Research Article

On Strong Coupled Coincidence Points of g -Couplings and an Application

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The purpose of this paper is to prove the existence and uniqueness of a strong coupled coincidence point of $F : X \times X \rightarrow X$ and $g : X \rightarrow X$, involving Banach and Chatterjea type g -couplings. We also give some examples and an application in support of the given concepts and our main results.

1. Introduction and Preliminaries

The concept of a coupled fixed point was first introduced by Guo et al. [1]. Later, many coupled fixed and coupled coincidence point results were given. For more details, see [2–22]. Kirk et al. [23] gave the concept of cyclic mappings. Recently, Choudhury et al. [24] introduced the concept of couplings, which are actually coupled cyclic mappings with respect to two given subsets of a metric space. In [24], they also proved the existence of strong coupled fixed points for Banach and Chatterjea couplings. Very recently, Aydi et al. [25, 26] proved some existence and uniqueness results of a strong coupled fixed point for nonlinear couplings in (partial) metric spaces. In this paper, we extend the concept of the Banach (resp., Chatterjea) type coupling to the Banach (resp., Chatterjea) type g -couplings. We generalize the results of Choudhury et al. [24].

First, we recall some known definitions.

Definition 1 (coupled fixed point, [27]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2 (strong coupled fixed point, [28]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a strong

coupled fixed point of the mapping $F : X \times X \rightarrow X$ if (x, y) is a coupled fixed point and $x = y$, that is, if $F(x, x) = x$.

Definition 3 (coupled Banach contraction mapping, [27]). Let (X, d) be a metric space. A mapping $F : X \times X \rightarrow X$ is called a coupled Banach contraction if there exists $k \in (0, 1)$ such that, for all $(x, y), (u, v) \in X \times X$, the following inequality is satisfied:

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)]. \quad (1)$$

Definition 4 (cyclic mapping, [23]). Let A and B be two nonempty subsets of a given set X . The mapping $f : X \rightarrow X$ is said to be cyclic (with respect to A and B) if

$$\begin{aligned} f(A) &\subset B, \\ f(B) &\subset A. \end{aligned} \quad (2)$$

Definition 5 (coupling, [28]). Let (X, d) be a metric space and let A, B be two nonempty subsets of X . Then a map $F : X \times X \rightarrow X$ is said to be a coupling with respect to A and B if

$$\begin{aligned} F(x, y) &\in B \\ F(y, x) &\in A, \end{aligned} \quad (3)$$

whenever $x \in A$ and $y \in B$.

Definition 6 (Banach type coupling, [24]). Let A and B be two nonempty subsets of a metric space (X, d) . A coupling $F : X \times X \rightarrow X$ is called a Banach type coupling with respect to A and B if it satisfies the following inequality:

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)]. \quad (4)$$

where $x, v \in A, y, u \in B$, and $k \in (0, 1)$.

Definition 7 (coupled coincidence point of F and g , [14]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition 8. Let U be a nonempty subset of a set X . U is said invariant by g if

$$g(U) \subseteq U. \quad (5)$$

2. Main Results

First of all, we introduce some definitions.

Definition 9 (strong coupled coincidence point of F and g). A coupled coincidence point $(x, y) \in X \times X$ of $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ is said to be a strong coupled coincidence point of F and g if $x = y$, i.e., $F(x, x) = g(x)$.

Definition 10 (g -coupling). Let (X, d) be a metric space and let A, B be two nonempty subsets of X . Given $F : X \times X \rightarrow X$ and $g : X \rightarrow X$, then F is said to be a g -coupling (with respect to A and B) if, for $x \in A$ and $y \in B$, one has

$$\begin{aligned} F(x, y) &\in g(A) \cap B, \\ F(y, x) &\in g(B) \cap A. \end{aligned} \quad (6)$$

Now, we give an example illustrating the concept of g -couplings.

Example 11. Let $X = \mathbb{R}$. $A = [0, \pi/2]$ and $B = [0, \pi/4]$ are two subsets of X . Let $F : X \times X \rightarrow X$ be defined by $F(x, y) = (x + y)/5$, for all $x, y \in X$. Define $g : X \rightarrow X$ by $g(x) = \sin(x)$. Then $g(A) = [0, 1]$ and $g(B) = [0, 1/\sqrt{2}]$. Further, $g(A) \cap B = [0, \pi/4]$ and $g(B) \cap A = [0, 1/\sqrt{2}]$. Now we show F is g -coupling with respect to A and B . For $x \in A$ and $y \in B$, we have

$$\begin{aligned} 0 \leq F(x, y) &\leq \frac{3\pi}{20} \implies \\ F(x, y) &\in g(A) \cap B. \\ 0 \leq F(y, x) &\leq \frac{3\pi}{20} \implies \\ F(y, x) &\in g(B) \cap A. \end{aligned} \quad (7)$$

This shows that F is a g -coupling with respect to A and B .

Remark 12. Every g -coupling is a coupling. For convenience, let $F : X \times X \rightarrow X$ be a g -coupling. Then by Definition 14, we

have $F(x, y) \in (g(A) \cap B) \subset B$ and $F(y, x) \in (g(B) \cap A) \subset A$. Then F is a coupling (with respect to A and B). But, the converse is not true in general.

Now, we give an example which shows that every coupling need not be a g -coupling.

