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Research Article

Algebra Properties in Fourier-Besov Spaces and Their Applications

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We estimate the norm of the product of two scale functions in Fourier-Besov spaces. As applications of these algebra properties, we establish the global well-posedness for small initial data and local well-posedness for large initial data of the generalized Navier-Stokes equations. Particularly, we give a blow-up criterion of the solutions in Fourier-Besov spaces as well as a space analyticity of Gevrey regularity.

1. Introduction

In this paper, we study the mild solutions to the generalized Navier-Stokes equations (NSE) in $\mathbb{R}^+ \times \mathbb{R}^n$

$$\begin{aligned} u_t + \mu (-\triangle)^\beta u &= Q(u,u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n; \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{1}$$

Here $u(t,x) = (u^1(t,x), \cdots, u^n(t,x))$, $\mu > 0$ is a constant, and the operator $(-\triangle)^\beta$ is the Fourier multiplier with symbol $|\xi|^{2\beta}$. The bilinear operator Q denotes the map of the form

$$Q^j(u,v) := \sum_{k,l,m=1}^{n} q_{k,l}^{j,m} \partial_m \left(u^k v^l\right), \quad j = 1, \cdots, n, \tag{2}$$

where $q_{k,l}^{j,m}(g) := \sum_{a,b=1}^{n} \alpha_{k,l}^{j,m,a,b} \mathcal{F}^{-1}((\xi_a \xi_b / |\xi|^2) \widehat{g}(\xi))$ and $\alpha_{k,l}^{j,m,a,b}$ are real numbers.

The incompressible NSE is a particular case of (1), by taking $Q_{NS} = -(1/2)\mathcal{P}(\operatorname{div}(u \otimes v) + \operatorname{div}(v \otimes u))$ with the Leray projector $\mathcal{P}$ defined as

$$\mathcal{F}(\mathcal{P} f)^j(\xi) = \sum_{k=1}^{n} \left(\delta_{j,k} - 1\right) \frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}_k(\xi), \tag{3}$$

where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$; that is,

$$\begin{aligned} u_t + \mu (-\triangle)^\beta u &= -\mathcal{P} \operatorname{div}(u \otimes u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n; \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{4}$$

If we set the initial data u_0 to be a divergence free vector field, the above system is exactly the (fractional) incompressible NSE.

For the classical incompressible NSE ($\beta = 1, Q = Q_{NS}$), the study of mild solution started by Fujita and Kato [1] in the space frame $\dot{H}^{1/2}(\mathbb{R}^3)$ and was extended by many mathematicians in different spaces [2–5]. As for the generalized case $\beta > 0, Q = Q_{NS}$, Lions [6] proved the global existence of classical solutions in 3-dimension when $\beta \geq 5/4$ (see also Wu [7] in n-dimension). For the important case $\beta < 5/4$, Wu [8, 9] studied the well-posedness in $\dot{B}_{p,q}^{1-2\beta+3/p}(\mathbb{R}^3)$. Inspired by Xiao [10] in the classical case ($\beta = 1$), Li and Zhai [11, 12] studied (1) in some critical Q-type spaces for $\beta \in (1/2, 1)$, and Zhai [13] showed the well-posedness in $BMO^{-(2\beta-1)}$ when $\beta \in (1/2,1)$. For the biggest critical space $\dot{B}_{\infty,\infty}^{-(2\beta-1)}$, Yu and Zhai [14] proved the well-posedness when $\beta \in (1/2, 1)$, and Cheskidov and Shvydkoy [15] showed the ill-posedness when $\beta \in [1, 5/4)$. Deng and Yao [16] studied (1) in Triebel-Lizorkin spaces $\dot{F}_{\alpha,r}^{-\beta}$ in 3-dimension and obtained the well-posedness in $\dot{F}^{-\beta}_{3/(\beta-1),2}$ and ill-posedness in $\dot{F}^{-\beta}_{3/(\beta-1),r}$ $(r > 2)$ in the case $\beta \in (1, 5/4)$.

We focus on the results in Fourier-Besov spaces. Early results came from Cannone and Karch [17] in pseudomeasure type spaces $\mathscr{PM}^{\alpha}$, in which they discuss the singular and regular solution of NSE. Iwabuchi [18] introduced a Besov type space $\mathscr{B}^s_q$ (named Fourier-Herz space in [19]) to study the well-posedness and ill-posedness of Keller-Segel system. Scapellato and Ragusa in [20] introduced a Morrey-type sapce (named mixed Morrey space) to study qualitative properties of partial differential equations with discontinuous coefficients. Lei and Lin [21] study the global well-posedness for NSE in spaces $\dot{\mathscr{X}}^{-1}$. All those spaces are special cases of Fourier-Besov spaces, which were first introduced by Konieczny and Yoneda [22], to study the dispersive effect of the Coriolis for NSE.

Our first result on the estimates of the product uv in Fourier-Besov spaces has the similar form with that in Sobolev spaces [23], which can be seen as follows.

Theorem 1. *Let $1 \le p, q \le \infty$ and $s_1, s_2 \in \mathbb{R}$ such that*

$$\begin{aligned} s_1 &< \frac{n}{p'}, \\ s_2 &< \frac{n}{p'}, \\ s_1 + s_2 &> \max\left\{\frac{n}{p'} - \frac{n}{p}, 0\right\}, \end{aligned} \tag{5}$$

where $1/p + 1/p' = 1$. Then for $u \in F\dot{B}^{s_1}_{p,q}$, $v \in F\dot{B}^{s_2}_{p,q}$, one has

$$\|uv\|_{F\dot{B}^{s_1+s_2-n/p'}_{p,q}} \le C\,\|u\|_{F\dot{B}^{s_1}_{p,q}}\,\|v\|_{F\dot{B}^{s_2}_{p,q}}. \tag{6}$$

Remark 2. An important case is $p = q = 2$, since by Plancherel's identity we know $F\dot{B}^{s}_{2,2} = \dot{H}^s$. This theorem gives that if $s_1 < n/2, s_2 < n/2, s_1 + s_2 > 0$, then $\|uv\|_{\dot{H}^{s_1+s_1-n/2}} \le C\|u\|_{\dot{H}^{s_1}}\|v\|_{\dot{H}^{s_2}}$. This is exactly the case in homogeneous Sobolev spaces [24]. A similar result in Besov space can be seen in p. 61 of [25].

As an application of this theorem, we study the Cauchy problem of (1) in Fourier-Besov spaces $F\dot{B}^{1-2\beta+n/p'}_{p,q}$.

Theorem 3. *Let $1 \le p, q \le \infty, q'/(1+q') < \beta < (q'/(1+q'))\min\{1 + n/p', 1 + n/2\}$. Then for any $u_0 \in F\dot{B}^{1-2\beta+n/p'}_{p,q}$ with $\nabla \cdot u_0 = 0$, the Cauchy problem (1) admits a unique mild solution u and*

$$\begin{aligned} u \in E_T &:= \mathscr{C}\left([0,T); F\dot{B}^{1-2\beta+n/p'}_{p,q}\right) \\ &\cap \mathscr{L}^{q}\left([0,T); F\dot{B}^{1-2\beta+n/p'+2\beta/q}_{p,q}\right) \\ &\cap \mathscr{L}^{2q}\left([0,T); F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q}\right). \end{aligned} \tag{7}$$

Moreover, let T^* denote the maximal time of existence of such a solution, then

(i) there is a constant C_0 such that if $\|u_0\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}} \le C_0\mu$, then

$$T^* = \infty; \tag{8}$$

(ii) if $T^* < \infty$, then

$$\|u\|_{\mathscr{L}^{2q}([0,T^*);F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} = \infty. \tag{9}$$

Particularly, our result also holds in the case $p > 1, q = 1, \beta = 1$.

