Research Article

A Subfamily of Univalent Functions Associated with $q$-Analogue of Noor Integral Operator

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The main objective of the present paper is to define a new subfamily of analytic functions using subordinations along with the newly defined $q$-Noor integral operator. We investigate a number of useful properties such as coefficient estimates, integral representation, linear combination, weighted and arithmetic means, and radius of starlikeness for this class.

1. Introduction and Definitions

In recent years, $q$-analysis ($q$-calculus) has motivated the researchers a lot due to its numerous applications in mathematics and physics. Jackson [1, 2] was the first to give some application of $q$-calculus and also introduced the $q$-analogue of derivative and integral operator. Later on, Aral and Gupta [3, 4] defined the $q$-Baskakov-Durrmeyer operator by using $q$-beta function while in papers [5, 6] the authors discussed the $q$-generalization of complex operators known as $q$-Picard and $q$-Gauss-Weierstrass singular integral operators. Using convolution of normalized analytic functions, Kanas and Raducanu [7] defined $q$-analogue of Ruscheweyh differential operator and studied some of its properties. The application of this differential operator was further studied by Aldweby and Darus [8] and Mahmood and Sokół [9]. The aim of the current paper is to define a $q$-analogue of the Noor integral operator involving convolution concepts and then give some interesting applications of this operator.

Let us denote the open unit disk by $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and the symbol $\mathfrak{A}$ denotes the family of those analytic functions $f$ which has the following Taylor series representation:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{D}). \quad (1)$$

For two functions $f$ and $g$ that are analytic in $\mathbb{D}$ and have the form (1), we define the convolution of these functions by

$$f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{D}). \quad (2)$$

For $0 < q < 1$, the $q$-derivative of a function $f \in \mathfrak{A}$ is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q - 1)}, \quad (z \neq 0). \quad (3)$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $z \in \mathbb{D}$

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] a_n z^{n-1}, \quad (4)$$

where

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{i=1}^{n-1} q^i, \quad [0, q] = 0. \quad (5)$$

For any nonnegative integer $n$, the $q$-number shift factorial is defined by

$$[n, q]! = \begin{cases} 1, & n = 0, \\ [1, q] [2, q] [3, q] \cdots [n, q], & n \in \mathbb{N}. \end{cases} \quad (6)$$
Also the $q$-generalized Pochhammer symbol for $x > 0$ is given by
\[
[x, q]_n = \begin{cases} 1, & n = 0, \\ [x, q] [x + 1, q] \cdots [x + n - 1, q], & n \in \mathbb{N}. \end{cases} \tag{7}
\]

For $\mu > -1$, we define the function $F_{\mu+1}^{\nu}(z)$ by
\[
\mathfrak{F}_{\nu+1}^{\mu+1}(z) = z \mathcal{D}_q f(z), \tag{8}
\]
where the function $\mathfrak{F}_{\nu+1}^{\mu+1}(z)$ is given by
\[
\mathfrak{F}_{\nu+1}^{\mu+1}(z) = z + \sum_{n=2}^{\infty} \frac{[\mu + 1, q]_{n-1}}{[n-1, q]} z^n, \quad (z \in \mathfrak{D}). \tag{9}
\]

It is quite clear that the series defined in (9) is convergent absolutely in $\mathfrak{D}$. Using the definition of $q$-derivative along with the idea of convolutions, we now define the integral operator $F_{\mu}^{\nu} : \mathfrak{A} \to \mathfrak{A}$ by
\[
F_{\mu}^{\nu} f(z) = \mathfrak{F}_{\nu+1}^{\mu+1}(z) * f(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n, \quad (z \in \mathfrak{D}), \tag{10}
\]
with
\[
\psi_{n-1} = \frac{[n, q]!}{[\mu + 1, q]_{n-1}}. \tag{11}
\]

From (10), we can easily get the identity
\[
[n, q] F_{\mu}^{\nu} f(z) = [\mu, q] F_{\nu+1}^{\mu+1} f(z)
+ \mu^\nu z \frac{d^\nu f(z)}{dz^\nu}. \tag{12}
\]

We note that $F_{\mu}^{0} f(z) = z \mathcal{D}_q f(z)$, $F_{\mu}^{1} f(z) = f(z)$, and
\[
\lim_{q \to 1^-} F_{\mu}^{\nu} f(z) = z + \sum_{n=2}^{\infty} (\mu + 1)_{n-1} a_n z^n. \tag{13}
\]

This shows that, by taking $q \to 1^-$, the operator defined in (10) reduces to the familiar Noor integral operator introduced in [10, 11]. Also for more details on the $q$-analogue of differential and integral operators, see the work [12–14].