Example 13. Let $X = \mathbb{R}$. Consider $A = [0, 3]$ and $B = [1, 5]$. We define $F : X \times X \rightarrow X$ by $F(x, y) = \cos(x + y) + 2$, for all $x, y \in X$. We first show that F is a coupling with respect to A and B . For $x \in A$ and $y \in B$, we have

$$1 \leq F(x, y) \leq 3 \implies F(x, y) \in B, \quad (8)$$

and

$$1 \leq F(y, x) \leq 3 \implies F(y, x) \in A. \quad (9)$$

Thus, F is a coupling with respect to A and B . Now, if we define $g : X \rightarrow X$ by $g(x) = \sin(x)$. Then $g(A) = [0, 1]$ and $g(B) = [-1, 1]$. Further, $g(A) \cap B = \{1\}$ and $g(B) \cap A = [0, 1]$. This proves that F is not a g -coupling with respect to A and B . Indeed, for $x = \pi/2 \in A$ and $y = \pi \in B$,

$$F(x, y) = F\left(\frac{\pi}{2}, \pi\right) = 2 \notin g(A) \cap B. \quad (10)$$

Definition 14 (Banach type g -coupling). Let A and B be two nonempty subsets of a metric space (X, d) . Then a g -coupling $F : X \times X \rightarrow X$ is said to be Banach type g -coupling (with respect to A and B) if

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(gx, gu) + d(gy, gv)], \quad (11)$$

whenever $x, v \in A$ and $y, u \in B$, where $k \in (0, 1)$.

Note that if $g = I$ (the identity mapping), then the Banach type g -coupling becomes a Banach type coupling. Our main result is as follows.

Theorem 15. Let A and B be any two subsets of a complete metric space (X, d) . If there exists a Banach type g -coupling, $F : X \times X \rightarrow X$ (with respect to A and B). Suppose that A and B are invariant by g , and $g(A), g(B)$ are closed subsets of (X, d) . Then

- (i) $A \cap B \neq \emptyset$,
- (ii) F and g have a coupled coincidence point in $A \times B$.

If, in addition, g is one to one on $A \cup B$, then F and g have a unique strong coupled coincidence point in $A \times B$.

Proof. Recall that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(gx, gu) + d(gy, gv)], \quad (12)$$

where $x, v \in A, y, u \in B$ and $k \in (0, 1)$.

Also, as A and B are invariant by g , so

$$\begin{aligned} gx, gv &\in g(A) \subseteq A, \\ gy, gu &\in g(B) \subseteq B \end{aligned} \quad (13)$$

Let $x_0 \in A$ and $y_0 \in B$, then by definition of a g -coupling, we have $F(x_0, y_0) \in g(A) \cap B$ and $F(y_0, x_0) \in g(B) \cap A$. In particular, $F(x_0, y_0) \in g(A)$ and $F(y_0, x_0) \in g(B)$. Hence, there exist $x_1 \in A$ and $y_1 \in B$ such that

$$\begin{aligned} F(x_0, y_0) &= g(x_1), \\ F(y_0, x_0) &= g(y_1). \end{aligned} \quad (14)$$

Continuing in this way, we get sequences $\{gx_n\}$ and $\{gy_n\}$ in $g(A)$ and $g(B)$, respectively, such that

$$\begin{aligned} g(x_{n+1}) &= F(x_n, y_n), \\ g(y_{n+1}) &= F(y_n, x_n). \end{aligned} \quad (15)$$

Now, using (12) and (15), we get

$$\begin{aligned} d(gx_1, gy_2) &= d(F(x_0, y_0), F(y_1, x_1)) \\ &\leq \frac{k}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)], \end{aligned} \quad (16)$$

and

$$\begin{aligned} d(gy_1, gx_2) &= d(F(y_0, x_0), F(x_1, y_1)) \\ &\leq \frac{k}{2} [d(gy_0, gx_1) + d(gx_0, gy_1)]. \end{aligned} \quad (17)$$

From the above two inequalities, we have

$$\begin{aligned} d(gx_1, gy_2) + d(gy_1, gx_2) &\leq \frac{k}{2} [d(gx_0, gy_1) \\ &+ d(gy_0, gx_1) + d(gy_0, gx_1) + d(gx_0, gy_1)], \\ &= k [d(gx_0, gy_1) + d(gy_0, gx_1)]; \end{aligned} \quad (18)$$

that is,

$$\begin{aligned} \frac{d(gx_1, gy_2) + d(gy_1, gx_2)}{2} \\ \leq \frac{k}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)]. \end{aligned} \quad (19)$$

Inductively, we have

$$d(gx_n, gy_{n+1}) \leq \frac{k^n}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)], \quad (20)$$

and

$$d(gy_n, gx_{n+1}) \leq \frac{k^n}{2} [d(gx_0, gy_1) + d(gy_0, gx_1)]. \quad (21)$$

Using (12) and (15), we have

$$\begin{aligned} d(gx_1, gy_1) &= d(F(x_0, y_0), F(y_0, x_0)), \\ &\leq \frac{k}{2} [d(gx_0, gy_0) + d(gy_0, gx_0)], \\ &= kd(gx_0, gy_0); \end{aligned} \quad (22)$$

that is,

$$d(gx_1, gy_1) \leq kd(gx_0, gy_0). \quad (23)$$

From (12), (15), and (23), we have

$$\begin{aligned} d(gx_2, gy_2) &= d(F(x_1, y_1), F(y_1, x_1)), \\ &\leq \frac{k}{2} [d(gx_1, gy_1) + d(gy_1, gx_1)], \\ &= kd(gx_1, gy_1), \\ &\leq k^2 d(gx_0, gy_0). \end{aligned} \quad (24)$$