This well-posedness result corresponds to the classical case for initial data $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$ [2] if we take $n = 3, p = q = 2, \beta = 1, \mu = 1$ in Theorem 3. Unfortunately, our result is not suitable for the case $p = q = 1, \beta = 1$, which has been proved in [26]. To address this special case, we also prove the following theorem.

Theorem 4. *Let $1 \le p \le q \le 2, \beta \in (1/(1 + 1/q'), (1 + n/2p')/(1 + 1/q'))$. Then for any $u_0 \in F\dot{B}^{1-2\beta+n/p'}_{p,q}$ with $\nabla \cdot u_0 = 0$, the Cauchy problem (1) admits a unique mild solution $u \in E_T$. Moreover, let T^* denote the maximal time of existence of such a solution, then*

(i) there is a constant C_0 such that if $\|u_0\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}} \le C_0\mu$, then

$$T^* = \infty; \tag{10}$$

(ii) if $T^* < \infty$, then

$$\|u\|_{\mathscr{L}^{2q}([0,T^*);F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} = \infty. \tag{11}$$

Particularly, our result also holds in the case $p = q = 1, \beta = 1$.

Now we focus on the space analyticity. Our main method is Gevrey estimate, which was introduced by Foias and Temam [27], since that Gevrey class technique has become an effective approach in the study of space analyticity of solutions. Ferrari and Titi [28] established Gevrey regularity for a very large class of parabolic equation with analytic nonlinearity. Grujić and Kukavica [29] prove the Gevrey regularity for NSE in L^p. More results on the analyticity of solution for NSE can be seen in Lemarie-Rieusset [5] and references therein. Biswas [30] established Gevrey class regularity of solutions to a large class of dissipative equations in Besov type spaces defined via caloric extension. Bae [31] proved the Gevrey estimate of solution for NSE in the spaces $\dot{\mathscr{X}}^{-1}$. Inspired by this, we establish the Gevrey class regularity for the generalized NSE in the Fourier-Besov spaces. We indicate that any order derivative of the solution u enjoys the same behavior with u in some sense. In fact, denote by $e^{\sqrt{t}|D|^{\beta}}$ the Fourier multiplier with symbol $e^{\sqrt{t}|\xi|^{\beta}}$, then we have the following result.

Theorem 5. *Let $1 \le p, q \le \infty, q'/(1 + q') < \beta < (q'/(1 + q'))\min\{1 + n/p', 1 + n/2\}, \beta \le 1$. Then for any $u_0 \in F\dot{B}^{1-2\beta+n/p'}_{p,q}$, the Cauchy problem (1) admits a unique mild* *solution $u \in E_T$ such that $e^{\sqrt{t}|D|^\beta} u \in E_T$. Particularly, our result also holds in the case $p > 1, q = 1, \beta = 1$.*

Theorem 6. *Let $1 \le p \le q \le 2, q'/(1+q') < \beta < (q'/(1+q'))(1+n/2p'), \beta \le 1$. Then for any $u_0 \in F\dot{B}^{1-2\beta+n/p'}_{p,q}$, the Cauchy problem (1) admits a unique mild solution $u \in E_T$ such that $e^{\sqrt{t}|D|^\beta} u \in E_T$. Particularly, our result also holds in the case $p = q = 1, \beta = 1$.*

Remark 7. In our later proof, we can also obtain that conclusions (i), (ii) in Theorems 3 and 4 are also valid for $e^{\sqrt{t}|D|^\beta} u$ in Theorems 5 and 6, respectively.

Throughout this paper, the notation $A \sim B$ means that there exist positive constants $C_1 \le C_2$ such that $C_1 A \le B \le C_2 A$. We use $\dot{B}^s_{p,q}$ to denote the classical homogeneous Besov spaces and $\dot{H}^s$ the homogeneous Sobolev spaces. Also, C denotes a positive constant which may differ in lines if not being specified, and p' is the number satisfying $1/p + 1/p' = 1$ for $1 \le p \le \infty$. The inverse Fourier transform is denoted by $\mathcal{F}^{-1}$.

We organize the paper as follows. In Section 2, we give some basic properties of Fourier-Besov spaces. Then we prove Theorem 1 as well as a corollary. In Section 3, we give the proof of Theorems 3 and 4. And in Section 4, we prove the space analyticity of Theorems 5 and 6.

2. Algebra Properties in Fourier-Besov Spaces

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a radial real-valued smooth function such that $0 \le \varphi(\xi) \le 1$ and

$$\operatorname{supp}\varphi \subset \left\{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\},$$
$$\sum_{j\in Z} \varphi\left(2^{-j}\xi\right) = 1 \quad \text{for any } \xi \ne 0. \tag{12}$$

We denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and $\mathbb{P}$ the set of all polynomials. The space of tempered distributions is denoted by S'.

Definition 8. For $s \in \mathbb{R}, 1 \le p, q \le \infty$, set

$$\|f\|_{F\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j\in Z} 2^{jsq} \left\|\varphi_j \widehat{f}\right\|^q_{L^p}\right)^{1/q}, & q < \infty; \\ \sup_{j\in Z} 2^{js} \left\|\varphi_j \widehat{f}\right\|_{L^p}, & q = \infty. \end{cases} \tag{13}$$

We define the homogeneous Fourier-Besov space $F\dot{B}^s_{p,q}$ as

$$F\dot{B}^s_{p,q} = \left\{ f \in \frac{S'}{\mathbb{P}} : \|f\|_{F\dot{B}^s_{p,q}} < \infty \right\}. \tag{14}$$

The Fourier-Besov spaces look similar to the classical Besov spaces, but without the inverse Fourier transform. In fact, there are close relationships between them [32]. These spaces are, also, similar to central Morrey spaces studied in [33]. In order to apply in PDE, we also need to derive the properties of Fourier-Besov spaces with space-time norm.

Definition 9. Let $s \in \mathbb{R}, 1 \le p, q \le \infty$ and $I = [0, T), T \in (0, \infty]$. The space-time norm is defined on $f(t, x)$ by

$$\|f(t,x)\|_{\mathscr{L}^r(I;F\dot{B}^s_{p,q})} := \left(\sum_{j\in Z} 2^{jsq} \left\|\varphi_j \widehat{f}\right\|^q_{L^r(I;L^p)}\right)^{1/q};$$
$$\|f(t,x)\|_{\mathscr{L}^r(I;\dot{B}^s_{p,q})} := \left(\sum_{j\in Z} 2^{jsq} \left\|\mathcal{F}^{-1}\varphi_j \widehat{f}\right\|^q_{L^r(I;L^p)}\right)^{1/q}. \tag{15}$$

Here $\dot{B}^s_{p,q}$ denotes the classical homogeneous Besov space. Besides, we also define the norm $\|f(t,x)\|_{L^r(I;X)}$ for some Banach space X with norm $\|\cdot\|_X$ by

$$\|f(t,x)\|_{L^r(I;X)} := \left(\int_I \|f(t,\cdot)\|^r_X \, dt\right)^{1/r}. \tag{16}$$

Remark 10. By the definition and Minkowski's inequality, we easily deduce that

$$\|f(t,x)\|_{\mathscr{L}^r(I;F\dot{B}^s_{p,q})} \le \|f(t,x)\|_{L^r(I;F\dot{B}^s_{p,q})}, \quad \text{for } r \le q;$$
$$\|f(t,x)\|_{L^r(I;F\dot{B}^s_{p,q})} \le \|f(t,x)\|_{\mathscr{L}^r(I;F\dot{B}^s_{p,q})}, \quad \text{for } r \ge q. \tag{17}$$

Proposition 11. *Let $s \in \mathbb{R}, 1 \le p, q, r \le \infty, I = [0, T), T \in (0, \infty]$. The following inclusions hold.*

(1) If $p = 2$, then $\mathscr{L}^r(I; F\dot{B}^s_{2,q}) = \mathscr{L}^r(I; \dot{B}^s_{2,q})$.

