b]$ will denote the commutator $ab - ba$. We say that a mapping f is cubic homogeneous if $f(\lambda x) = \lambda^3 f(x)$ for all vectors x and all scalars λ, and a cubic homogeneous mapping f is called a cubic Lie derivation if $f([x, y]) = [f(x), y^3] + [x^3, f(y)]$ for all vectors x, y [23, 24].

In this article, we first investigate generalized Hyers–Ulam stability of the equation

$$\begin{aligned} &f(3x-y)+f(x+y) \\ &\quad = 2f(2x-y)+12f(x)+2f(y), \end{aligned} \tag{6}$$

in ρ-complete convex modular algebras without using the Fatou property and Δ_2-condition and then present alternatively generalized Hyers–Ulam stability of (6) using necessarily Δ_2-condition without the Fatou property in ρ-complete convex modular algebras.

2. Generalized Hyers–Ulam Stability of (6)

First of all, we remark that (6) is equivalent to the original cubic functional equation, and so every solution of (6) is a cubic mapping.

For notational convenience, we let the difference operators CE_f of cubic equation (6) and CD_f of cubic derivation be as follows:

$$\begin{aligned} CE_f(\lambda x, \lambda y) &:= f(3\lambda x - \lambda y) + f(\lambda x + \lambda y) \\ &\quad - 2\lambda^3 f(2x-y) - 12\lambda^3 f(x) \\ &\quad - 2\lambda^3 f(y), \\ CD_f(x,y) &:= f([x,y]) - \left[f(x), y^3\right] - \left[x^3, f(y)\right] \end{aligned} \tag{7}$$

for all x, y in a linear space X and $\lambda \in \Lambda := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. In the following, we present a generalized Hyers–Ulam stability via direct method of the system $CE_f = 0$ and $CD_f = 0$ in ρ-complete convex modular algebras without using both the Fatou property and Δ_2-condition.

Theorem 3. *Suppose that a mapping $f : \chi_\rho \longrightarrow \chi_\rho$ satisfies*

$$\begin{aligned} &\rho\left(CE_f(\lambda x, \lambda y)\right) \le \phi_1(x, y, z), \\ &\rho\left(CD_f(x, y)\right) \le \phi_2(x, y) \end{aligned} \tag{8}$$

and $\phi_1 : \chi_\rho^3 \longrightarrow [0, \infty)$, $\phi_2 : \chi_\rho^2 \longrightarrow [0, \infty)$ are mappings such that

$$\begin{aligned} &\Phi(x,y,z) := \sum_{j=0}^{\infty}\frac{\phi_1\left(2^j x, 2^j y, 2^j z\right)}{2^{3j}} < \infty, \\ &\lim_{n\longrightarrow\infty}\frac{\phi_2\left(2^n x, 2^n y\right)}{8^{2n}} = 0 \end{aligned} \tag{9}$$

for all $x, y, z \in \chi_\rho$ and $\lambda \in \Lambda$. If for each $x \in \chi_\rho$ the mapping $r \longrightarrow f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique cubic Lie derivation $F_1 : \chi_\rho \longrightarrow \chi_\rho$ which satisfies equation (6) and

$$\rho\left(f(x) - F_1(x)\right) \le \frac{1}{16}\Phi(x, x, 0) \tag{10}$$

for all $x \in \chi_\rho$.

Proof. Putting $y = x$ and $\lambda = 1$ in (8), we obtain

$$\rho\left(2f\left(2x\right) - 16f\left(x\right)\right) \le \phi_1\left(x, x, 0\right), \tag{11}$$

which yields

$$\begin{aligned}
\rho\left(f\left(2x\right) - 8f\left(x\right)\right) &= \frac{1}{2}\rho\left(2f\left(2x\right) - 16f\left(x\right)\right) \\
&\le \frac{1}{2}\phi_1\left(x, x, 0\right), \\
\rho\left(f\left(x\right) - \frac{f\left(2x\right)}{8}\right) &\le \frac{1}{8}\rho\left(f\left(2x\right) - 8f\left(x\right)\right) \\
&\le \frac{1}{16}\phi_1\left(x, x, 0\right)
\end{aligned} \tag{12}$$

for all $x \in \chi_\rho$. Since $\sum_{j=0}^{n-1}(1/8^{j+1}) \le 1$, we prove the following functional inequality:

$$\begin{aligned}
&\rho\left(f\left(x\right) - \frac{f\left(2^n x\right)}{2^{3n}}\right) \\
&= \rho\left[\sum_{j=0}^{n-1}\left(\frac{f\left(2^j x\right)}{2^{3j}} - \frac{f\left(2^{j+1}x\right)}{2^{3(j+1)}}\right)\right] \\
&= \rho\left[\sum_{j=0}^{n-1}\frac{1}{2^{3(j+1)}}\left(8f\left(2^j x\right) - f\left(2^{j+1}x\right)\right)\right] \\
&\le \sum_{j=0}^{n-1}\frac{1}{2^{3(j+1)}}\rho\left(8f\left(2^j x\right) - f\left(2^{j+1}x\right)\right) \\
&\le \frac{1}{16}\sum_{j=0}^{n-1}\frac{\phi_1\left(2^j x, 2^j x, 0\right)}{2^{3j}}
\end{aligned} \tag{13}$$

for all $x \in \chi_\rho$ by using the property of convex modular ρ.

