Research Article

The Schur Multiplicative and Harmonic Convexities for Three Classes of Symmetric Functions

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We investigate the Schur harmonic convexity for two classes of symmetric functions and the Schur multiplicative convexity for a class of symmetric functions by using a new method and generalizing previous result. As applications, we establish some inequalities by use of the theory of majorization, in particular, and we give some new geometric inequalities in the n -dimensional space.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}^n$ the n -dimensional Euclidean space, $\mathbb{R} = \mathbb{R}^1 = (-\infty, +\infty)$ and $\mathbb{R}_+^n = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, 2, \dots, n\}$. For $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, we denote by

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n), \\ xy &= (x_1 y_1, x_2 y_2, \dots, x_n y_n), \\ \alpha x &= (\alpha x_1, \alpha x_2, \dots, \alpha x_n), \\ \alpha + x &= (\alpha + x_1, \alpha + x_2, \dots, \alpha + x_n), \\ \frac{1}{x} &= \left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n} \right). \end{aligned} \quad (1)$$

Furthermore, for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$, we denote by

$$\begin{aligned} \log x &= (\log x_1, \log x_2, \dots, \log x_n), \\ x^\alpha &= (x_1^\alpha, x_2^\alpha, \dots, x_n^\alpha). \end{aligned} \quad (2)$$

Recently, the Schur convexity, Schur multiplicative convexity, and Schur harmonic convexity for some special functions and their applications have been investigated by many authors; see, for instance, [1–12] and the references therein.

For $x = (x_1, \dots, x_n) \in [0, 1]^n = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i < 1, i = 1, 2, \dots, n\}$ and $r \in \{1, 2, \dots, n\}$, the symmetric function $F_n(x, r)$ was defined by Guan [13] as

$$\begin{aligned} F_n(x, r) &= F_n(x_1, x_2, \dots, x_n; r) \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r \frac{x_{i_j}}{1 - x_{i_j}}, \end{aligned} \quad (3)$$

where $i_1, i_2, \dots, i_r$ are positive integers. In [13], Guan discussed the Schur convexity and Schur multiplicative convexity for $F_n(x, r)$, established some interesting inequalities by use of the theory of majorization, and proposed an open problem as follows.

Open Problem A. If $n \geq 5$ and $r \in \{3, \dots, n-2\}$, then the function $F_n(x, r)$ is Schur convex, or Schur concave.

Chu et al. [2] solved the Open Problem A and established the following Theorem A.

Theorem A. If $n \geq 2$ and $r \in \{2, \dots, n\}$, then the function $F_n(x, r)$ is Schur concave in $[0, (r-1)/(2(n-1))]^n$ and Schur convex in $[(r-1)/(2(n-1)), 1]^n$.

Xia and Chu [14] defined the following symmetric function:

$$\begin{aligned} G_n(x, r) &= G_n(x_1, x_2, \dots, x_n; r) \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r \frac{1 - x_{i_j}}{x_{i_j}}, \end{aligned} \quad (4)$$

where $x = (x_1, \dots, x_n) \in (0, 1]^n$, $r \in \{1, 2, \dots, n\}$, and $i_1, i_2, \dots, i_n$ are positive integers. In [14], they discussed the Schur convexity and Schur multiplicative convexity for $G_n(x, r)$ and established some inequalities.

For $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ and $r \in \{1, 2, \dots, n\}$, the symmetric function $M_n(x, r)$ was defined by Xia et al. [15] as

$$M_n(x, r) = M_n(x_1, x_2, \dots, x_n; r) \\ = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r \frac{x_{i_j}}{1 + x_{i_j}}, \quad (5)$$

where $i_1, i_2, \dots, i_n$ are positive integers. In [15], Xia et al. proved that $M_n(x, r)$ is Schur concave and Schur harmonic convex in $\mathbb{R}_+^n$ (see [15, Theorems 3.1-3.2]) and Schur multiplicatively concave in $[1, +\infty)^n$, while $M_n(x, 1)$ is Schur multiplicatively convex in $(0, 1]^n$, and $M_n(x, n)$ is Schur multiplicatively concave in $[1, +\infty)^n$ (see [15, Theorem 3.3]).