Inductively,

$$d(gx_n, gy_n) \leq k^n d(gx_0, gy_0), \quad n \geq 0. \quad (25)$$

Now, by (20), (21), and (25) and by triangular inequality, we have, for all $n \geq 0$,

$$\begin{aligned} &d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \\ &\leq d(gx_n, gy_n) + d(gy_n, gx_{n+1}) + d(gy_n, gx_n) \\ &\quad + d(gx_n, gy_{n+1}), \\ &= 2d(gx_n, gy_n) \\ &\quad + [d(gy_n, gx_{n+1}) + d(gx_n, gy_{n+1})], \\ &\leq 2k^n d(gx_0, gy_0) \\ &\quad + k^n [d(gx_0, gy_1) + d(gy_0, gx_1)]. \end{aligned} \quad (26)$$

Since $k \in (0, 1)$, it follows that $\sum d(gx_n, gx_{n+1}) + \sum d(gy_n, gy_{n+1}) < \infty$. Obviously, the sequences $\{gx_n\}$ and $\{gy_n\}$ are Cauchy in $g(A)$ and $g(B)$, respectively. As $g(A)$ and $g(B)$ are closed subsets in the complete metric space (X, d) , $\{g(x_n)\}$ and $\{g(y_n)\}$ are convergent in $g(A)$ and $g(B)$, respectively. Therefore, there exist $u \in g(A)$ and $v \in g(B)$ such that

$$\begin{aligned} gx_n &\longrightarrow u, \\ gy_n &\longrightarrow v \\ &\text{as } n \longrightarrow \infty. \end{aligned} \quad (27)$$

Using (25), we have

$$d(gx_n, gy_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (28)$$

Hence, from (27)

$$u = v. \quad (29)$$

As $u \in g(A)$ and $v \in g(B) \implies u \in g(A) \cap g(B)$, now, since $u \in g(A)$ and $v \in g(B)$, there exist $a \in A$ and $b \in B$ such that $u = g(a)$ and $v = g(b)$. By (27) and (29), we have

$$\begin{aligned} gx_n &\longrightarrow g(a), \\ gy_n &\longrightarrow g(b), \end{aligned} \quad (30)$$

and

$$g(a) = g(b). \quad (31)$$

Therefore,

$$g(a) = g(b) \in A \cap B. \quad (32)$$

Part (i) is completed. Now, by (12), (15), (30), (31), and triangular inequality, we have

$$\begin{aligned} d(g(a), F(a, b)) &\leq d(g(a), gy_{n+1}) + d(gy_{n+1}, F(a, b)), \\ &= d(g(a), gy_{n+1}) + d(F(y_n, x_n), F(a, b)), \\ &\leq d(g(a), gy_{n+1}) \\ &\quad + \frac{k}{2} [d(gy_n, g(a)) + d(gx_n, g(b))], \\ &\longrightarrow 0 \end{aligned} \quad (33)$$

as $n \longrightarrow \infty$.

Thus, we have

$$F(a, b) = g(a). \quad (34)$$

Similarly,

$$\begin{aligned} d(g(b), F(b, a)) &\leq d(g(b), gx_{n+1}) + d(gx_{n+1}, F(b, a)), \\ &= d(g(b), gx_{n+1}) + d(F(x_n, y_n), F(b, a)), \\ &\leq d(g(b), gx_{n+1}) \\ &\quad + \frac{k}{2} [d(gx_n, g(b)) + d(gy_n, g(a))], \\ &\longrightarrow 0 \end{aligned} \quad (35)$$

as $n \longrightarrow \infty$.

We deduce that

$$F(b, a) = g(b). \quad (36)$$

Using (34) and (36), we get $F(a, b) = g(a)$ and $F(b, a) = g(b)$. Thus $(a, b) \in A \times B$ is a coupled coincidence point of F and g . (ii) is completed.

In view of (31) and the fact that g is one to one, we have $a = b$, so F and g have a strong coupled coincidence point;

that is, $F(a, a) = g(a)$. Suppose there exist two strong coupled coincidence points $l, m \in A \cap B$ of F and g , then

$$\begin{aligned} F(l, l) &= g(l), \\ F(m, m) &= g(m). \end{aligned} \quad (37)$$

From (12), we have

$$\begin{aligned} d(g(l), g(m)) &= d(F(l, l), F(m, m)), \\ &\leq \frac{k}{2} [d(g(l), g(m)) + d(g(l), g(m))], \\ &= k [d(g(l), g(m))]. \end{aligned} \quad (38)$$

Since $k \in (0, 1)$, we deduce that $d(g(l), g(m)) = 0$, i.e., $g(l) = g(m)$, so $l = m$ due to the fact that g is one to one. Hence F and g have a unique strong coupled coincidence point in $A \cap B$. $\square$

We present the following examples.

Example 16. Let $X = \mathbb{R}$ be endowed by $d(x, y) = |x - y|$, where $x, y \in X$. Let $A = [0, 1/2]$ and $B = [0, 1]$. We define $F : X \times X \longrightarrow X$ by $F(x, y) = (x^2 + y^2)/6$, where $x, y \in X$ and $g : X \longrightarrow X$ is defined by $g(x) = x^2$. Then $g(A) = [0, 1/4]$ and $g(B) = [0, 1]$, so $g(A)$ and $g(B)$ are closed subsets of X . Also $g(A) \subseteq A$ and $g(B) \subseteq B$, so A and B are invariant by g . Now, we show that F is a g -coupling.