Now, replacing x by $2^m x$ in (13), we have

$$\begin{aligned}
&\rho\left(\frac{f\left(2^m x\right)}{2^{3m}} - \frac{f\left(2^{m+n}x\right)}{2^{3(m+n)}}\right) \\
&\qquad \le \frac{1}{16}\sum_{j=m}^{m+n-1}\frac{\phi_1\left(2^j x, 2^j x, 0\right)}{2^{3j}}
\end{aligned} \tag{14}$$

which converges to zero as $m \longrightarrow \infty$ by assumption (9). Thus the above inequality implies that the sequence $\{f(2^n x)/2^{3n}\}$ is ρ-Cauchy for all $x \in \chi_\rho$ and so it is convergent in χ_ρ since the space χ_ρ is ρ-complete. Thus, we may define a mapping $F_1 : \chi_\rho \longrightarrow \chi_\rho$ as

$$\begin{aligned}
&F_1\left(x\right) \coloneqq \rho - \lim_{n\longrightarrow\infty}\frac{f\left(2^n x\right)}{2^{3n}} \Longleftrightarrow \\
&\lim_{n\longrightarrow\infty}\rho\left(\frac{f\left(2^n x\right)}{2^{3n}} - F_1\left(x\right)\right) = 0,
\end{aligned} \tag{15}$$

for all $x \in \chi_\rho$.

Claim 1. F_1 is a cubic mapping satisfying approximation (10). In fact, if we put $(x, y, z) \coloneqq (2^n x, 2^n y, 0)$ in (8) and then divide the resulting inequality by 2^{3n}, one obtains

$$\begin{aligned}
\rho\left(\frac{CE_f\left(2^n\lambda x, 2^n\lambda y\right)}{R\cdot 2^{3n}}\right) &\le \frac{\rho\left(CE_f\left(2^n\lambda x, 2^n\lambda y\right)\right)}{R\cdot 2^{3n}} \\
&\le \frac{\phi_1\left(2^n x, 2^n y, 0\right)}{R\cdot 2^{3n}} \longrightarrow 0
\end{aligned} \tag{16}$$

for all $x, y \in \chi_\rho$, where $R \ge 16|\lambda| + 3$ is a fixed positive real. Thus we figure out by use of the first remark

$$\begin{aligned}
&\rho\left(\frac{1}{R}CE_{F_1}\left(\lambda x, \lambda y\right)\right) = \rho\left(\frac{1}{R}CE_{F_1}\left(\lambda x, \lambda y\right)\right. \\
&\left. - \frac{CE_f\left(2^n\lambda x, 2^n\lambda y\right)}{R\cdot 2^{3n}} + \frac{CE_f\left(2^n\lambda x, 2^n\lambda y\right)}{R\cdot 2^{3n}}\right) \le \frac{1}{R} \\
&\cdot \rho\left(F_1\left(3\lambda x - \lambda y\right) - \frac{f\left(2^n\left(3\lambda x - \lambda y\right)\right)}{2^{3n}}\right) + \frac{1}{R} \\
&\cdot \rho\left(F_1\left(\lambda x + \lambda y\right) - \frac{f\left(2^n\left(\lambda x + \lambda y\right)\right)}{2^{3n}}\right) + \frac{2\left|\lambda\right|^3}{R} \\
&\cdot \rho\left(F_1\left(2x - y\right) - \frac{f\left(2^n\left(2x - y\right)\right)}{2^{3n}}\right) + \frac{12\left|\lambda\right|^3}{R} \\
&\cdot \rho\left(F_1\left(x\right) - \frac{f\left(2^n x\right)}{2^{3n}}\right) + \frac{2\left|\lambda\right|^3}{R}\rho\left(F_1\left(y\right)\right. \\
&\left. - \frac{f\left(2^n y\right)}{2^{3n}}\right) + \frac{1}{R}\rho\left(\frac{CE_f\left(2^n\lambda x, 2^n\lambda y\right)}{2^{3n}}\right)
\end{aligned} \tag{17}$$