It is natural to ask whether the symmetric functions $F_n(x, r)$ and $G_n(x, r)$ are Schur harmonic convex, or Schur harmonic concave in $\mathbb{R}_+^n$, while the function $M_n(x, r)$ is Schur multiplicatively convex or Schur multiplicatively concave for $x \in (0, 1)^n$ and $r \in \{2, \dots, n-1\}$, which have not been studied in existing literatures.

The main purpose of this paper is to discuss the Schur harmonic convexity of $F_n(x, r)$ and $G_n(x, r)$ in $\mathbb{R}_+^n$ and solve the problem of the Schur multiplicative convexity or concavity of $M_n(x, r)$ in $(0, 1)^n$ ($r = 2, \dots, n-1$) by using a new method, the main difficulty in proving Theorem 3 is to apply some techniques of differential calculus and combinatorics, in particular, and construct one complicate equality (29) in proof of Theorem 3. As applications, we establish some inequalities by use of the theory of majorization, in particular, we give some new geometric inequalities in the n -dimensional space.

Our main results are the following three theorems.

Theorem 1. For $x \in (0, 1)^n \cup (1, +\infty)^n$, $n \geq 2$ and $r \in \{1, 2, \dots, n\}$.

(1) The function $F_n(x, r)$ is Schur harmonic convex in $(0, 1)^n$;

(2) If r is even integer (or odd integer, respectively), then $F_n(x, r)$ is Schur harmonic convex (or concave, respectively) in $(1, +\infty)^n$.

Theorem 2. For $x \in (0, +\infty)^n$, $n \geq 2$ and $r \in \{1, 2, \dots, n\}$.

(1) The function $G_n(x, r)$ is Schur harmonic concave in $(0, 1]^n$;

(2) if r is even integer (or odd integer, respectively), then $G_n(x, r)$ is Schur harmonic concave (or convex, respectively) in $(1, +\infty)^n$.

Theorem 3. For $x \in (0, +\infty)^n$, $n \geq 3$, and $r \in \{2, \dots, n-1\}$, $M_n(x, r)$ is Schur multiplicatively convex in $(0, (n-r)/(n-1))^n$ and Schur multiplicatively concave in $((n-r)/(n-1), +\infty)^n$.

Remark 4. From the proof of Theorem 3.3 in [15], it is easy to see that Theorem 3 is valid for $r = 1$ and $r = n$, and Theorems 3 is a generalization of the Theorem 3.3 in [15], which is the main result in [15], thus, by using a new method we generalize

and solve the problem of the Schur multiplicative convexity or concavity of $M_n(x, r)$ in $(0, 1)^n$ ($r = 2, \dots, n-1$) from [15].

This paper, except for the introduction, is divided into three sections. In Section 2, we recall some definitions and lemmas. By using the results of Section 2, we give the proof of the main results in Section 3. Finally, some applications are given by use of the theory of majorization.

2. Definitions and Lemmas

For convenience, we recall some definitions.

Definition 5. Let $\Omega \subseteq \mathbb{R}^n$ be a set; a real-valued function f on Ω is said to be Schur convex if

$$f(x_1, x_2, \dots, x_n) \leq f(y_1, y_2, \dots, y_n) \quad (6)$$

for each pair of n -tuples $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in Ω , such that $x < y$, that is

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, \quad k = 1, 2, \dots, n-1 \quad (7)$$

and

$$\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}, \quad (8)$$

where $x_{[i]}$ denotes the i th largest component in x . f is called Schur concave if $-f$ is Schur convex.

The above notion of Schur convexity was first introduced by Schur in 1923 and has many important applications in analytic inequalities [16–20], isoperimetric problem for polytopes [21], linear regression [22], combinatorial optimization [23], graphs and matrices [24], gamma and digamma functions [25], information-theoretic topics [26], stochastic orderings [27], and other related fields. The Definition 5 can be found in many references such as [20, 28]. Following Schur, the following definition of the Schur multiplicative convex function was introduced in [29, 30].