As $g(A) \cap B = [0, 1/4]$ and $g(B) \cap A = [0, 1/2]$, so for $x \in A$ and $y \in B$, we have

$$0 \leq F(x, y) \leq \frac{5}{24}, \implies \quad (39)$$

$$F(x, y) \in g(A) \cap B,$$

and

$$0 \leq F(y, x) \leq \frac{5}{24}, \implies \quad (40)$$

$$F(y, x) \in g(B) \cap A.$$

This shows that F is a g -coupling with respect to A and B . Again, for $x, v \in A$ and $y, u \in B$, we have

$$\begin{aligned} d(F(x, y), F(u, v)) &= \left| \frac{x^2 + y^2}{6} - \frac{u^2 + v^2}{6} \right|, \\ &= \frac{1}{6} |x^2 - u^2 + y^2 - v^2|, \\ &\leq \frac{1}{6} [|x^2 - u^2| + |y^2 - v^2|], \\ &= \frac{k}{2} [d(gx, gu) + d(gy, gv)], \end{aligned} \quad (41)$$

$$\text{where } k = \frac{1}{3} \in (0, 1).$$

This shows that F is a Banach type g -coupling (with respect to A and B). Thus all the conditions of Theorem 15 are satisfied; therefore there exists $(a, b) \in A \times B$ such that $F(a, b) = g(a)$ and $F(b, a) = g(b)$. Here, $(a, b) = (0, 0)$.

Example 17. Let $X = \mathbb{R}$ be endowed with the metric $d(x, y) = |x - y|$. Take $A = [0, 2]$ and $B = [0, 3]$. Let F be defined as $F(x, y) = (x + y)/10$, where $x, y \in X$. Consider $g : X \rightarrow X$ as $g(x) = x/2$. Then $g(A) = [0, 1]$ and $g(B) = [0, 3/2]$, so $g(A)$ and $g(B)$ are closed subsets of X . Also, $g(A) \subseteq A$ and $g(B) \subseteq B$, so A and B are invariant by g . Now, we shall show that F is a g -coupling. As $g(A) \cap B = [0, 1]$ and $g(B) \cap A = [0, 1]$, so for all $x \in A$ and $y \in B$, we have $0 \leq F(x, y) \leq 1/2$ and $0 \leq F(y, x) \leq 1/2$; that is, $F(x, y) \in g(A) \cap B$ and $F(y, x) \in g(B) \cap A$. Thus, F is a g -coupling with respect to A and B . Again, for $x, v \in A$ and $y, u \in B$, we have

$$\begin{aligned} d(F(x, y), F(u, v)) &= \left| \frac{x+y}{10} - \frac{u+v}{10} \right|, \\ &= \frac{1}{5} \left| \frac{x-u}{2} + \frac{y-v}{2} \right|, \\ &\leq \frac{1}{5} \left[\left| \frac{x-u}{2} \right| + \left| \frac{y-v}{2} \right| \right], \\ &= \frac{1}{5} \left[\left| \frac{x}{2} - \frac{u}{2} \right| + \left| \frac{y}{2} - \frac{v}{2} \right| \right], \\ &= \frac{k}{2} [d(gx, gu) + d(gy, gv)], \end{aligned} \quad (42)$$

$$\text{where } k = \frac{2}{5} \in (0, 1).$$

This shows that F is a Banach type g -coupling (with respect to A and B). Thus all the conditions of Theorem 15 are satisfied, so there exists $(a, b) \in A \times B$ such that $F(a, b) = g(a)$ and $F(b, a) = g(b)$, i.e.,

$$\begin{aligned} \frac{a+b}{10} &= \frac{a}{2}, \\ \frac{b+a}{10} &= \frac{b}{2}. \end{aligned} \quad (43)$$

Then $a = b = 0$. The mapping g is one to one, so $(0, 0)$ is the unique strong g -coupled coincidence point of F and g .

Corollary 18. Let $g : X \rightarrow X$ be a Banach contraction. Then every Banach type g -coupling (with respect to A and B) is a Banach type coupling (with respect to A and B), where A and B are subsets of X .

Proof. Since g is a Banach contraction, there exists $\alpha \in (0, 1)$ such that

$$d(gx, gy) \leq \alpha d(x, y) \quad \forall x, y \in X. \quad (44)$$

Let $F : X \times X \rightarrow X$ be a Banach type g -coupling (with respect to A and B), i.e.,

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(gx, gu) + d(gy, gv)], \quad (45)$$

where $x, v \in A$, $y, u \in B$, and $k \in (0, 1)$. By Definition 5, F is a coupling (with respect to A and B). Now, using (44) in (45), we get

$$\begin{aligned} d(F(x, y), F(u, v)) &\leq \frac{k}{2} [\alpha d(x, u) + \alpha d(y, v)] \\ &\leq \frac{\alpha k}{2} [d(x, u) + d(y, v)]. \end{aligned} \quad (46)$$

Since $k, \alpha \in (0, 1)$, we have $k\alpha = k_1 \in (0, 1)$, so

$$d(F(x, y), F(u, v)) \leq \frac{k_1}{2} [d(x, u) + d(y, v)], \quad (47)$$

where $x, v \in A$, $y, u \in B$, and $k_1 \in (0, 1)$. Hence, F is a Banach type coupling (with respect to A and B). $\square$

Now, we give some examples where Theorem 15 works, but the result of Choudhury et al. [24] is not applicable.