for all $x, y \in \chi_\rho$, $\lambda \in \Lambda$ and all positive integers n. Taking the limit as $n \longrightarrow \infty$, one obtains $\rho((1/R)CE_{F_1}(\lambda x, \lambda y)) = 0$, and so $CE_{F_1}(\lambda x, \lambda y) = 0$ for all $x, y \in \chi_\rho$. Hence, taking $\lambda = 1$ in $CE_{F_1}(x, y) = 0$, it follows that F_1 satisfies (6) and so it is cubic. On the other hand, since $\sum_{i=0}^{n}(1/2^{3(i+1)} + 1/2^3) \le 1$ for all $n \in \mathbb{N}$, it follows from (12) and the first remark that

$$\begin{aligned}
&\rho(f(x) - F_1(x)) \\
&= \rho\left(\sum_{i=0}^{n}\frac{1}{2^{3(i+1)}}\left(2^3 f\left(2^i x\right) - f\left(2^{i+1}x\right)\right) + \frac{f\left(2^{n+1}x\right)}{2^{3(n+1)}} - \frac{F_1(2x)}{2^3}\right) \le \frac{1}{2} \\
&\cdot \sum_{i=0}^{n}\frac{1}{2^{3(i+1)}}\rho\left(CE_f\left(2^i x, 2^i x\right)\right) + \frac{1}{2^3}\rho\left(\frac{f\left(2^{n+1}x\right)}{2^{3n}} - F_1(2x)\right) \le \frac{1}{16}\sum_{i=0}^{n}\frac{1}{2^{3i}}\phi_1\left(2^i x, 2^i x, 0\right) + \frac{1}{2^3} \\
&\cdot \rho\left(\frac{f\left(2^n \cdot 2x\right)}{2^{3n}} - F_1(2x)\right),
\end{aligned} \tag{18}$$

without applying the Fatou property of the modular ρ for all $x \in \chi_\rho$ and all $n \in \mathbb{N}$, from which we obtain the approximation of f by the cubic mapping F_1 as follows:

$$\begin{aligned}
\rho(f(x) - F_1(x)) &\le \frac{1}{16}\sum_{i=0}^{\infty}\frac{1}{2^{3i}}\phi_1\left(2^i x, 2^i x, 0\right) \\
&= \frac{1}{16}\Phi(x, x, 0)
\end{aligned} \tag{19}$$

for all $x \in \chi_\rho$ by taking $n \longrightarrow \infty$ in the last inequality.

Claim 2. F_1 is cubic homogeneous. By (17), we have $CE_{F_1}(\lambda x, \lambda x) = 0$, which yields $F_1(2\lambda x) = 8\lambda^3 F_1(x)$ for all $x \in \chi_\rho$ and $\lambda \in \Lambda$. From the assumption that for each $x \in \chi_\rho$ the mapping $r \longrightarrow f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, it follows that $F_1(\lambda x) = \lambda^3 F_1(x)$ for all $x \in \chi_\rho$ and $\lambda \in \mathbb{R}$ by the same argument as in the paper [4, 25]. Thus, for any nonzero $\lambda \in \mathbb{C}$

$$\begin{aligned}
F_1(\lambda x) &= F_1\left(2\frac{\lambda}{|\lambda|}\frac{|\lambda|}{2}x\right) = 8\left(\frac{\lambda}{|\lambda|}\right)^3 F_1\left(\frac{|\lambda|}{2}x\right) \\
&= 8\left(\frac{\lambda}{|\lambda|}\right)^3\left(\frac{|\lambda|}{2}\right)^3 F_1(x) = \lambda^3 F_1(x)
\end{aligned} \tag{20}$$

for all $x \in \chi_\rho$ and $\lambda \in \mathbb{C}$, which concludes that F_1 is cubic homogeneous.

Claim 3. F_1 is a cubic Lie derivation. From the second inequality in (9) and the second condition in (8), we arrive at

$$\begin{aligned}
\rho\left(\frac{1}{4}CD_{F_1}(x, y)\right) &= \rho\left(\frac{1}{4}CD_{F_1}(x, y) - \frac{CD_f\left(2^n x, 2^n y\right)}{4 \cdot 8^{2n}} + \frac{CD_f\left(2^n x, 2^n y\right)}{4 \cdot 8^{2n}}\right) \le \frac{1}{4} \\
&\cdot \rho\left(F_1([x, y]) - \frac{f\left(2^{2n}[x, y]\right)}{8^{2n}}\right) + \frac{1}{4} \\
&\cdot \rho\left(\frac{\left[x^3, f\left(2^n y\right)\right]}{8^n} - \left[x^3, F_1(y)\right]\right) + \frac{1}{4} \\
&\cdot \rho\left(\frac{\left[f\left(2^n x\right), y^3\right]}{8^n} - \left[F_1(x), y^3\right]\right) + \frac{1}{4 \cdot 8^{2n}} \\
&\cdot \rho\left(CD_f\left(2^n x, 2^n y\right)\right)
\end{aligned} \tag{21}$$

for all $x, y \in \chi_\rho$, which tends to zero as n tends to ∞. Therefore, one obtains $\rho((1/4)CD_{F_1}(x, y)) = 0$, and so F_1 is a cubic Lie derivation.