(2) If $p < 2$, then $\mathscr{L}^r(I; F\dot{B}^s_{p,q}) \subset \mathscr{L}^r(I; \dot{B}^s_{p',q})$.

(3) If $p > 2$, then $\mathscr{L}^r(I; \dot{B}^s_{p,q}) \subset \mathscr{L}^r(I; F\dot{B}^s_{p',q})$.

(4) If $q_1 \le q_2$, then $\mathscr{L}^r(I; F\dot{B}^s_{p,q_1}) \subset \mathscr{L}^r(I; F\dot{B}^s_{p,q_2})$.

(5) If $1 \le q \le \infty, 1 \le p_1 \le p_2 \le \infty, s_1, s_2 \in R$ satisfy $s_1 + n/p_1 = s_2 + n/p_2$, then

$$\mathscr{L}^r\left(I; F\dot{B}^{s_2}_{p_2,q}\right) \subset \mathscr{L}^r\left(I; F\dot{B}^{s_1}_{p_1,q}\right). \tag{18}$$

(6) If $1 \le p_1, p_2, q_1, q_2, r_1, r_2 \le \infty$ satisfy $s = (1-\theta)s_1 + \theta s_2, 1/p = (1-\theta)/p_1 + \theta/p_2, 1/q = (1-\theta)/q_1 + \theta/q_2$ and $1/r = (1-\theta)/r_1 + \theta/r_2$ for $0 \le \theta \le 1$, then

$$\|f\|_{\mathscr{L}^r(I;F\dot{B}^s_{p,q})} \le \|f\|^{1-\theta}_{\mathscr{L}^{r_1}(I;F\dot{B}^{s_1}_{p_1,q_1})} \|f\|^{\theta}_{\mathscr{L}^{r_2}(I;F\dot{B}^{s_2}_{p_2,q_2})}. \tag{19}$$

The special case $p = q$ has an interesting equivalent norm, which can be seen by the following proposition (see [22] for the proof).

Proposition 12. *Define the spaces $\dot{\mathcal{X}}_{s,p}$ as*

$$\dot{\mathcal{X}}_{s,p} = \left\{ f \in \frac{S'}{\mathbb{P}} : \left(\int_{R^n} |\xi|^{sp} \left|\widehat{f}\right|^p d\xi\right)^{1/p} < \infty \right\}. \tag{20}$$

Then we have $\dot{\mathcal{X}}_{s,p} = F\dot{B}^s_{p,p}$ and the norms are equivalent

$$\|f\|_{F\dot{B}^s_{p,p}} \sim \left(\int_{R^n} |\xi|^{sp} \left|\widehat{f}\right|^p d\xi\right)^{1/p}. \tag{21}$$

Now we give the proof of Theorem 1. The tools we use are the paraproduct and Bony's decomposition, which can be found in [8, 17].

Proof of Theorem 1. We will use the technique of the paraproduct. Set

$$\begin{aligned} \dot{\Delta}_j u &= \left(\mathscr{F}^{-1}\varphi_j\right) * u, \\ \dot{S}_j &= \sum_{k\le j-1} \dot{\Delta}_k u, \\ \tilde{\dot{\Delta}}_j u &= \sum_{|k-j|\le 1} \dot{\Delta}_k u, \\ &\qquad \text{for } \forall j \in Z. \end{aligned} \tag{22}$$

By Bony's decomposition, we have for fixed j

$$\begin{aligned} \dot{\Delta}_j (uv) &= \sum_{|k-j|\le 4} \dot{\Delta}_j \left(\dot{S}_{k-1} u \dot{\Delta}_k v\right) \\ &\quad + \sum_{|k-j|\le 4} \dot{\Delta}_j \left(\dot{S}_{k-1} v \dot{\Delta}_k u\right) \\ &\quad + \sum_{k\ge j-3} \dot{\Delta}_j \left(\dot{\Delta}_k u \tilde{\dot{\Delta}}_k v\right) := I_j + II_j + III_j. \end{aligned} \tag{23}$$

Thus we can divide the norm by

$$\begin{aligned} \|uv\|_{F\dot{B}^{s_1+s_2-n/p'}_{p,q}} &\le \left(\sum_j 2^{(s_1+s_2-n/p')jq} \left\|\widehat{I_j}\right\|^q_{L^p}\right)^{1/q} \\ &\quad + \left(\sum_j 2^{(s_1+s_2-n/p')jq} \left\|\widehat{II_j}\right\|^q_{L^p}\right)^{1/q} \\ &\quad + \left(\sum_j 2^{(s_1+s_2-n/p')jq} \left\|\widehat{III_j}\right\|^q_{L^p}\right)^{1/q}. \end{aligned} \tag{24}$$

The terms I_j and II_j are symmetrical. Using Young's inequality and Hölder's inequality

$$\begin{aligned} \left\|\widehat{I_j}\right\|_{L^p} &\le \sum_{|k-j|\le 4} \left\|\widehat{\dot{S}_{k-1} u \dot{\Delta}_k v}\right\|_{L^p} \le \sum_{|k-j|\le 4} \left\|\widehat{v_k}\right\|_{L^p} \\ &\quad \cdot \sum_{l\le k-2} \left\|\widehat{u_l}\right\|_{L^1} \le \sum_{|k-j|\le 4} \left\|\widehat{v_k}\right\|_{L^p} \\ &\quad \cdot \left(\sum_{l\le k-2} 2^{(s_1-n/p')lq} \left\|\widehat{u_l}\right\|^q_{L^1}\right)^{1/q} \\ &\quad \cdot \left(\sum_{l\le k-2} 2^{(n/p'-s_1)lq'}\right)^{1/q'} \le C \sum_{|k-j|\le 4} 2^{(n/p'-s_1)k} \left\|\widehat{v_k}\right\|_{L^p} \\ &\quad \cdot \|u\|_{F\dot{B}^{s_1-n/p'}_{1,q}}. \end{aligned} \tag{25}$$

Using the inclusion $F\dot{B}^{s_1}_{p,q} \subset F\dot{B}^{s_2-n/p'}_{1,q}$, we have

$$\left(\sum_j 2^{(s_1+s_2-n/p')jq} \left\|\widehat{I_j}\right\|^q_{L^p}\right)^{1/q} \le C \|u\|_{F\dot{B}^{s_1}_{p,q}} \|v\|_{F\dot{B}^{s_2}_{p,q}}. \tag{26}$$