Example 19. Let $X = \mathbb{R}$ be endowed with the metric $d(x, y) = |x - y|$. Take $A = (0, 1)$ and $B = (0, 3)$. Let $F : X \times X \rightarrow X$ be defined as

$$F(x, y) = \begin{cases} \frac{1}{2}, & 0 < x, y < 3, \\ \frac{x^2 + y}{3}, & \text{elsewhere.} \end{cases} \quad (48)$$

Consider $g : X \rightarrow X$ by

$$g(x) = \begin{cases} \frac{1}{2}, & -\infty < x < 1, \\ 1, & 1 \leq x < 2, \\ \frac{3}{2}, & 2 \leq x < \infty. \end{cases} \quad (49)$$

Since A and B are not closed in X , we can not apply the result of Choudhury et al. [24]. Clearly, $g(A) = \{1/2\}$ and $g(B) = \{1/2, 1, 3/2\}$, so $g(A)$ and $g(B)$ are closed subsets of X . Also, $g(A) \subseteq A$ and $g(B) \subseteq B$; hence A and B are invariant by g . Now, we shall show that F is a g -coupling. As $g(A) \cap B = \{1/2\}$ and $g(B) \cap A = \{1/2\}$, so for all $x \in A$ and $y \in B$, we have $F(x, y) = 1/2$ and $F(y, x) = 1/2$; that is, $F(x, y) \in g(A) \cap B$ and $F(y, x) \in g(B) \cap A$. Thus, F is a g -coupling with respect to A and B . Again, for $x, v \in A$ and $y, u \in B$, we have $F(x, y) = 1/2$ and $F(u, v) = 1/2$, so $d(F(x, y), F(u, v)) = 0$. Thus, for all $x, v \in A$ and $y, u \in B$, with $k \in (0, 1)$, we have

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(gx, gu) + d(gy, gv)]. \quad (50)$$

This shows that F is a Banach type g -coupling (with respect to A and B). Thus all the conditions of Theorem 15 are satisfied, so there exists $(a, b) \in A \times B$ such that $F(a, b) = g(a)$ and $F(b, a) = g(b)$. Clearly, for $a = 1/4 \in A$ and $b = 1/3 \in B$, we have

$$F\left(\frac{1}{4}, \frac{1}{3}\right) = \frac{1}{2} = g\left(\frac{1}{4}\right) \quad (51)$$

and

$$F\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{1}{2} = g\left(\frac{1}{3}\right). \quad (52)$$

Thus $(1/4, 1/3)$ is a coupled coincidence point of F and g in $A \times B$.

Example 20. Let $X = [-2, 5]$ be endowed with the metric $d(x, y) = |x - y|$. Take $A = (0, \pi/2)$ and $B = (0, \pi)$. Let $F : X \times X \rightarrow X$ be defined as

$$F(x, y) = \begin{cases} 1, & 0 < x, y < \pi, \\ \frac{\sin x + \cos y}{5}, & \text{elsewhere.} \end{cases} \quad (53)$$

We define $g : X \rightarrow X$ by

$$g(x) = \begin{cases} 1, & -2 \leq x < 2, \\ \frac{\pi}{2}, & 2 \leq x \leq 5. \end{cases} \quad (54)$$

Since A and B are not closed in X , we can not apply the result of Choudhury et al. [24]. Clearly, $g(A) = \{1\}$ and $g(B) = \{1, \pi/2\}$, so $g(A)$ and $g(B)$ are closed subsets of X . Also, $g(A) \subseteq A$ and $g(B) \subseteq B$; hence A and B are invariant by g . Now, we shall show that F is a g -coupling. As $g(A) \cap B = \{1\} = g(B) \cap A$, so for all $x \in A$ and $y \in B$, we have $F(x, y) = F(y, x) = 1$; that is, $F(x, y) \in g(A) \cap B$ and $F(y, x) \in g(B) \cap A$. Thus, F is a g -coupling with respect to A and B . Again, for $x, v \in A$ and $y, u \in B$, we have $F(x, y) = F(u, v) = 1$, so $d(F(x, y), F(u, v)) = 0$. This shows that F is a Banach type g -coupling (with respect to A and B). Thus all the conditions of Theorem 15 are satisfied, so there exists $(a, b) \in A \times B$ such that $F(a, b) = g(a)$ and $F(b, a) = g(b)$. Clearly, for $a = \pi/6 \in A$ and $b = \pi/4 \in B$, we have

$$F\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = 1 = g\left(\frac{\pi}{6}\right) \quad (55)$$

and

$$F\left(\frac{\pi}{4}, \frac{\pi}{6}\right) = 1 = g\left(\frac{\pi}{4}\right). \quad (56)$$

Thus, $(\pi/6, \pi/4)$ is a coupled coincidence point of F and g in $A \times B$.

Example 21. Let $X = [-2\pi, 2\pi]$ be endowed with the metric $d(x, y) = |x - y|$. Take $A = (-\pi, \pi)$ and $B = (-3\pi/4, 3\pi/4)$. Let $F : X \times X \rightarrow X$ be defined as $F(x, y) = (\sin x + \sin y)/3$ and $g : X \rightarrow X$ is defined as $g(x) = \sin x$. Since A and B are not closed in X , we can not apply the result of Choudhury et al. [24]. We have $g(A) = g(B) = [-1, 1]$. Also, $g(A) \subseteq A$ and $g(B) \subseteq B$; hence A and B are invariant by g . Now, we shall show that F is a g -coupling. As $g(A) \cap B = g(B) \cap A = [-1, 1]$, so for all $x \in A$ and $y \in B$, we have