Claim 4. F_1 is a unique cubic Lie derivation. To show the uniqueness of F_1, let us assume that there exists a cubic Lie derivation $G_1 : \chi_\rho \longrightarrow \chi_\rho$ which satisfies the inequality

$$\begin{aligned}
\rho(f(x) - G_1(x)) &\le \frac{1}{16}\sum_{j=0}^{\infty}\frac{\phi_1\left(2^j x, 2^j x\right)}{2^{3j}} \\
&= \frac{1}{16}\Phi(x, x)
\end{aligned} \tag{22}$$

for all $x \in \chi_\rho$, but suppose $F_1(x_0) \ne G_1(x_0)$ for some $x_0 \in X$. Then there exists a positive constant $\varepsilon > 0$ such that $\varepsilon < \rho(F_1(x_0) - G_1(x_0))$. For such given $\varepsilon > 0$, it follows from (9) that there is a positive integer $n_0 \in \mathbb{N}$ such that $\sum_{j=n_0}^{\infty}(\phi_1(2^j x_0, 2^j x_0)/2^{3(j+1)}) < \varepsilon$. Since F_1 and G_1 are cubic mappings, we see from the equality $F_1(2^{n_0}x_0) = 2^{3n_0}F_1(x_0)$ and $G_1(2^{n_0}x_0) = 2^{3n_0}G_1(x_0)$ that

$$\begin{aligned}
\varepsilon &< \rho(F_1(x_0) - G_1(x_0)) \\
&= \rho\left(\frac{F_1\left(2^{n_0}x_0\right) - f\left(2^{n_0}x_0\right)}{2^{3n_0}} + \frac{f\left(2^{n_0}x_0\right) - G_1\left(2^{n_0}x_0\right)}{2^{3n_0}}\right) \le \frac{1}{2^{3n_0}}\rho\left(F_1\left(2^{n_0}x_0\right) - f\left(2^{n_0}x_0\right)\right) + \frac{1}{2^{3n_0}}\rho\left(f\left(2^{n_0}x_0\right) - G_1\left(2^{n_0}x_0\right)\right) \\
&\le \frac{1}{2^{3n_0}}\frac{1}{8}\sum_{j=0}^{\infty}\frac{\phi_1\left(2^{j+n_0}x_0, 2^{j+n_0}x_0\right)}{2^{3j}} = \frac{1}{8} \\
&\cdot \sum_{j=n_0}^{\infty}\frac{\phi_1\left(2^j x_0, 2^j x_0\right)}{2^{3j}} < \varepsilon,
\end{aligned} \tag{23}$$

which leads a contradiction. Hence the mapping F_1 is a unique cubic Lie derivation near f satisfying approximation (10) on the modular algebra χ_ρ. □

As a corollary of Theorem 3, we obtain the following stability result of cubic equation (6) associated with cubic Lie derivation on the Banach algebra χ_ρ, which may be considered as endowed with modular $\rho = \|\cdot\|$.

Corollary 4. *Suppose χ_ρ is a Banach algebra with norm $\|\cdot\|$. For given real numbers $\theta, \theta_i, \vartheta_i \geq 0$, $r_i < 3$ $(i = 1, 2)$, and $a_1 + b_1 < 3$, $a_2 + b_2 < 6$, suppose that a mapping $f : \chi_\rho \longrightarrow \chi_\rho$ satisfies*

$$\begin{aligned} \left\|CE_f(\lambda x, \lambda y)\right\| &\leq \theta_1 \|x\|^{r_1} + \theta_2 \|y\|^{r_2} \\ &\quad + \vartheta_1 \|x\|^{a_1} \|y\|^{b_1}, \qquad (24) \\ \left\|CD_f(x, y)\right\| &\leq \vartheta_2 \|x\|^{a_2} \|y\|^{b_2} \end{aligned}$$

for all $x, y \in \chi_\rho$ and $\lambda \in \Lambda$, where $x, y \neq 0$ whenever $r_i, a_i, b_i < 0$ and that for each $x \in \chi_\rho$ the mapping $r \longrightarrow f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous. Then there exists a unique cubic Lie derivation $F_1 : \chi_\rho \longrightarrow \chi_\rho$ such that

$$\begin{aligned} \rho\left(f(x) - F_1(x)\right) &\leq \frac{\theta_1 \|x\|^{r_1}}{2\left(2^3 - 2^{r_1}\right)} + \frac{\theta_2 \|x\|^{r_2}}{2\left(2^3 - 2^{r_2}\right)} \\ &\quad + \frac{\theta \|x\|^{a_1 + b_1}}{2\left(2^3 - 2^{a_1 + b_1}\right)} \end{aligned} \qquad (25)$$

for all $x \in \chi_\rho$, where $x \neq 0$ whenever $r_i, a_1 + b_1 < 0$.