Definition 6. Let $\Omega \subseteq \mathbb{R}_+^n$ be a set; a real-valued function f on Ω is said to be Schur multiplicatively convex if

$$f(x_1, x_2, \dots, x_n) \leq f(y_1, y_2, \dots, y_n) \quad (9)$$

for each pair of n -tuples $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in Ω , such that $\log x < \log y$. f is called Schur multiplicatively concave if $1/f$ is Schur multiplicatively convex.

Recently, Chu and Lv [31] introduced the notion of Schur harmonic convexity.

Definition 7. Let $\Omega \subseteq \mathbb{R}_+^n$ be a set; a real-valued function f on Ω is said to be Schur harmonic convex function if

$$f(x_1, x_2, \dots, x_n) \leq f(y_1, y_2, \dots, y_n), \quad (10)$$

for each pair of n -tuples $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in Ω , such that $1/x < 1/y$. f is called Schur harmonic concave if inequality (10) is reversed.

In order to establish our main result, we need two lemmas.

Lemma 8 (see [29, 30]). Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric multiplicatively convex set with nonempty interior in $\text{int } \Omega$ and $f: \Omega \rightarrow \mathbb{R}_+^1$ be a continuous symmetry function on Ω . If f is differentiable in $\text{int } \Omega$, then f is Schur multiplicatively convex on Ω if and only if

$$(\log x_1 - \log x_2) \left(x_1 \frac{\partial f(x)}{\partial x_1} - x_2 \frac{\partial f(x)}{\partial x_2} \right) \geq 0 \quad (11)$$

for all $x = (x_1, \dots, x_n) \in \text{int } \Omega$. And f is Schur multiplicatively concave on Ω if and only if inequality (11) is reversed. Here Ω is a multiplicatively convex set which means that $x^{1/2} y^{1/2} \in \Omega$ for $x, y \in \Omega$.

Lemma 9 (see [31]). Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric harmonic convex set with nonempty interior in $\text{int } \Omega$ and $f: \Omega \rightarrow \mathbb{R}_+^1$ be a continuous symmetry function on Ω . If f is differentiable on $\text{int } \Omega$, then f is Schur harmonic convex on Ω if and only if

$$(x_1 - x_2) \left(x_1^2 \frac{\partial f(x)}{\partial x_1} - x_2^2 \frac{\partial f(x)}{\partial x_2} \right) \geq 0 \quad (12)$$

for all $x = (x_1, \dots, x_n) \in \text{int } \Omega$. And f is Schur harmonic concave in Ω if and only if inequality (12) is reversed. Here Ω is a harmonic convex set which means that $2xy/(x+y) \in \Omega$ for $x, y \in \Omega$.

3. Proofs of the Main Results

Proof of Theorem 1. (1) According to Lemma 9, we only need to prove that

$$P_n(x, r) = (x_1 - x_2) \left(x_1^2 \frac{\partial F_n(x, r)}{\partial x_1} - x_2^2 \frac{\partial F_n(x, r)}{\partial x_2} \right) \geq 0 \quad (13)$$

for all $x \in (0, 1)^n$ and $r \in \{1, 2, \dots, n\}$. The proof is divided into three cases.

Case 1. If $r = 1$, then from (3) we have

$$F_n(x, 1) = \sum_{i=1}^n \frac{x_i}{1 - x_i}, \quad (14)$$

$$\frac{\partial F_n(x, 1)}{\partial x_i} = \frac{1}{(1 - x_i)^2}, \quad i = 1, 2 \quad (15)$$

By (15), we obtain

$$\begin{aligned} P_n(x, 1) &= (x_1 - x_2) \left(x_1^2 \frac{\partial F_n(x, 1)}{\partial x_1} - x_2^2 \frac{\partial F_n(x, 1)}{\partial x_2} \right) \\ &= \frac{(x_1 - x_2)^2 [x_1(1 - x_2) + x_2(1 - x_1)]}{(1 - x_1)^2 (1 - x_2)^2} \\ &\geq 0. \end{aligned} \quad (16)$$