Motivated from the work studied in [7, 15–17], we now define subfamilies of the set $\mathfrak{A}$ by using the operator $F_{\mu}^{\nu}$ as follows.

**Definition 1.** Let $-1 \leq B < A \leq 1$ and $0 < q < 1$. Then the function $f \in \mathfrak{A}$ is in the class $\mathcal{O}_q(\mu, A, B)$ if it satisfies
\[
\frac{z \frac{d}{dz} \left( F_{\mu}^{\nu} f(z) \right)}{F_{\mu}^{\nu} f(z)} < \frac{1 + A z}{1 + B z}, \quad (z \in \mathfrak{D}), \tag{14}
\]
where the notation "$<$" denotes the familiar subordinations.

Equivalently, a function $f \in \mathfrak{A}$ is in the class $\mathcal{O}_q(\mu, A, B)$, if and only if
\[
\frac{z \frac{d}{dz} \left( F_{\mu}^{\nu} f(z) \right)}{F_{\mu}^{\nu} f(z)} - 1 < \frac{1}{A - B \left( z \frac{d}{dz} \left( F_{\mu}^{\nu} f(z) \right) / F_{\mu}^{\nu} f(z) \right)}, \quad (z \in \mathfrak{D}). \tag{15}
\]
We will assume throughout our discussion, unless otherwise stated, that
\[
\mu > -1, \\
-1 \leq B < A \leq 1, \\
0 < q < 1
\]
and all coefficients $a_n$ are positive.

We need the following result in the proof of a result.

**Lemma 2** (see [18]). Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then
\[
\frac{1 + A_1 z}{1 + B_1 z} < \frac{1 + A_2 z}{1 + B_2 z}. \tag{17}
\]

**2. Main Results**

**Theorem 3.** Let $f \in \mathfrak{A}$ be given by (1). Then the function $f$ is in the family $\mathcal{O}_q(\mu, A, B)$, if and only if
\[
\sum_{n=2}^{\infty} \left[ [n, q] (1 - B) - 1 + A \right] a_n [n-1]_q^{-1} (A - B). \tag{18}
\]

**Proof.** Let us assume first that inequality (18) holds. To show $f \in \mathcal{O}_q(\mu, A, B)$, we only need to prove the inequality (15). For this, consider
\[
\frac{z \frac{d}{dz} \left( F_{\mu}^{\nu} f(z) \right)}{F_{\mu}^{\nu} f(z)} - 1 < \frac{1}{A - B \left( z \frac{d}{dz} \left( F_{\mu}^{\nu} f(z) \right) / F_{\mu}^{\nu} f(z) \right)}, \tag{19}
\]
where we have used (4), (10), and (18) and this completes the direct part. Conversely, let $f \in \mathcal{O}_q(\mu, A, B)$ be of the form (1). Then from (15) along with (10), we have, for $z \in \mathfrak{D}$,
\[
\frac{z \frac{d}{dz} \left( F_{\mu}^{\nu} f(z) \right)}{F_{\mu}^{\nu} f(z)} - 1 < \frac{1}{A - B \left( z \frac{d}{dz} \left( F_{\mu}^{\nu} f(z) \right) / F_{\mu}^{\nu} f(z) \right)}, \tag{20}
\]
where
\[
\Re \left\{ \sum_{n=2}^{\infty} \psi_{n-1} [n, q] [n-1]_q^{-1} a_n z^n \right\} < 1. \tag{21}
\]
Now we choose values of $z$ on the real axis such that $z \partial_q (\mathfrak{A}_q^q f(z))/\mathfrak{A}_q^q f(z)$ is real. Upon clearing the denominator in (21) and letting $z \to 1^-$ through real values, we obtain the required inequality (18).

**Theorem 4.** Let $f \in \mathcal{O}_q(\mu, A, B)$. Then

$$
\mathfrak{A}_q^q f(z) = \exp \int_0^z \frac{1}{t} \left( \frac{1 - A\phi(t)}{1 - B\phi(t)} \right) dt,
$$

with $|\phi(z)| < 1$ and $z \in \mathfrak{D}$.