In a similar way, we can prove

$$\left(\sum_j 2^{(s_1+s_2-n/p')jq} \left\|\widehat{II_j}\right\|^q_{L^p}\right)^{1/q} \le C \|u\|_{F\dot{B}^{s_1}_{p,q}} \|v\|_{F\dot{B}^{s_2}_{p,q}}. \tag{27}$$

For the remaining term, we first consider the case $p \le 2$, in which $s_1 + s_2 > 0$. By Hölder's inequality with $1/p = 1/p' + 1/p - 1/p'$ and by Young's inequality with $1 + 1/p - 1/p' = 1/p + 1/p$, we have

$$\begin{aligned} &2^{(s_1+s_2-n/p')j} \left\|\widehat{III_j}\right\|_{L^p} \\ &\quad \le C \sum_{k\ge j-3} 2^{(s_1+s_2-n/p')j} 2^{(n/p')j} \left\|\widehat{u_k} * \sum_{|l-k|\le 1} \widehat{v_l}\right\|_{L^{pp'/(p'-p)}} \\ &\quad \le C \sum_{k\ge j-3} 2^{(s_1+s_2)j} \left\|\widehat{u_k}\right\|_{L^p} \sum_{|l-k|\le 1} \left\|\widehat{v_l}\right\|_{L^p} \\ &\quad \le C \sum_{k\ge j-3} 2^{(s_1+s_2)(j-k)} 2^{s_1 k} \left\|\widehat{u_k}\right\|_{L^p} \sum_{|l-k|\le 1} 2^{s_2 l} \left\|\widehat{v_l}\right\|_{L^p}. \end{aligned} \tag{28}$$

When $q > 2$, we take l^q-norm of both sides of (28) and use Young's inequality with $1 + 1/q = 1/q' + 2/q$ to get

$$\begin{aligned} &\left(\sum_j 2^{(s_1+s_2-n/p')jq} \left\|\widehat{III_j}\right\|^q_{L^p}\right)^{1/q} \\ &\quad \le C \left\| 2^{s_1 k} \left\|\widehat{u_k}\right\|_{L^p} \sum_{|l-k|\le 1} 2^{s_2 l} \left\|\widehat{v_l}\right\|_{L^p} \right\|_{l^{q/2}(k)} \\ &\quad \le C \|u\|_{F\dot{B}^{s_1}_{p,q}} \|v\|_{F\dot{B}^{s_2}_{p,q}}. \end{aligned} \tag{29}$$

When $q \le 2$, then $F\dot{B}^{s_2}_{p,q} \subset F\dot{B}^{s_2}_{p,q'}$; by definition, taking l^q-norm of both sides of (28) and using Young's inequality with $1 + 1/q = 1 + 1/q$, we get

$$\begin{aligned} &\left(\sum_j 2^{(s_1+s_2-n/p')jq} \left\|\widehat{III_j}\right\|^q_{L^p}\right)^{1/q} \\ &\quad \le C \left\| 2^{s_1 k} \left\|\widehat{u_k}\right\|_{L^p} \sum_{|l-k|\le 1} 2^{s_2 l} \left\|\widehat{v_l}\right\|_{L^p} \right\|_{l^1(k)} \\ &\quad \le C \|u\|_{F\dot{B}^{s_1}_{p,q}} \|v\|_{F\dot{B}^{s_2}_{p,q'}} \le C \|u\|_{F\dot{B}^{s_1}_{p,q}} \|v\|_{F\dot{B}^{s_2}_{p,q}}. \end{aligned} \tag{30}$$

For the case $p > 2$, we have $s_1 + s_2 > n/p' - n/p$. By Hölder's inequality, there holds

$$\begin{aligned} 2^{(s_1+s_2-n/p')j} \left\| \widehat{III_j} \right\|_{L^p} &\le C \sum_{k \ge j-3} 2^{(s_1+s_2-n/p')j} 2^{(n/p)j} \\ &\cdot \left\| \widehat{u_k} * \sum_{|l-k|\le 1} \widehat{v_l} \right\|_{L^\infty} \le C \sum_{k\ge j-3} 2^{(s_1+s_2-n/p'+n/p)j} \left\|\widehat{u_k}\right\|_{L^{p'}} \\ &\cdot \sum_{|l-k|\le 1} \left\|\widehat{v_l}\right\|_{L^p} \le C \sum_{k\ge j-3} 2^{(s_1+s_2-n/p'+n/p)(j-k)} \\ &\cdot 2^{s_1 k} \left\|\widehat{u_k}\right\|_{L^p} \sum_{|l-k|\le 1} 2^{s_2 l} \left\|\widehat{v_l}\right\|_{L^p}. \end{aligned} \tag{31}$$

Following the same steps as in the case $p \le 2$, we obtain the same estimate. Collecting the above estimates we finish our proof. □

By a slight modification of the proof, we can also obtain the following.

Corollary 13. *Let $1 \le p \le \infty$ and $s_1, s_2 \in \mathbb{R}$ such that*

$$\begin{aligned} s_1 &\le \frac{n}{p'}, \\ s_2 &\le \frac{n}{p'}, \\ s_1 + s_2 &> \max\left\{\frac{n}{p'} - \frac{n}{p}, 0\right\}, \end{aligned} \tag{32}$$

where $1/p + 1/p' = 1$. Then for $u \in F\dot{B}^{s_1}_{p,1}$, $v \in F\dot{B}^{s_2}_{p,1}$, one has

$$\|uv\|_{F\dot{B}^{s_1+s_1-n/p'}_{p,1}} \le C \|u\|_{F\dot{B}^{s_1}_{p,1}} \|v\|_{F\dot{B}^{s_2}_{p,1}}. \tag{33}$$

3. The Well-Posedness

To prove the well-posedness, we invoke the fix point principle. We consider the mild solution which means the equivalent integral equation

$$\begin{aligned} u(t,x) &= e^{-\mu t(-\triangle)^\beta} u_0 \\ &\quad + \int_0^t e^{-\mu(t-\tau)(-\triangle)^\beta} Q(u,u)(\tau,x)\, d\tau \\ &:= e^{-\mu t(-\triangle)^\beta} u_0 + B(u,u). \end{aligned} \tag{34}$$

Lemma 14 (linear estimate). *Let $I = [0,T)$, $0 < T \le \infty$, $s \in \mathbb{R}$, $1 \le p, q \le \infty$. Set $Lu_0 = e^{-\mu t(-\triangle)^\beta} u_0$; we have*

$$\begin{aligned} &\|Lu_0\|_{\mathscr{L}^\infty(I;F\dot{B}^s_{p,q})} + \mu^{1/q} \|Lu_0\|_{\mathscr{L}^q(I;F\dot{B}^{s+2\beta/q}_{p,q})} \\ &\quad \le C \|u_0\|_{F\dot{B}^s_{p,q}}. \end{aligned} \tag{35}$$

Proof. By the Fourier transform, we have

$$(Lu_0)^\wedge(\xi) = e^{-\mu t|\xi|^{2\beta}} \widehat{u_0}. \tag{36}$$

Multiplying φ_j and taking the L^p-norm imply

$$\left\|\varphi_j (Lu_0)^\wedge\right\|_{L^p} \le e^{-\mu t 2^{2j\beta}(3/4)^{2\beta}} \left\|\widehat{u_{0j}}\right\|_{L^p}, \tag{37}$$

where we denote $\widehat{u_j} = \varphi_j \hat{u}$. Multiplying 2^{js} and taking l^q-norm

$$\|Lu_0\|_{\mathscr{L}^\infty(I;F\dot{B}^s_{p,q})} \le \|u_0\|_{F\dot{B}^s_{p,q}}. \tag{38}$$