Case 2. If $n \geq 2$ and $r = n$, then from (3) we obtain

$$F_n(x, n) = \prod_{i=1}^n \frac{x_i}{1 - x_i}, \quad (17)$$

$$\frac{\partial F_n(x, n)}{\partial x_i} = \frac{F_n(x, n)}{x_i(1 - x_i)}, \quad i = 1, 2 \quad (18)$$

By (18), we obtain

$$\begin{aligned} P_n(x, n) &= (x_1 - x_2) \left(x_1^2 \frac{\partial F_n(x, n)}{\partial x_1} - x_2^2 \frac{\partial F_n(x, n)}{\partial x_2} \right) \\ &= \frac{(x_1 - x_2)^2 F_n(x, n)}{(1 - x_1)(1 - x_2)} \geq 0. \end{aligned} \quad (19)$$

Case 3. If $n \geq 3$ and $2 \leq r \leq n - 1$, noting that the r -th order elementary symmetric function $E_n(x, r)$ (see [32]) is defined as

$$\begin{aligned} E_n(x, r) &= E_n(x_1, x_2, \dots, x_n; r) \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r x_{i_j}, \end{aligned} \quad (20)$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$, $r \in \{1, 2, \dots, n\}$, and $i_1, i_2, \dots, i_r$ are positive integers and $E_n(x, 0) = E_n(x_1, x_2, \dots, x_n; 0) = 1$.

Let $u_i = x_i/(1 - x_i)$, $i = 1, \dots, n$, and then from (3), we have

$$\begin{aligned} F_n(x, r) &= \frac{x_1 x_2}{(1 - x_1)(1 - x_2)} E_{n-2}(u_3, u_4, \dots, u_n; r - 2) \\ &\quad + \frac{x_1}{1 - x_1} E_{n-2}(u_3, u_4, \dots, u_n; r - 1) \\ &\quad + E_{n-1}(u_2, \dots, u_n; r) \\ &= \frac{x_1 x_2}{(1 - x_1)(1 - x_2)} E_{n-2}(u_3, u_4, \dots, u_n; r - 2) \\ &\quad + \frac{x_2}{1 - x_2} E_{n-2}(u_3, u_4, \dots, u_n; r - 1) \\ &\quad + E_{n-1}(u_1, u_3, \dots, u_n; r), \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\partial F_n(x, r)}{\partial x_1} &= \frac{x_2}{(1 - x_1)^2 (1 - x_2)} E_{n-2}(u_3, u_4, \dots, u_n; r - 2) \\ &\quad + \frac{1}{(1 - x_1)^2} E_{n-2}(u_3, u_4, \dots, u_n; r - 1), \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{\partial F_n(x, r)}{\partial x_2} &= \frac{x_1}{(1 - x_1)(1 - x_2)^2} E_{n-2}(u_3, u_4, \dots, u_n; r - 2) \\ &\quad + \frac{1}{(1 - x_2)^2} E_{n-2}(u_3, u_4, \dots, u_n; r - 1), \end{aligned} \quad (23)$$

From (22) and (23), we obtain

$$\begin{aligned} P_n(x, r) &= (x_1 - x_2) \left(x_1^2 \frac{\partial F_n(x, r)}{\partial x_1} - x_2^2 \frac{\partial F_n(x, r)}{\partial x_2} \right) \\ &= \frac{(x_1 - x_2)^2}{(1 - x_1)^2 (1 - x_2)^2} \{ x_1 x_2 E_{n-2}(u_3, u_4, \dots, u_n; r - 2) + [x_1(1 - x_2) + x_2(1 - x_1)] \\ &\quad \cdot E_{n-2}(u_3, u_4, \dots, u_n; r - 1) \} \geq 0, \end{aligned} \quad (24)$$

From Cases 1–3 we obtain that (13) holds for all $x \in (0, 1)^n$ and $r \in \{1, 2, \dots, n\}$; therefore, $F_n(x, r)$ is schur harmonic convex in $(0, 1)^n$.