$$-\frac{2}{3} \leq F(x, y) \leq \frac{2}{3} \implies \quad (57)$$

$$F(x, y) \in g(A) \cap B,$$

and

$$-\frac{2}{3} \leq F(y, x) \leq \frac{2}{3} \implies \quad (58)$$

$$F(y, x) \in g(B) \cap A.$$

Thus, F is a g -coupling with respect to A and B . Again, for $x, v \in A$ and $y, u \in B$, we have

$$\begin{aligned} d(F(x, y), F(u, v)) &= \left| \frac{\sin x + \sin y}{3} - \frac{\sin u + \sin v}{3} \right|, \\ &= \frac{1}{3} |(\sin x - \sin u) + (\sin y - \sin v)|, \\ &\leq \frac{1}{3} [|\sin x - \sin u| + |\sin y - \sin v|], \\ &= \frac{k}{2} [d(gx, gu) + d(gy, gv)], \end{aligned} \quad (59)$$

where $k = 2/3 \in (0, 1)$. This shows that F is a Banach type g -coupling (with respect to A and B). Thus all the conditions of Theorem 15 are satisfied, so there exists $(a, b) \in A \times B$ such that $F(a, b) = g(a)$ and $F(b, a) = g(b)$. Note that $(0, 0)$ is a (strong) coupled coincidence point of F and g in $A \times B$.

Now, we introduce the concept of Chatterjea type g -couplings and we prove existence of strong coupled coincidence points.

Definition 22. Let A and B be any two nonempty subsets of a metric space (X, d) and $g : X \rightarrow X$ be a self-mapping. Let $F : X \times X \rightarrow X$ be a g -coupling with respect to A and B . Then F is said to be a Chatterjea type g -coupling with respect to A and B , if it satisfies

$$\begin{aligned} d(F(x, y), F(u, v)) &\leq k [d(g(y), F(x, y)) + d(g(v), F(u, v))] \end{aligned} \quad (60)$$

where $x, v \in A$ and $y, u \in B$, where $k \in (0, 1/2)$.

Theorem 23. Let A and B be any two nonempty subsets of a complete metric space (X, d) and let $g : X \rightarrow X$ be a self-mapping. Let $F : X \times X \rightarrow X$ be a Chatterjea type g -coupling with respect to A and B . Further, suppose that A and B are invariant by g and $g(A), g(B)$ are closed subsets of (X, d) . Then

$$(i) \ A \cap B \neq \emptyset,$$

$$(ii) \ F \text{ and } g \text{ have a coupled coincidence point in } A \times B.$$

If, in addition, g is one to one, then F and g have a unique strong coupled coincidence point in $A \times B$.

Proof. Let $x_0 \in A$ and $y_0 \in B$. Following the same lines as proof of Theorem 15, we construct $\{g(x_n)\}$ and $\{g(y_n)\}$ in $g(A)$ and $(g(B))$, respectively, such that, for all $n \geq 0$,

$$\begin{aligned} g(x_{n+1}) &= F(x_n, y_n), \\ g(y_{n+1}) &= F(y_n, x_n). \end{aligned} \quad (61)$$

Now, by using (60) and (61), we get

$$\begin{aligned} d(g(x_1), g(y_2)) &= d(F(x_0, y_0), F(y_1, x_1)) \\ &\leq k [d(g(y_0), F(x_0, y_0)) + d(g(x_1), F(y_1, x_1))] \\ &= k [d(g(y_0), g(x_1)) + d(g(x_1), g(y_2))]. \end{aligned} \quad (62)$$

Therefore,

$$d(g(x_1), g(y_2)) \leq td(g(y_0), g(x_1)), \quad (63)$$

where $t = k/(1 - k) \in (0, 1)$. Similarly,

$$\begin{aligned} d(g(y_1), g(x_2)) &= d(F(y_0, x_0), F(x_1, y_1)) \\ &\leq k[d(g(x_0), F(y_0, x_0)) + d(g(y_1), F(x_1, y_1))] \\ &= k[d(g(x_0), g(y_1)) + d(g(y_1), g(x_2))]. \end{aligned} \quad (64)$$

We deduce

$$d(g(y_1), g(x_2)) \leq td(g(x_0), g(y_1)). \quad (65)$$

Again,

$$\begin{aligned} d(g(x_2), g(y_3)) &= d(F(x_1, y_1), F(y_2, x_2)) \\ &\leq k[d(g(y_1), F(x_1, y_1)) + d(g(x_2), F(y_2, x_2))] \\ &= k[d(g(y_1), g(x_2)) + d(g(x_2), g(y_3))]. \end{aligned} \quad (66)$$

Using (65),

$$d(g(x_2), g(y_3)) \leq t^2 d(g(x_0), g(y_1)). \quad (67)$$

Also,

$$\begin{aligned} d(g(y_2), g(x_3)) &= d(F(y_1, x_1), F(x_2, y_2)) \\ &\leq k[d(g(x_1), F(y_1, x_1)) + d(g(y_2), F(x_2, y_2))] \\ &= k[d(g(x_1), g(y_2)) + d(g(y_2), g(x_3))]. \end{aligned} \quad (68)$$

Using (63),

$$d(g(y_2), g(x_3)) \leq t^2 d(g(y_0), g(x_1)). \quad (69)$$

By induction, as (63), (65), (67), and (69), we have

$$\begin{aligned} d(g(x_n), g(y_{n+1})) &\leq t^n d(g(x_0), g(y_1)) \\ &\text{for all } n \geq 0. \end{aligned} \quad (70)$$

$$\begin{aligned} d(g(y_n), g(x_{n+1})) &\leq t^n d(g(y_0), g(x_1)) \\ &\text{for all } n \geq 0. \end{aligned} \quad (71)$$