Similarly, multiplying $2^{j(s+2\beta/q)}$ and taking L^q-norm with respect to time on I,

$$\begin{aligned} &2^{j(s+2\beta/q)} \left\|\varphi_j (Lu_0)^\wedge\right\|_{L^q(I;L^p)} \\ &\quad \le \left(\int_I e^{-\mu q t 2^{2j\beta}(3/4)^{2\beta}} 2^{2j\beta} dt\right)^{1/q} 2^{js} \left\|\widehat{u_{0j}}\right\|_{L^p}. \end{aligned} \tag{39}$$

Since $\int_0^\infty e^{-\mu q t 2^{2j\beta}(3/4)^{2\beta}} 2^{2j\beta} dt = \mu q (3/4)^{2\beta}$, taking l^q-norm we get

$$\|Lu_0\|_{\mathscr{L}^q(I;F\dot{B}^{s+2\beta/q}_{p,q})} \le C\mu^{-1/q} \|u_0\|_{F\dot{B}^s_{p,q}}. \tag{40}$$

□

Lemma 15 (nonlinear estimate). *Let $I = [0,T)$, $0 < T \le \infty$, $s \in \mathbb{R}$, $1 \le p, q \le \infty$. Set $Gf = \int_0^t e^{-\mu(t-\tau)(-\triangle)^\beta} f(\tau,x) d\tau$; we have*

$$\begin{aligned} &\mu^{1/q'} \|Gf\|_{\mathscr{L}^\infty(I;F\dot{B}^s_{p,q})} + \mu \|Gf\|_{\mathscr{L}^q(I;F\dot{B}^{s+2\beta/q}_{p,q})} \\ &\quad \le C \|f\|_{\mathscr{L}^q(I;F\dot{B}^{s-2\beta/q'}_{p,q})}. \end{aligned} \tag{41}$$

Proof. Fourier transform gives

$$(Gf)^\wedge(\xi) = \int_0^t e^{-\mu(t-\tau)|\xi|^{2\beta}} \widehat{f(\tau)} d\tau. \tag{42}$$

Multiplying φ_j and taking the L^p-norm

$$\left\|\varphi_j (Gf)^\wedge\right\|_{L^p} \le \int_0^t e^{-\mu(t-\tau)2^{2j\beta}(3/4)^{2\beta}} \left\|\widehat{f_j}\right\|_{L^p} d\tau, \tag{43}$$

where we denote $\widehat{u_j} = \varphi_j \hat{u}$. Multiplying 2^{js} and using Hölder's inequality, we get

$$\begin{aligned} &2^{js} \left\|\varphi_j (Gf)^\wedge\right\|_{L^p} \\ &\quad \le \int_0^t e^{-\mu(t-\tau)2^{2j\beta}(3/4)^{2\beta}} 2^{j(2\beta/q')} 2^{j(s-2\beta/q')} \left\|\widehat{f_j}\right\|_{L^p} d\tau \\ &\quad \le \left(\int_I e^{-\mu q' t 2^{2j\beta}(3/4)^{2\beta}} 2^{2j\beta} dt\right)^{1/q'} 2^{j(s-2\beta/q')} \left\|\widehat{f_j}\right\|_{L^q(I;L^p)} \end{aligned} \tag{44}$$

Using $\int_0^\infty e^{-\mu q' t 2^{2j\beta}(3/4)^{2\beta}} 2^{2j\beta} dt = \mu q' (3/4)^{2\beta}$ and taking l^q-norm, we conclude

$$\|Gf\|_{\mathscr{L}^\infty(I;F\dot{B}^s_{p,q})} \le C\mu^{-1/q'} \|f\|_{\mathscr{L}^q(I;F\dot{B}^{s-2\beta/q'}_{p,q})}. \tag{45}$$

Similarly, multiplying $2^{j(s+2\beta/q)}$

$$\begin{aligned} &2^{j(s+2\beta/q)} \left\|\varphi_j (Gf)^\wedge\right\|_{L^p} \\ &\quad \leq \int_0^t e^{-\mu(t-\tau)2^{2j\beta}(3/4)^{2\beta}} 2^{j(2\beta/q+2\beta/q')} 2^{j(s-2\beta/q')} \left\|\widehat{f_j}\right\|_{L^p} d\tau. \end{aligned} \tag{46}$$

Taking L^q-norm with respect to time on I and using Young's inequality,

$$\begin{aligned} &2^{j(s+2\beta/q)} \left\|\varphi_j (Gf)^\wedge\right\|_{L^q(I;L^p)} \\ &\quad \leq \int_I e^{-\mu t 2^{2j\beta}(3/4)^{2\beta}} 2^{2j\beta} dt 2^{j(s-2\beta/q')} \left\|\widehat{f_j}\right\|_{L^q(I;L^p)}. \end{aligned} \tag{47}$$

Now taking l^q-norm, we obtain our desired inequality

$$\|Gf\|_{\mathscr{L}^q(I;F\dot{B}^{s+2\beta/q}_{p,q})} \leq C\mu^{-1} \|f\|_{\mathscr{L}^q(I;F\dot{B}^{s-2\beta/q'}_{p,q})}. \tag{48}$$

□

Lemma 16 (bilinear estimate). *Let $1 \leq p, q \leq \infty$, $q'/(1+q') < \beta < (q'/(1+q'))\min\{1+n/p', 1+n/2\}$. We have*

$$\begin{aligned} &\|Q(u,v)\|_{\mathscr{L}^q(I;F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q})} \\ &\quad \leq C \|u\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} \|v\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}. \end{aligned} \tag{49}$$

Particularly, the result also holds in the case $p > 1, q = 1, \beta = 1$.

Proof. By Remark 7 and Hölder's inequality, it is sufficient to prove

$$\begin{aligned} &\|Q(u,v)\|_{F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q}} \\ &\quad \leq C \|u\|_{F\dot{B}^{1-2\beta+n/p'+2\beta/q}_{p,q}} \|v\|_{F\dot{B}^{1-2\beta+n/p'+2\beta/q}_{p,q}}. \end{aligned} \tag{50}$$

Note that all indices in the definition of $Q(u,v)$ are finite; we have

$$\begin{aligned} &\|Q(u,v)\|_{F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q}} \\ &\quad \leq C \sum \left\| \mathscr{F}^{-1} \frac{\xi_a \xi_b}{|\xi|^2} \mathscr{F}\partial_m \left(u^k v^l\right) \right\|_{F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q}}. \end{aligned} \tag{51}$$

On the other hand, by Definition 8, we know

$$\begin{aligned} &\left\| \mathscr{F}^{-1} \frac{\xi_a \xi_b}{|\xi|^2} \mathscr{F}\partial_m \left(u^k v^l\right) \right\|_{F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q}} \\ &\quad \leq C \left\|u^k v^l\right\|_{F\dot{B}^{2-2\beta+n/p'-2\beta/q'}_{p,q}}. \end{aligned} \tag{52}$$

Thus, we just need to consider the estimate of the product $\|uv\|_{F\dot{B}^{2-2\beta+n/p'-2\beta/q'}_{p,q}}$. However, set $s_1 = s_2 = 1-2\beta+n/p'+\beta/q$, then by Theorem 1 and Corollary 13, we obtain our desired result. □

Next we introduce an abstract lemma on the existence of fixed point solutions [14, 19].