We observe that if the modular ρ satisfies the Δ_2-condition, then $\kappa \geq 1$ for nontrivial modular ρ, and $\kappa \geq 2$ for nontrivial convex modular ρ. See [13–16]. In the following theorem, we prove generalized Hyers–Ulam stability of the system $CD_f = 0$ and $CE_f = 0$ using necessarily Δ_2-condition, which permits the existence of ρ-Cauchy sequence in χ_ρ.

Theorem 5. *Let χ_ρ be a ρ-complete convex modular space with Δ_2-condition. Suppose there exist two functions $\varphi_1, \varphi_2 : \chi_\rho^2 \longrightarrow [0, \infty)$ for which a mapping $f : \chi_\rho \longrightarrow \chi_\rho$ satisfies*

$$\begin{aligned} &\rho\left(CE_f(\lambda x, \lambda y)\right) \leq \varphi_1(x, y), \\ &\Psi(x, y) := \sum_{j=1}^{\infty} \frac{k^{4j}}{2^j} \varphi_1\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \end{aligned} \qquad (26)$$

$$\begin{aligned} &\rho\left(CD_f(x, y)\right) \leq \varphi_2(x, y), \\ &\lim_{n \longrightarrow \infty} k^{6n} \varphi_2\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned} \qquad (27)$$

for all $x, y \in \chi_\rho$ and $\lambda \in \Lambda$. If for each $x \in \chi_\rho$ the mapping $r \longrightarrow f(rx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique cubic Lie derivation $F_2 : \chi_\rho \longrightarrow \chi_\rho$ satisfying (6) and

$$\rho\left(f(x) - F_2(x)\right) \leq \frac{1}{4\kappa^2} \Psi(x, x) \qquad (28)$$

for all $x \in \chi_\rho$.

Proof. First, we remark that since $\sum_{j=1}^{\infty} (\kappa^{4j}/2^j) \varphi_1(0,0) = \Psi(0,0) < \infty$ and $\rho(CE_f(0,0)) \leq \varphi_1(0,0)$, we lead to $\varphi_1(0,0) = 0$, $CE_f(0,0) = 0$ and so $f(0) = 0$. Thus, it follows from (12) that

$$\rho\left(f(x) - 8f\left(\frac{x}{2}\right)\right) \leq \frac{1}{2} \varphi_1\left(\frac{x}{2}, \frac{x}{2}\right) \qquad (29)$$

for all $x \in \chi_\rho$. Thus, one obtains the following inequality by the convexity of the modular ρ and Δ_2-condition:

$$\begin{aligned} &\rho\left(f(x) - 8^2 f\left(\frac{x}{2^2}\right)\right) \\ &\quad \leq \frac{1}{2} \rho\left(2f(x) - 2 \cdot 8f\left(\frac{x}{2}\right)\right) \\ &\qquad + \frac{1}{2^2} \rho\left(2^2 \cdot 8f\left(\frac{x}{2}\right) - 2^2 \cdot 8^2 f\left(\frac{x}{2^2}\right)\right) \\ &\quad \leq \frac{\kappa}{2^2} \varphi_1\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{\kappa^5}{2^3} \varphi_1\left(\frac{x}{2^2}, \frac{x}{2^2}\right) \end{aligned} \qquad (30)$$

for all $x \in \chi_\rho$. Then using the repeated process for any $n \geq 2$, we prove the following functional inequality:

$$\rho\left(f(x) - 8^n f\left(\frac{x}{2^n}\right)\right) \leq \frac{1}{2\kappa^3} \sum_{j=1}^{n} \frac{\kappa^{4j}}{2^j} \varphi_1\left(\frac{x}{2^j}, \frac{x}{2^j}\right) \qquad (31)$$

for all $x \in \chi_\rho$. In fact, it is true for $n = 2$. Assume that inequality (31) holds true for n. Thus, using the convexity of the modular ρ, we deduce

$$\begin{aligned} &\rho\left(f(x) - 8^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right) \\ &\quad = \rho\left(\frac{1}{2}\left(2f(x) - 2 \cdot 8f\left(\frac{x}{2}\right)\right)\right. \\ &\qquad \left.+ \frac{1}{2}\left(2 \cdot 8f\left(\frac{x}{2}\right) - 2 \cdot 8^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right)\right) \leq \frac{\kappa}{2} \\ &\qquad \cdot \rho\left(f(x) - 8f\left(\frac{x}{2}\right)\right) + \frac{\kappa^4}{2} \rho\left(f\left(\frac{x}{2}\right)\right. \\ &\qquad \left.- 8^n f\left(\frac{x}{2^{n+1}}\right)\right) \leq \frac{\kappa}{2} \frac{1}{2} \varphi_1\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{\kappa^4}{2} \cdot \frac{1}{2\kappa^3} \\ &\qquad \cdot \sum_{j=1}^{n} \frac{\kappa^{4j}}{2^j} \varphi_1\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) = \frac{1}{2\kappa^3} \sum_{j=1}^{n+1} \frac{\kappa^{4j}}{2^j} \varphi_1\left(\frac{x}{2^j}, \frac{x}{2^j}\right), \end{aligned} \qquad (32)$$