**Proof.** Let $f \in \mathcal{O}_q(\mu, A, B)$ and setting

$$
z \partial_q \left( \mathfrak{A}_q^q f(z) \right) / \mathfrak{A}_q^q f(z) = v(z),
$$

with

$$
v(z) < \frac{1 + Az}{1 + Bz},
$$

equivalently, we can write

$$
\left| v(z) - \frac{1}{A - Bv(z)} \right| < 1,
$$

or in other way, we have

$$
v(z) - 1 = \frac{\phi(z)}{A - Bv(z)},
$$

$$
|\phi(z)| < 1,
$$

$$(z \in \mathfrak{D}).
$$

Thus we can rewrite

$$
z \partial_q \left( \mathfrak{A}_q^q f(z) \right) / \mathfrak{A}_q^q f(z) = \left( \frac{1 - A\phi(t)}{1 - B\phi(t)} \right),
$$

and further by simple computation of integration, the proof is completed.

**Theorem 5.** Let $f_i \in \mathcal{O}_q(\mu, A, B)$ and have the form

$$
f_i(z) = z + \sum_{k=1}^{\infty} a_{i,k} z^k, \quad \text{for } i = 1, 2, \ldots, l.
$$

Then $F \in \mathcal{O}_q(\mu, A, B)$, where

$$
F(z) = \sum_{i=1}^{l} c_i f_i(z) \quad \text{with} \quad \sum_{i=1}^{l} c_i = 1.
$$

**Proof.** By the virtue of Theorem 3, one can write

$$
\sum_{n=2}^{\infty} \left( \frac{[n, q] (1 - B) - 1 + A}{A - B} \right) \psi_{n-1} a_{n,j} \varphi_{n,j} < 1.
$$

Therefore

$$
F(z) = \sum_{i=2}^{l} c_i \left( z + \sum_{n=2}^{\infty} a_{n,i} z^n \right) = z + \sum_{i=2}^{l} \left( \sum_{n=2}^{\infty} c_i a_{n,i} \right) z^n;
$$

however

$$
\sum_{n=2}^{\infty} \left( \frac{[n, q] (1 - B) - 1 + A}{A - B} \right) \psi_{n-1} a_{n,j} \varphi_{n,j} \leq 1;
$$

then $F \in \mathcal{O}_q(\mu, A, B)$. Hence the proof is complete.

**Theorem 6.** If $f$ and $g$ belong to $\mathcal{O}_q(\mu, A, B)$, then their weighted mean $h_j$ is also in $\mathcal{O}_q(\mu, A, B)$, where $h_j$ is defined by

$$
h_j(z) = \left\{ \frac{(1 - j) f(z) + (1 + j) g(z)}{2} \right\}.
$$

**Proof.** From (33), we can easily write

$$
h_j(z) = z + \sum_{n=2}^{\infty} \left( \frac{[n, q] (1 - B) - 1 + A}{A - B} \right) \left( \frac{1}{2} \right) \sum_{n=2}^{\infty} (1 - j) a_n + (1 + j) b_n \varphi_{n-1} < 1.
$$

To prove that $h_j \in \mathcal{O}_q(\mu, A, B)$, we need to show that

$$
\sum_{n=2}^{\infty} \left( \frac{[n, q] (1 - B) - 1 + A}{A - B} \right) \left( \frac{1}{2} \right) \sum_{n=2}^{\infty} (1 - j) a_n + (1 + j) b_n \psi_{n-1} < 1.
$$

For this, consider

$$
\sum_{n=2}^{\infty} \left( \frac{[n, q] (1 - B) - 1 + A}{A - B} \right) \left( \frac{1}{2} \right) \sum_{n=2}^{\infty} (1 - j) a_n + (1 + j) b_n \psi_{n-1} = \left( \frac{1 - j}{2} \right)
$$

and

$$
\sum_{n=2}^{\infty} \left( \frac{[n, q] (1 - B) - 1 + A}{A - B} \right) \left( \frac{1}{2} \right) \sum_{n=2}^{\infty} (1 - j) a_n + (1 + j) b_n \psi_{n-1} \leq \left( \frac{1 - j}{2} \right) \sum_{n=2}^{\infty} \left( \frac{[n, q] (1 - B) - 1 + A}{A - B} \right) \left( \frac{1}{2} \right)
$$

where we have used inequality (18). Hence the result follows.
Theorem 7. Let $f_i$ with $i = 1, 2, \ldots, \lambda$ belong to the class $\mathcal{O}_q(\mu, A, B)$. Then the arithmetic mean $h$ of $f_i$ is given by

$$h(z) = \frac{1}{\lambda} \sum_{i=1}^{\lambda} f_i(z)$$

and is also in the class $\mathcal{O}_q(\mu, A, B)$.