Lemma 17. *Let X be a Banach space with norm $\|\cdot\|_X$ and $B : X \times X \longmapsto X$ be a bounded bilinear operator satisfying*

$$\|B(u,v)\|_X \leq \eta \|u\|_X \|v\|_X, \tag{53}$$

for all $u, v \in X$ and a constant $\eta > 0$. Then for any fixed $y \in X$ satisfying $\|y\|_X < \epsilon < 1/4\eta$, the equation $x := y + B(x,x)$ has a solution $\overline{x}$ in X such that $\|\overline{x}\|_X \leq 2\|y\|_X$. Also, the solution is unique in $\overline{B}(0, 2\epsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X < \epsilon, x' = y' + B(x',x'), \|x'\|_X < 2\epsilon$, then

$$\left\|\overline{x} - x'\right\|_X \leq \frac{1}{1-4\epsilon\eta} \left\|y - y'\right\|_X. \tag{54}$$

This lemma allows us to solve the Cauchy problem (1) with bounded bilinear form and small data. Now we begin our proof.

Proof of Theorem 3. We first seek the solution in the spaces $\mathscr{L}^{2q}(I; F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})$. By (6) of Proposition 11

$$\begin{aligned} &\|f\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} \\ &\quad \leq \|f\|^{1/2}_{\mathscr{L}^{\infty}(I;F\dot{B}^{1-2\beta+n/p'}_{p,q})} \|f\|^{1/2}_{\mathscr{L}^{q}(I;F\dot{B}^{1-2\beta+n/p'+2\beta/q}_{p,q})}. \end{aligned} \tag{55}$$

By Lemma 15 with $s = 1-2\beta+n/p'$ and Lemma 16

$$\begin{aligned} &\|B(u,v)\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} \\ &\quad \leq C\mu^{-1/2-1/2q'} \|u\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} \\ &\quad \cdot \|v\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}. \end{aligned} \tag{56}$$

By Lemma 17, we know that if $\|e^{-\mu t(-\triangle)^\beta} u_0\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} < R$ with $R = \mu^{1/2+1/2q'}/4C$, then (34) has a unique solution in $B(0, 2R)$, where

$$\begin{aligned} B(0,2R) := \Big\{ &x \\ &\in \mathscr{L}^{2q}\left(I; F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q}\right) : \|x\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} \\ &\leq 2R \Big\}. \end{aligned} \tag{57}$$

Now we need to derive $\|e^{-\mu t(-\triangle)^\beta} u_0\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} < R$. First, we consider small initial data. Lemma 14 and (55) imply that

$$\begin{aligned} &\left\|e^{-\mu t(-\triangle)^\beta} u_0\right\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+2\beta/q}_{p,q})} \\ &\quad \leq C\mu^{-1/2q} \|u_0\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}}. \end{aligned} \tag{58}$$

Thus we can take u_0 such that $\|u_0\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}} < C_0\mu$ with $C_0 = (4C^2)^{-1}$. Next, for the large initial data u_0, we divide u_0 by $u_0 = \mathscr{F}^{-1}\chi_{\{|\xi|\le\gamma\}}\widehat{u_0} + \mathscr{F}^{-1}\chi_{\{|\xi|>\gamma\}}\widehat{u_0} := u_1^0 + u_2^0$, where $\gamma = \gamma(u_0) > 0$ is a large real number determined later. Since $\mathscr{F}^{-1}\chi_{\{|\xi|>\gamma\}}\widehat{u_0}$ converges to 0 in $F\dot{B}^{1-2\beta+n/p'}_{p,q}$ as $\gamma \longrightarrow +\infty$, by (58) there exists some γ large enough such that

$$\begin{aligned}\left\|e^{\mu t(-\triangle)^\beta}u_2^0\right\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} &\le C\mu^{-1/2q}\left\|u_2^0\right\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}}\\ &\le \frac{\mu^{1/2+1/2q'}}{8C}.\end{aligned}\tag{59}$$

Now for u_0^1, there holds

$$\begin{aligned}&\left\|e^{\mu t(-\triangle)^\beta}u_1^0\right\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}\\ &= \left(\sum_{j\in Z}2^{j(1-2\beta+n/p'+\beta/q)q}\cdot\left\|\varphi_j e^{-\mu t|\xi|^{2\beta}}\chi_{\{|\xi|\le\gamma\}}\widehat{u_0}\right\|^q_{L^{2q}(I;L^p)}\right)^{1/q}\\ &\le \left(\sum_{j\in Z}2^{j(1-2\beta+n/p'+\beta/q)q}\cdot\left\|\sup_{|\xi|\le\gamma}e^{-\mu t|\xi|^{2\beta}}|\xi|^{\beta/q}\right\|^q_{L^{2q}([0,T))}\left\|\varphi_j|\xi|^{-\beta/q}\widehat{u_0}\right\|^q_{L^p}\right)^{1/q}\\ &\le C\gamma^{\beta/q}T^{1/2q}\|u_0\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}}.\end{aligned}\tag{60}$$

Thus we can choose T small enough such that

$$T \le \left(\frac{\mu^{1/2+1/2q'}}{8C^2\gamma^{\beta/q}\|u_0\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}}}\right)^{2q}.\tag{61}$$

We now conclude that (34) has a unique solution $u \in \mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})$. By (34), Lemmas 14 and 15, and Theorem 1, we conclude

$$\begin{aligned}u &\in \mathscr{L}^{\infty}\left(I;F\dot{B}^{1-2\beta+n/p'}_{p,q}\right)\\ &\cap \mathscr{L}^{q}\left(I;F\dot{B}^{1-2\beta+n/p'+2\beta/q}_{p,q}\right).\end{aligned}\tag{62}$$

The continuity with respect to time is standard and thus we prove Theorem 3 up to the blow-up criterion. Next we prove the blow-up criterion. Suppose T^* is the maximal time of existence of mild solution associated with u_0. If we have a solution of (1) on $[0,T^*)$ such that

$$\|u\|_{\mathscr{L}^{2q}([0,T^*);F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} < \infty,\tag{63}$$

then the integral equation (34), Lemmas 14 and 15, and Theorem 1 imply that for all $t \in [0,T^*)$

$$\|u(t)\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}} < \infty.\tag{64}$$

Using the integral equation (34), and by similar method with the proof of Lemmas 14, 15, and 16

$$\begin{aligned}\left\|u(t')-u(t)\right\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}} &\le \left\|\left(I-e^{-\mu(t-t')(-\triangle)^\beta}\right)\right.\\ &\left.\cdot u_0\right\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}} + \|Q(u,u)\|_{\mathscr{L}^q([t,t');F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q})}\\ &+\left\|\left(I-e^{-\mu(t-t')(-\triangle)^\beta}\right)\right.\\ &\left.\cdot Q\left(u(t'),u(t')\right)\right\|_{\mathscr{L}^q_{t'}([0,t');F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q})}.\end{aligned}\tag{65}$$