which proves (31) for $n+1$. Now, replacing x by $2^{-m}x$ in (31), we have

$$\begin{aligned}
&\rho\left(2^{3m} f\left(\frac{x}{2^m}\right)-2^{3(m+n)} f\left(\frac{x}{2^{m+n}}\right)\right) \\
&\quad\leq \kappa^{3m} \rho\left(f\left(\frac{x}{2^m}\right)-2^{3n} f\left(\frac{x}{2^{m+n}}\right)\right) \\
&\quad\leq \frac{\kappa^{3m}}{2\kappa^3}\sum_{j=1}^{n}\frac{\kappa^{4j}}{2^j}\varphi_1\left(\frac{x}{2^{j+m}},\frac{x}{2^{j+m}}\right) \\
&\quad\leq \frac{\kappa^{3m}}{2\kappa^3}\sum_{j=1}^{n}\frac{\kappa^{4j}}{2^j}\varphi_1\left(\frac{x}{2^{j+m}},\frac{x}{2^{j+m}}\right)\cdot\frac{\kappa^m}{2^m} \\
&\quad= \frac{1}{2\kappa^3}\sum_{j=1}^{n}\frac{\kappa^{4(j+m)}}{2^{j+m}}\varphi_1\left(\frac{x}{2^{j+m}},\frac{x}{2^{j+m}}\right) \\
&\quad= \frac{1}{2\kappa^3}\sum_{j=m+1}^{m+n}\frac{\kappa^{4j}}{2^j}\varphi_1\left(\frac{x}{2^j},\frac{x}{2^j}\right),
\end{aligned} \tag{33}$$

which converges to zero as $m \longrightarrow \infty$ by assumption (27). Thus, the sequence $\{8^n f(x/2^n)\}$ is ρ-Cauchy for all $x \in \chi_\rho$ and so it is ρ-convergent in χ_ρ since the space χ_ρ is ρ-complete. Thus, we may define a mapping $F_2 : \chi_\rho \longrightarrow \chi_\rho$ as

$$\begin{aligned}
&F_2(x) := \rho - \lim_{n\longrightarrow\infty} 8^n f\left(\frac{x}{2^n}\right) \Longleftrightarrow \\
&\lim_{n\longrightarrow\infty}\rho\left(8^n f\left(\frac{x}{2^n}\right)-F_2(x)\right)=0,
\end{aligned} \tag{34}$$

for all $x \in \chi_\rho$.

Claim 1. F_2 is a cubic mapping with estimation (28) near f. By Δ_2-condition without using the Fatou property, we can see the following inequality:

$$\begin{aligned}
&\rho\left(f(x)-F_2(x)\right) \leq \frac{1}{2}\rho\left(2f(x)-2\cdot 8^n f\left(\frac{x}{2^n}\right)+2\right. \\
&\quad\left.\cdot 8^n f\left(\frac{x}{2^n}\right)-2F_2(x)\right) \leq \frac{\kappa}{2}\rho\left(f(x)-8^n f\left(\frac{x}{2^n}\right)\right) \\
&\quad+\frac{\kappa}{2}\rho\left(8^n f\left(\frac{x}{2^n}\right)-F_2(x)\right) \leq \frac{\kappa}{2}\cdot\frac{1}{2\kappa^3} \\
&\quad\cdot\sum_{j=1}^{n}\frac{\kappa^{4j}}{2^j}\varphi_1\left(\frac{x}{2^j},\frac{x}{2^j}\right)+\frac{\kappa}{2}\rho\left(8^n f\left(\frac{x}{2^n}\right)-F_2(x)\right) \\
&\quad\leq \frac{1}{4\kappa^2}\sum_{j=1}^{\infty}\frac{\kappa^{4j}}{2^j}\varphi_1\left(\frac{x}{2^j},\frac{x}{2^j}\right)=\frac{1}{4\kappa^2}\Psi(x,x)
\end{aligned} \tag{35}$$

by taking $n \longrightarrow \infty$, which yields approximation (28).