Proof. From (37), we can write

$$h(z) = \frac{1}{\lambda} \sum_{i=1}^{\lambda} \left( z + \sum_{n=2}^{\infty} a_n z^n \right) = z + \sum_{n=2}^{\infty} \left( \frac{1}{\lambda} \sum_{i=1}^{\lambda} a_n \right) z^n.$$ (38)

Since $f_i \in \mathcal{O}_q(\mu, A, B)$ for every $i = 1, 2, \ldots, \lambda$, using (38) and (18), we have

$$\sum_{n=2}^{\infty} a_n \left( [n, q] (1 - B) - 1 + A \right) \left( \frac{1}{\lambda} \sum_{i=1}^{\lambda} a_n \right) \leq \frac{1}{\lambda} \sum_{i=1}^{\lambda} (A - B) = (A - B),$$ (39)

and this completes the proof. \hfill \Box

Theorem 8. Let $f \in \mathcal{O}_q(\mu, A, B)$. Then $f$ is in the family $\mathcal{D}^*(\beta)$ of starlike functions of order $\beta$ ($0 \leq \beta < 1$) for $|z| < r_1$, where

$$r_1 = \left( \frac{1 - \beta}{1 - \beta \beta} \right) \left( \frac{[n, q] (1 - B) - 1 + A}{[n, q] (A - B) [\mu + 1, q]_{n-1}} \right)^{1/(n-1)}. \quad (40)$$

Proof. Let $f \in \mathcal{O}_q(\mu, A, B)$. To prove $f \in \mathcal{D}^*(\beta)$, we only need to show

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < 1.$$ (41)

Using (1) along with some simple computation yields

$$\sum_{n=2}^{\infty} \left( \frac{n - \beta}{1 - \beta} \right) |a_n| |z|^{n-1} < 1.$$ (42)

Since $f \in \mathcal{O}_q(\mu, A, B)$, from (18), we can easily obtain

$$\sum_{n=2}^{\infty} \left( \frac{[n, q]!}{[\mu + 1, q]_{n-1}} \left( \frac{[n, q] (1 - B) - 1 + A}{(A - B)} \right) |a_n| \right) < 1.$$ (43)

Now inequality (42) will be true, if the following holds:

$$\sum_{n=2}^{\infty} \left( \frac{n - \beta}{1 - \beta} \right) |a_n| |z|^{n-1}$$

$$< \sum_{n=2}^{\infty} \left( \frac{[n, q]!}{[\mu + 1, q]_{n-1}} \left( \frac{[n, q] (1 - B) - (1 - A)}{A - B} \right) \right) |a_n|,$$ (44)

which implies that

$$|z|^{n-1} < \frac{(1 - \beta)([n, q] (1 - B) - 1 + A)}{(n - \beta)(A - B) [\mu + 1, q]_{n-1}},$$ (45)

and thus we get the needed result. \hfill \Box

Theorem 9. Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$ and $\mathcal{F}_q^{\mu+1} f(z) \neq 0$ in $\mathcal{D}$, and this satisfies

$$\left[ \frac{[\mu + 1, q]_{n-1}}{[\mu + 1, q]} - \frac{[\mu, q]_{n-1}}{[\mu, q]} \right] < \frac{1 + A_1 z}{1 + B_1 z}.$$ (46)

Then $f \in \mathcal{O}_q(\mu + 1, A_2, B_2)$.

Proof. Since $\mathcal{F}_q^{\mu+1} f(z) \neq 0$ in $\mathcal{D}$, therefore let us define the function $p(z)$ by

$$\frac{z \partial_q \left( \mathcal{F}_q^{\mu+1} f(z) \right)}{\mathcal{F}_q^{\mu+1} f(z)} = p(z) \quad (z \in \mathcal{D}). \quad (47)$$

By the virtue of identity (12), we obtain

$$\frac{1}{q^n} \left\{ \frac{[\mu + 1, q]_{n-1}}{[\mu + 1, q]} \mathcal{F}_q^{\mu+1} f(z) - \frac{[\mu, q]_{n-1}}{[\mu, q]} \mathcal{F}_q^{\mu-1} f(z) \right\} = p(z). \quad (48)$$

Therefore, using (46), we have

$$\frac{z \partial_q \left( \mathcal{F}_q^{\mu+1} f(z) \right)}{\mathcal{F}_q^{\mu+1} f(z)} = p(z) < \frac{1 + A_1 z}{1 + B_1 z},$$ (49)

and now, using Lemma 2, we have $f \in \mathcal{O}_q(\mu + 1, A_2, B_2)$. \hfill \Box

Conflicts of Interest

The authors agree with the contents of the manuscript and there are no conflicts of interest among the authors.

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