Since $\|(I-e^{-\mu(t-t')(-\triangle)^\beta})u_0\|_{F\dot{B}^{1-2\beta+n/p'}_{p,q}}$ converges to 0 as $t' \longrightarrow t$, and from (63) we know that

$$\|Q(u,u)\|_{\mathscr{L}^q(I;F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q})} \le \|u\|^2_{\mathscr{L}^{2q}(I;F\dot{B}^{1-\beta+n/p'-\beta/q}_{p,q})}.\tag{66}$$

Thus we can conclude that the other two terms also converge to 0 as $t' \longrightarrow t$, so $u(t)$ satisfies the Cauchy criterion at T^*. Thus there exists an element u^* in $F\dot{B}^{1-2\beta+n/p'}_{p,q}$ such that $u(t) \longrightarrow u^*$ as $t \longrightarrow T^*$. Now set $u(T^*) = u^*$, and consider the fractional Navier-Stokes equations starting by u^*. Using Theorem 1 we obtain that a solution exists on a larger time interval than $[0,T^*)$, which is a contradiction. □

Proof of Theorem 4. Using the same method in proving Theorem 3, to prove Theorem 4, it is sufficient to re-estimate $B(u,v)$, and by Lemma 15 we only need to prove the following lemma. □

Lemma 18. *Let* $1 \le p \le q \le 2, \beta \in (1/(1+1/q'), (1+n/2p')/(1+1/q')]$. *Then there exists some constant* $C = C(\beta,p,q) > 0$ *depending on* β, p, q *such that*

$$\begin{aligned}&\|Q(u,v)\|_{\mathscr{L}^q(I;F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q})}\\ &\le C\|u\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}\|v\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}.\end{aligned}\tag{67}$$

Particularly, it is true for the case: $p = q = 1, \beta = 1$.

Proof. Similarly to Lemma 16, we only need to estimate $\|uv\|_{\mathscr{L}^q(I:F\dot{B}^{2-2\beta+3/p'-2\beta/q'}_{p,q})}$. Using Bony's decomposition

$$\begin{aligned}
&\|uv\|_{\mathscr{L}^q(I:F\dot{B}^{2-2\beta+3/p'-2\beta/q'}_{p,q})} \\
&\quad\le \frac{8}{3}\left[\left(\sum_j 2^{(2-2\beta+n/p'-2\beta/q')jq}\left\|\widehat{I_j}\right\|^q_{L^q(I;L^p)}\right)^{1/q}\right. \\
&\quad+\left(\sum_j 2^{(2-2\beta+n/p'-2\beta/q')jq}\left\|\widehat{II_j}\right\|^q_{L^q(I;L^p)}\right)^{1/q} \\
&\quad\left.+\left(\sum_j 2^{(2-2\beta+n/p'-2\beta/q')jq}\left\|\widehat{III_j}\right\|^q_{L^q(I;L^p)}\right)^{1/q}\right].
\end{aligned} \tag{68}$$

The terms I_j and II_j are symmetrical. Using Young's inequality and Hölder's inequality we have

$$\begin{aligned}
\left\|\widehat{I_j}\right\|_{L^q(I;L^p)} &\le \sum_{|k-j|\le 4}\left\|\widehat{\dot{S}_{k-1}u\dot{\Delta}_k v}\right\|_{L^q(I;L^p)} \\
&\le \sum_{|k-j|\le 4}\left\|\widehat{v_k}\right\|_{L^{2q}(I;L^p)}\sum_{l\le k-2}\left\|\widehat{u_l}\right\|_{L^{2q}(I;L^1)} \\
&\le \sum_{|k-j|\le 4}\left\|\widehat{v_k}\right\|_{L^{2q}(I;L^p)} \\
&\quad\cdot\left(\sum_{l\le k-2}2^{(1-2\beta+\beta/q)lq}\left\|\widehat{u_l}\right\|^q_{L^{2q}(I;L^1)}\right)^{1/q} \\
&\quad\cdot\left(\sum_{l\le k-2}2^{(2\beta-1-\beta/q)lq'}\right)^{1/q'} \le C\sum_{|k-j|\le 4}2^{(2\beta-1-\beta/q)k} \\
&\quad\cdot\left\|\widehat{v_k}\right\|_{L^{2q}(I;L^p)}\|u\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+\beta/q}_{1,q})}.
\end{aligned} \tag{69}$$

Recalling $\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q}) \subset \mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+\beta/q}_{1,q})$ in Proposition 11, there holds

$$\begin{aligned}
&\left(\sum_j 2^{(2-2\beta+n/p'-2\beta/q')jq}\left\|\widehat{I_j}\right\|^q_{L^q(I;L^p)}\right)^{1/q} \\
&\quad\le C\|u\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}\|v\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}.
\end{aligned} \tag{70}$$

In a similar way, we can prove

$$\begin{aligned}
&\left(\sum_j 2^{(2-2\beta+n/p'-2\beta/q')jq}\left\|\widehat{II_j}\right\|^q_{L^q(I;L^p)}\right)^{1/q} \\
&\quad\le C\|u\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}\|v\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}.
\end{aligned} \tag{71}$$

For the remaining term, we invoke Minkowski's inequality. In fact when $q \ge p$

$$\begin{aligned}
&\left(\sum_j 2^{(2-2\beta+n/p'-2\beta/q')jq}\left\|\widehat{III_j}\right\|^q_{L^q(I;L^p)}\right)^{1/q} \le \left\|\sum_{k>j-3}\left\|2^{(2-2\beta+n/p'-2\beta/q')j}\varphi_j(\xi)\left[\widehat{u_k}*\sum_{|l-k|\le 1}\widehat{v_l}\right]\right\|_{L^q(I;L^p)}\right\|_{l^q(j)} \\
&\le \sum_k\left(\sum_{j<k+3}\left\|2^{(2-2\beta+n/p'-2\beta/q')j}\varphi_j(\xi)\left[\widehat{u_k}*\sum_{|l-k|\le 1}\widehat{v_l}\right]\right\|^q_{L^q(I;L^p)}\right)^{1/q} \\
&\le \sum_k\left\|\left(\sum_{j<k+3}\left\|2^{(2-2\beta+n/p'-2\beta/q')j}\varphi_j(\xi)\left[\widehat{u_k}*\sum_{|l-k|\le 1}\widehat{v_l}\right]\right\|^q_{L^p_\xi}\right)^{1/q}\right\|_{L^q_t} \\
&\le \sum_k\left\|\left\|\int_{R^3}2^{(2-2\beta+n/p'-2\beta/q')pj}\left|\varphi_j(\xi)\right|^p\left[\widehat{u_k}*\sum_{|l-k|\le 1}\widehat{v_l}\right]^p d\xi\right\|^{1/p}_{l^{q/p}(j<k+3)}\right\|_{L^q_t} \\
&\le \sup_\xi\left(\sum_j\varphi_j(\xi)^q\right)^{1/q}\sum_k 2^{(2-2\beta+n/p'-2\beta/q')(k+3)}\left\|\widehat{u_k}*\sum_{|l-k|\le 1}\widehat{v_l}\right\|_{L^q(I;L^p)} \\
&\le C\sum_k 2^{(1-2\beta+\beta/q)k}\left\|\widehat{u_k}\right\|_{L^{2q}(I:L^1)}\sum_{|l-k|\le 1}2^{(1-2\beta+n/p'+\beta/q)(k-l)}2^{(1-2\beta+n/p'+\beta/q)l}\left\|\widehat{v_l}\right\|_{L^{2q}(I;L^p)} \\
&\le C\|u\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}\|v\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}.
\end{aligned} \tag{72}$$