Now, setting $(x,y) := (2^{-n}x, 2^{-n}y)$ in (26) and multiplying the resulting inequality by 8^n, we get

$$\begin{aligned}
&\rho\left(2^{3n}CE_f\left(2^{-n}\lambda x, 2^{-n}\lambda y\right)\right) \leq \kappa^{3n}\varphi_1\left(2^{-n}x,2^{-n}y\right) \\
&\quad\leq \kappa^{3n}\varphi_1\left(2^{-n}x,2^{-n}y\right)\cdot\frac{\kappa^n}{2^n}=\frac{\kappa^{4n}}{2^n}\varphi_1\left(2^{-n}x,2^{-n}y\right),
\end{aligned} \tag{36}$$

which tends to zero as $n \longrightarrow \infty$ for all $x, y \in \chi_\rho$. Thus, it follows from the first remark that

$$\begin{aligned}
&\rho\left(\frac{1}{R}CE_{F_2}(\lambda x,\lambda y)\right)=\rho\left(\frac{1}{R}CE_{F_2}(\lambda x,\lambda y)\right. \\
&\quad\left.-\frac{2^{3n}}{R}CE_f\left(\frac{\lambda x}{2^n},\frac{\lambda y}{2^n}\right)+\frac{2^{3n}}{R}CE_f\left(\frac{\lambda x}{2^n},\frac{\lambda y}{2^n}\right)\right) \\
&\quad\leq \frac{1}{R}\rho\left(F_2(3\lambda x-\lambda y)-2^{3n}f\left(\frac{3\lambda x-\lambda y}{2^n}\right)\right) \\
&\quad+\frac{2|\lambda|^3}{R}\rho\left(F_2(y)-2^{3n}f\left(\frac{y}{2^n}\right)\right)+\frac{1}{R} \\
&\quad\cdot\rho\left(F_2(\lambda x+\lambda y)-2^{3n}f\left(\frac{\lambda x+\lambda y}{2^n}\right)\right)+\frac{2|\lambda|^3}{R} \\
&\quad\cdot\rho\left(F_2(2x-y)-2^{3n}f\left(\frac{2x-y}{2^n}\right)\right)+\frac{12|\lambda|^3}{R} \\
&\quad\cdot\rho\left(F_2(x)-2^{3n}f\left(\frac{x}{2^n}\right)\right)+\frac{1}{R} \\
&\quad\cdot\rho\left(2^{3n}CE_f\left(\frac{\lambda x}{2^n},\frac{\lambda y}{2^n}\right)\right)
\end{aligned} \tag{37}$$

for all $x, y \in \chi_\rho$, $\lambda \in \Lambda$, and all positive integers n, where $R \geq 16|\lambda|^3+3$ is a fixed real number. Taking the limit as $n \longrightarrow \infty$, one obtains $\rho((1/R)CE_{F_2}(x,y)) = 0$, and thus $CE_{F_2}(x,y) = 0$ for all $x, y \in \chi_\rho$. Hence $F_2 : \chi_\rho \longrightarrow \chi_\rho$ satisfies (6), and so it is cubic.

Claim 2. F_2 is a cubic Lie derivation. By the same proof of Theorem 3, the mapping F_2 is a cubic homogeneous mapping. From the last inequality in (27) and the last condition in (26), one obtains that

$$\begin{aligned}
&\rho\left(\frac{1}{4}CD_{F_2}(x,y)\right)=\rho\left(\frac{1}{4}CD_{F_2}(x,y)\right. \\
&\quad\left.-8^{2n}\frac{CD_f\left(2^{-n}x,2^{-n}y\right)}{4}+8^{2n}\frac{CD_f\left(2^{-n}x,2^{-n}y\right)}{4}\right) \\
&\quad\leq\frac{1}{4}\rho\left(F_2([x,y])-8^{2n}f\left(2^{-2n}[x,y]\right)\right)+\frac{1}{4} \\
&\quad\cdot\rho\left(8^n\left[x^3,f\left(2^{-n}y\right)\right]-\left[x^3,F_2(y)\right]\right)+\frac{1}{4} \\
&\quad\cdot\rho\left(8^n\left[f\left(2^{-n}x\right),y^3\right]-\left[F_2(x),y^3\right]\right)+\frac{1}{4} \\
&\quad\cdot\rho\left(8^{2n}CD_f\left(2^{-n}x,2^{-n}y\right)\right)\leq\frac{1}{4}\rho\left(F_2([x,y])\right. \\
&\quad\left.-8^{2n}f\left(2^{-2n}[x,y]\right)\right)+\frac{1}{4}\rho\left(8^n\left[x^3,f\left(2^{-n}y\right)\right]\right. \\
&\quad\left.-\left[x^3,F_2(y)\right]\right)+\frac{1}{4}\rho\left(8^n\left[f\left(2^{-n}x\right),y^3\right]\right. \\
&\quad\left.-\left[F_2(x),y^3\right]\right)+\frac{\kappa^{6n}}{4}\varphi_2\left(2^{-n}x,2^{-n}y\right)
\end{aligned} \tag{38}$$

for all $x, y \in \chi_\rho$, from which $CD_{F_2}(x, y) = 0$ by taking $n \longrightarrow \infty$ and so F_2 is a cubic Lie derivation.