By (60), we have

$$\begin{aligned} d(g(x_{n+1}), g(y_{n+1})) &= d(F(x_n, y_n), F(y_n, x_n)) \\ &\leq k[d(g(y_n), F(x_n, y_n)) + d(g(x_n), F(y_n, x_n))] \\ &= k[d(g(y_n), g(x_{n+1})) + d(g(x_n), g(y_{n+1}))]. \end{aligned} \quad (72)$$

Using (70) and (71), we get

$$\begin{aligned} d(g(x_{n+1}), g(y_{n+1})) &\leq kt^n [d(g(x_0), g(y_1)) + d(g(y_0), g(x_1))] \\ &\text{for all } n \geq 0. \end{aligned} \quad (73)$$

By triangular inequality, using (70), (71), and (73),

$$\begin{aligned} &d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) \\ &\leq \sum [d(g(x_n), g(y_n)) + d(g(y_n), g(x_{n+1}))] \\ &\quad + \sum [d(g(y_n), g(x_n)) + d(g(x_n), g(y_{n+1}))] \\ &= 2d(g(x_n), g(y_n)) + d(g(y_n), g(x_{n+1})) \\ &\quad + d(g(x_n), g(y_{n+1})) \\ &\leq 2kt^{n-1} [d(g(x_0), g(y_1)) + d(g(y_0), g(x_1))] \\ &\quad + t^n [d(g(x_0), g(y_1)) + d(g(y_0), g(x_1))]. \end{aligned} \quad (74)$$

Since $0 < t < 1$, it follows that

$$\begin{aligned} &\sum d(g(x_n), g(x_{n+1})) + \sum d(g(y_n), g(y_{n+1})) \\ &< \infty. \end{aligned} \quad (75)$$

This implies that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in $g(A)$ and $g(B)$, respectively. Since $g(A)$ and $g(B)$ are closed subsets of a complete metric space, $\{g(x_n)\}$ and $\{g(y_n)\}$ are convergent in $g(A)$ and $g(B)$, respectively. Thus there exist $u = g(a) \in g(A)$ and $v = g(b) \in g(B)$, such that

$$\begin{aligned} g(x_n) &\longrightarrow u = g(a), \\ g(y_n) &\longrightarrow v = g(b). \end{aligned} \quad (76)$$

By (73), we have

$$d(g(x_n), g(y_n)) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (77)$$

From (76) and (77), we get

$$g(a) = g(b) \in A \cap B. \quad (78)$$

Proceeding similarly to the proof of Theorem 15, we get that $F(a, b) = g(a)$ and $F(b, a) = g(b)$. Hence (a, b) is a coupled coincidence point of F and g in $A \times B$.

Again, if g is one to one, it is obvious that (a, a) is the unique strong coupled coincidence point of F and g . $\square$

Now, we report some examples where again the result of Choudhury et al. [24] can not be applied.

Example 24. Let $X = \mathbb{R}$ be endowed with the metric $d(x, y) = |x - y|$. Take $A = (0, 2)$ and $B = (0, 3)$. Let $F : X \times X \longrightarrow X$ be defined as $F(x, y) = 1 + [\sin((x + y)/4)]/3$, where $[x]$ denotes the first integer less than or equal to x and $g : X \longrightarrow X$ is defined as

$$g(x) = \begin{cases} 1, & 0 < x < 3, \\ x^2 + 3x, & \text{elsewhere.} \end{cases} \quad (79)$$

Since A and B are not closed in X , we can not apply the result of Choudhury et al. [24]. We have $g(A) = g(B) = \{1\}$. Clearly, A and B are invariant by g . Now, we shall show that F is a g -coupling. As $g(A) \cap B = g(B) \cap A = \{1\}$, so for all

$x \in A$ and $y \in B$, we have $F(x, y) = F(u, v) = 1$. Also, $F(x, y) \in g(A) \cap B$ and $F(u, v) \in g(B) \cap A$. Thus, F is a g -coupling with respect to A and B . Also, F is a Chatterjea type g -coupling with respect to A and B . Thus, all the conditions of Theorem 23 are satisfied, so there exists $(a, b) \in A \times B$ such that $F(a, b) = g(a)$ and $F(b, a) = g(b)$. For example, take $a = 1/2$ and $b = 1/3$.

Example 25. Let $X = \mathbb{R}^2$ be endowed with the Euclidean metric d . It is known that (X, d) is complete. Let $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : \sqrt{(x_1^2 + x_2^2)} < 1\}$ and $B = \{x = (x_1, x_2) \in \mathbb{R}^2 : \sqrt{(x_1^2 + x_2^2)} < 2\}$. Consider $F : X \times X \rightarrow X$ as

$$F(x, y) = F((x_1, x_2), (y_1, y_2)) = \left(\left[\frac{|x_1| + |x_2|}{4}, \frac{|y_1| + |y_2|}{4} \right] \right), \quad (80)$$

where $[x]$ denotes the first integer less than or equal to x . Since A and B are not closed in X , we can not apply the result of Choudhury et al. [24]. Now, we define $g : X \rightarrow X$ by $g(x) = g(x_1, x_2) = ([|x_1|], [|x_2|])$. We have $g(A) = \{(0, 0)\}$ and $g(B) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$; i.e., $g(A)$ and $g(B)$ are the sets of lattice points in A and B with nonnegative integers, respectively. Clearly, $g(A)$ and $g(B)$ are closed subsets in X . Also A and B are invariant by g . As $g(A) \cap B = g(B) \cap A = \{(0, 0)\}$, so for all $x \in A$ and $y \in B$, we have $F(x, y) = F(u, v) = (0, 0)$. Also, $F(x, y) \in g(A) \cap B$ and $F(u, v) \in g(B) \cap A$. Thus, F is a g -coupling with respect to A and B . Also, F is a Chatterjea type g -coupling with respect to A and B . Thus all the conditions of Theorem 23 are satisfied, so there exists $(a, b) \in A \times B$ such that $F(a, b) = g(a)$ and $F(b, a) = g(b)$. For example, take $a = (1/4, 1/3)$ and $b = (1/2, 1/6)$.