In the last inequality we use $\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q}) \subset \mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q'})$ for $q \in [1,2]$ by (4) of Proposition 11. Collecting the above estimates we conclude our desired result. □

4. Space Analyticity

The proof is similar with the proof of Theorem 3. Let

$$U(t,x) := e^{\mu\sqrt{t}|D|^{\beta}} u(t,x). \tag{73}$$

By the integral equation (34), we have

$$\begin{aligned} U(t,x) &= e^{\mu(\sqrt{t}-(1/2)t)|D|^{\beta}} e^{-(1/2)\mu t)|D|^{\beta}} u_0 \\ &+ \int_0^t e^{\mu(\sqrt{t}-\sqrt{\tau}-(1/2)(t-\tau))|D|^{\beta}} e^{-(1/2)(t-\tau)|D|^{\beta}} e^{\sqrt{\tau}|D|^{\beta}} Q(u, \\ & u)\, d\tau. \end{aligned} \tag{74}$$

Since $e^{\mu(\sqrt{t}-(1/2)t)|\xi|^{\beta}}$ and $e^{\mu(\sqrt{t}-\sqrt{\tau}-(1/2)(t-\tau))|\xi|^{\beta}}$ are uniformly bounded on $t \in (0,\infty)$ and $\tau \in [0,t]$, together with Lemmas 14 and 15 we only need to consider the estimate of $\|e^{\sqrt{\tau}|D|^{\beta}}Q(u,v)\|_{\mathscr{L}^{q}(I;F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q})}$. To apply Lemma 17, we need a bilinear estimate. To this end, we write $Q(u,v)$ as $Q(e^{-\sqrt{\tau}|D|^{\beta}}U, e^{-\sqrt{\tau}|D|^{\beta}}V)$ with $V := e^{\sqrt{\tau}|D|^{\beta}}v$ and we prove the following lemma.

Lemma 19. *Let $1 \le p,q \le \infty, q'/(1+q') < \beta < (q'/(1+q'))\min\{1+n/p', 1+n/2\}, \beta \le 1$. Then*

$$\begin{aligned} &\left\| \left\| e^{\sqrt{\tau}|D|^{\beta}} Q(u,v) \right\|_{\mathscr{L}^{q}(I;F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q})} \right\| \\ &\quad \le C\|U\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} \|V\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}. \end{aligned} \tag{75}$$

Particularly, our result also holds in the case $p > 1, q = 1, \beta = 1$.

Proof. Using the same discussion in the proof of Lemma 16, it is sufficient to consider the estimate of $\|e^{\sqrt{\tau}|D|^{\beta}}(uv)\|_{\mathscr{L}^{q}(I;F\dot{B}^{2-2\beta+n/p'-2\beta/q'}_{p,q})}$. By Bony's decomposition, we have for fixed j

$$\begin{aligned} \dot{\Delta}_j e^{\sqrt{\tau}|D|^{\beta}}(uv) &= \sum_{|k-j|\le 4} \dot{\Delta}_j e^{\sqrt{\tau}|D|^{\beta}} \left(\dot{S}_{k-1}u\dot{\Delta}_k v\right) \\ &+ \sum_{|k-j|\le 4} \dot{\Delta}_j e^{\sqrt{\tau}|D|^{\beta}} \left(\dot{S}_{k-1}v\dot{\Delta}_k u\right) \\ &+ \sum_{k\ge j-3} \dot{\Delta}_j e^{\sqrt{\tau}|D|^{\beta}} \left(\dot{\Delta}_k u \tilde{\dot{\Delta}}_k v\right) \\ &:= R_j^1 + R_j^2 + R_j^3. \end{aligned} \tag{76}$$

The idea is that the L^p-norm of $\mathscr{F}\dot{\Delta}_j e^{\sqrt{\tau}|D|^{\beta}}(uv)$ can be bounded by that of $\mathscr{F}\dot{\Delta}_j(UV)$. In fact,

$$\begin{aligned} \left\|\widehat{R_j^1}\right\|_{L^p} &= \left\| \sum_{|k-j|\le 4} \varphi_j e^{\sqrt{\tau}|\xi|^{\beta}} \left[\left(\sum_{l\le k-2} e^{-\sqrt{\tau}|\xi|^{\beta}} \widehat{U_l} \right) \right.\right. \\ &\left.\left. * e^{-\sqrt{\tau}|\xi|^{\beta}} \widehat{V_k} \right] \right\|_{L^p} = \left\| \sum_{|k-j|\le 4} \varphi_j \right. \\ &\cdot \int_{R^n} e^{\sqrt{\tau}(|\xi|^{\beta}-|\xi-\eta|^{\beta}-|\eta|^{\beta})} \left(\sum_{l\le k-2} \widehat{U_l} \right)(\xi-\eta) \\ &\left. \cdot \widehat{V_k}(\eta)\, d\eta \right\|_{L^p}. \end{aligned} \tag{77}$$

Since $e^{\sqrt{\tau}(|\xi|^{\beta}-|\xi-\eta|^{\beta}-|\eta|^{\beta})}$ is uniformly bounded on τ if $\beta \in [0,1]$, we can eliminate this term. Thus, we obtain

$$\left\|\widehat{R_j^1}\right\|_{L^p} \le \left\| \sum_{|k-j|\le 4} \varphi_j \left| \left(\sum_{l\le k-2} \widehat{U_l} \right) \right| * \left|\widehat{V_k}\right| \right\|_{L^p}. \tag{78}$$

Similarly, we can also obtain that

$$\begin{aligned} \left\|\widehat{R_j^2}\right\|_{L^p} &\le \left\| \sum_{|k-j|\le 4} \varphi_j \left| \left(\sum_{l\le k-2} \widehat{V_l} \right) \right| * \left|\widehat{U_k}\right| \right\|_{L^p}; \\ \left\|\widehat{R_j^3}\right\|_{L^p} &\le \left\| \sum_{|k-j|\le 4} \varphi_j \left|\widehat{U_k}\right| * \left| \sum_{|l-k|\le 1} \widehat{V_l} \right| \right\|_{L^p}. \end{aligned} \tag{79}$$

Following the same step in the proof of Theorem 1 and Lemma 16, it is easy to conclude our desired result. □

By this lemma and the above observations, we can obtain Theorem 5 by following the proof of Theorem 3 line by line. Similarly, to prove Theorem 6, it is sufficient to obtain the following lemma.

Lemma 20. *Let $1 \le p \le q \le 2, q'/(1+q') < \beta < (q'/(1+q'))(1+n/2p'), \beta \le 1$. Then*

$$\begin{aligned} &\left\| \left\| e^{\sqrt{\tau}|D|^{\beta}} Q(u,v) \right\|_{\mathscr{L}^{q}(I;F\dot{B}^{1-2\beta+n/p'-2\beta/q'}_{p,q})} \right\| \\ &\quad \le C\|U\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})} \|V\|_{\mathscr{L}^{2q}(I;F\dot{B}^{1-2\beta+n/p'+\beta/q}_{p,q})}. \end{aligned} \tag{80}$$

Particularly, our result also holds in the case $p = q = 1, \beta = 1$.

Proof. Using the discussion in Lemma 19 and following the same steps in the proof of Lemma 18, it is easy to conclude our desired result. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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