Claim 3. F_2 is unique. To show the uniqueness of F_2, let us assume that there exists a cubic Lie derivation $G_2 : \chi_\rho \longrightarrow \chi_\rho$ which satisfies the approximation (28). Since F_2 and G_2 are cubic mappings, we see from the equalities $2^{3n}F_2(2^{-n}x) = F_2(x)$ and $2^{3n}G_2(2^{-n}x) = G_2(x)$ that

$$\begin{aligned} &\rho\left(G_2(x) - F_2(x)\right) \\ &= \rho\left(\frac{2^{3(n+1)}}{2^3}\left(G_2\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right)\right. \\ &\left.+ \frac{2^{3(n+1)}}{2^3}\left(f\left(\frac{x}{2^n}\right) - F_2\left(\frac{x}{2^n}\right)\right)\right) \le \frac{\kappa^{3(n+1)}}{2^3} \\ &\cdot \rho\left(G_2\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right) + \frac{\kappa^{3(n+1)}}{2^3}\rho\left(f\left(\frac{x}{2^n}\right)\right. \\ &\left.- F_2\left(\frac{x}{2^n}\right)\right) \le \frac{\kappa^{3(n+1)}}{2^3} \cdot \frac{1}{2\kappa^2}\sum_{j=1}^{\infty}\frac{\kappa^{4j}}{2^j}\varphi_1\left(\frac{x}{2^{j+n}}, \frac{x}{2^{j+n}}\right) \\ &\cdot \frac{\kappa^n}{2^n} \le \frac{\kappa}{2^4}\sum_{j=1}^{\infty}\frac{\kappa^{4(j+n)}}{2^{j+n}}\varphi_1\left(\frac{x}{2^{j+n}}, \frac{x}{2^{j+n}}\right) = \frac{\kappa}{2^4} \\ &\cdot \sum_{j=n+1}^{\infty}\frac{\kappa^{4j}}{2^j}\varphi_1\left(\frac{x}{2^j}, \frac{x}{2^j}\right) \end{aligned} \tag{39}$$

which tends to zero as $n \longrightarrow \infty$ for all $x \in \chi_\rho$. Hence the mapping F_2 is a unique cubic Lie derivation satisfying (28).

□

Remark. In Theorem 5 if χ_ρ is a Banach algebra with norm ρ, and so $\rho(2x) = 2\rho(x)$, $\kappa := 2$, then we see from (26) and (27) that there exists a unique cubic Lie derivation $F_2 : \chi_\rho \longrightarrow \chi_\rho$, defined as $F_2(x) = \lim_{n\longrightarrow\infty}2^{3n}f(x/2^n)$, $x \in \chi_\rho$, which satisfies (6) and

$$\rho\left(f(x) - F_2(x)\right) \le \frac{1}{16}\sum_{j=1}^{\infty}2^{3j}\varphi_1\left(\frac{x}{2^j}, \frac{x}{2^j}\right) \tag{40}$$

for all $x \in \chi_\rho$.

As a corollary of Theorem 5, we obtain the following stability result of (6), which generalizes stability result on Banach algebras.

Corollary 6. *Suppose χ_ρ is a Banach algebra with norm $\|\cdot\|$ and $\kappa = 2$. For given real numbers $\theta_i, \vartheta_i \ge 0$, $r_i > 3$ $(i = 1, 2)$, $a_1 + b_1 > 3$, and $6 < a_2 + b_2$, if a mapping $f : \chi_\rho \longrightarrow \chi_\rho$ satisfies*

$$\begin{aligned} \left\|CE_f(x, y)\right\| &\le \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \vartheta_1\|x\|^{a_1}\|y\|^{b_1}, \\ \left\|CD_f(x, y)\right\| &\le \vartheta_2\|x\|^{a_2}\|y\|^{b_2} \end{aligned} \tag{41}$$

for all $x, y \in \chi_\rho$ and $\lambda \in \Lambda$, then there exists a unique cubic Lie derivation $F_2 : \chi_\rho \longrightarrow \chi_\rho$ such that

$$\begin{aligned} \left\|f(x) - F_2(x)\right\| &\le \frac{\theta_1\|x\|^{r_1}}{2\left(2^{r_1} - 2^3\right)} + \frac{\theta_2\|x\|^{r_2}}{2\left(2^{r_2} - 8\right)} \\ &+ \frac{\vartheta_1\|x\|^{a_1+b_1}}{2\left(2^{a_1+b_1} - 8\right)} \end{aligned} \tag{42}$$

for all $x \in \chi_\rho$.

3. Conclusion

We introduce modular algebras with modular ρ over $\mathbb{K}$ and obtain stability results of a cubic equation associated with cubic derivations on ρ-complete modular algebras, which generalizes stability results on Banach algebras.

Data Availability

Previously reported data were used to support this study and are available at https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-017-1422-z and https://www.hindawi.com/journals/jfs/2015/461719/. These prior studies (and datasets) are cited at relevant places within the text as [13–17].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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