3. Application to Integral Equations

In this section, we study an existence and uniqueness problem of a solution for a class of nonlinear integral equations by using our obtained results. First, let $X = C([0, 1], \mathbb{R})$ be the set of all continuous functions defined on $[0, 1]$ and equipped by the metric $d(x, y) = \sup\{|x(t) - y(t)|\}$ for all $t \in [0, 1]$. Clearly, (X, d) is complete.

Consider the following system of nonlinear integral equations:

$$\begin{aligned} x(p) &= h(p) \\ &+ \int_0^1 k(p, q) [f_1(q, x(q)) + f_2(q, y(q))] dq, \\ p &\in [0, 1] \\ y(p) &= h(p) \\ &+ \int_0^1 k(p, q) [f_1(q, y(q)) + f_2(q, x(q))] dq, \\ p &\in [0, 1], \end{aligned} \quad (81)$$

where $h \in C([0, 1], \mathbb{R})$, $k : [0, 1] \times [0, 1] \rightarrow [0, \infty)$, and $f_1, f_2 : [0, 1] \times C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$.

Consider two nonempty subsets $A, B \subseteq X$. Let $x(t) \in A$ and $y(t) \in B$, where $t \in [0, 1]$. Suppose there exist $0 < \alpha, \beta \leq 1/2$ such that, for each $t, p, q \in [0, 1]$, we have

$$|f_1(p, x(t)) - f_1(p, y(t))| \leq \alpha |x(t) - y(t)|; \quad (82)$$

$$|f_2(p, x(t)) - f_2(p, y(t))| \leq \beta |x(t) - y(t)|; \quad (83)$$

$$k(p, q) \leq \frac{1}{2}. \quad (84)$$

Theorem 26. Under conditions (82)-(84), system (81) has a unique common solution in $C([0, 1], \mathbb{R})$.

Proof. We define the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ as

$$\begin{aligned} F(x, y) p &= h(p) \\ &+ \int_0^1 k(p, q) [f_1(q, x(q)) + f_2(q, y(q))] dq, \\ p &\in [0, 1], \end{aligned} \quad (85)$$

and

$$g(x) p = \begin{cases} x(p), & x(p) \in A \\ 2x(p) + 3, & \text{elsewhere,} \end{cases} \quad (86)$$

where $A = \{x(p) \in X : x(p) = h(p) + s, s \in \mathbb{R}\}$ is a closed subset in X . Taking $B = A$, we have $g(A) \subset A$ and $g(B) \subseteq B$. Also, $g(A)$ and $g(B)$ are closed subsets in X . Now, we show that F is a g -coupling. For $x(p) \in A$ and $y(p) \in B$, we have

$$\begin{aligned} F(x, y) p &= h(p) \\ &+ \int_0^1 k(p, q) [f_1(q, x(q)) + f_2(q, y(q))] dq \\ &= h(p) + r_1 \in g(A) \cap B = A; \quad r_1 \in \mathbb{R} \end{aligned} \quad (87)$$

and

$$\begin{aligned} F(y, x) p &= h(p) \\ &+ \int_0^1 k(p, q) [f_1(q, y(q)) + f_2(q, x(q))] dq \\ &= h(p) + r_2 \in g(B) \cap A = B; \quad r_2 \in \mathbb{R}. \end{aligned} \quad (88)$$

This shows that F is a g -coupling with respect to A and B . We shall show that F is a Banach type g -coupling. For this, let $x, v \in A$ and $y, u \in B$. Using (82), (83), and (84),

$$\begin{aligned} |F(x, y) p - F(u, v) p| &= \left| \int_0^1 k(p, q) [f_1(q, x(q)) + f_2(q, y(q))] dq \right. \\ &\quad \left. - \int_0^1 k(p, q) [f_1(q, u(q)) + f_2(q, v(q))] dq \right| \end{aligned}$$

$$\begin{aligned}
& - \int_0^1 k(p, q) [f_1(q, u(q)) + f_2(q, v(q))] dq \\
& \leq \int_0^1 k(p, q) [|f_1(q, x(q)) - f_1(q, u(q))| \\
& + |f_2(q, y(q)) - f_2(q, v(q))|] \leq \frac{1}{2} \cdot [\alpha |x(q) \\
& - u(q)| + \beta |v(q) - y(q)|] \leq \frac{1}{2} \max\{\alpha, \beta\} [|x(q) \\
& - u(q)| + |v(q) - y(q)|] \leq \frac{1}{4} [d(x, u) + d(y, v)] \\
& = \frac{k}{2} [d(x, u) + d(y, v)], \quad k = \frac{1}{2}.
\end{aligned} \tag{89}$$

Thus all the conditions of Theorem 15 are satisfied. Thus F and g have a unique common coupled fixed point (u, u) , satisfying $F(u, u) = gu = u$ in A . So (u, u) is the unique solution of the system of integral equations (81). $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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