(2) Here we give only the proof in the case of r being even integer, since the proof in the case of r being odd integer is similar. According to Lemma 9, we only need to prove that (13) holds for all $x \in (1, +\infty)^n$. From the proofs of Cases 2–3 in (1), it is easy to see that $P_n(x, r) \geq 0$ for $x \in (1, +\infty)^n$; thus we derive that (13) holds for all $x \in (1, +\infty)^n$ in the case of r being even integer; therefore, $F_n(x, r)$ is Schur harmonic convex in $(1, +\infty)^n$.

The proof of Theorem 1 is completed. $\square$

Similar to the proof of Theorem 1, we can use Lemma 9 and (20) to prove Theorem 2; therefore we omit the details of the proof. Next we give the proof of Theorem 3.

Proof of Theorem 3. According to Lemma 8, we only need to discuss the nonnegativity and nonpositivity of $Q_n(x, r) = (\log x_1 - \log x_2)(x_1(\partial M_n(x, r)/\partial x_1) - x_2(\partial M_n(x, r)/\partial x_2))$ for all $x \in (0, +\infty)^n$ and $r \in \{2, \dots, n-1\}$.

Let $v_i = x_i/(1 + x_i)$, $i = 1, \dots, n$; then from (5) and (20) we have

$$\begin{aligned} M_n(x, r) &= \frac{x_1 x_2}{(1 + x_1)(1 + x_2)} E_{n-2}(v_3, v_4, \dots, v_n; r - 2) \\ &\quad + \frac{x_1}{1 + x_1} E_{n-2}(v_3, v_4, \dots, v_n; r - 1) \\ &\quad + E_{n-1}(v_2, \dots, v_n; r) \\ &= \frac{x_1 x_2}{(1 + x_1)(1 + x_2)} E_{n-2}(v_3, v_4, \dots, v_n; r - 2) \\ &\quad + \frac{x_2}{1 + x_2} E_{n-2}(v_3, v_4, \dots, v_n; r - 1) \\ &\quad + E_{n-1}(v_1, v_3, \dots, v_n; r), \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{\partial E_n(x, r)}{\partial x_1} &= \frac{1 + x_2}{x_1 f(x_1) f(x_2)} E_{n-2}(v_3, v_4, \dots, v_n; r - 2) \\ &\quad + \frac{1}{x_1 f(x_1)} E_{n-2}(v_3, v_4, \dots, v_n; r - 1), \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{\partial E_n(x, r)}{\partial x_2} &= \frac{1 + x_1}{x_2 f(x_1) f(x_2)} E_{n-2}(v_3, v_4, \dots, v_n; r - 2) \\ &\quad + \frac{1}{x_2 f(x_2)} E_{n-2}(v_3, v_4, \dots, v_n; r - 1), \end{aligned} \quad (27)$$

where $f(x) = (1 + x)^2/x$. From (26) and (27) we obtain

$$\begin{aligned} Q_n(x, r) &= (\log x_1 - \log x_2) \left(x_1 \frac{\partial M_n(x, r)}{\partial x_1} - x_2 \frac{\partial M_n(x, r)}{\partial x_2} \right) \\ &= -\frac{\log x_1 - \log x_2}{f(x_1) f(x_2)} [g(x_1, r) - g(x_2, r)], \end{aligned} \quad (28)$$

where $g(x, r) = x E_{n-2}(v_3, v_4, \dots, v_n; r - 2) + f(x) E_{n-2}(v_3, v_4, \dots, v_n; r - 1)$. Note that

$$\begin{aligned} E_{n-2}(v_3, v_4, \dots, v_n; r - 1) &= \frac{1}{r - 1} \\ &\quad \cdot \sum_{3 \leq i_3 < \dots < i_r \leq n} \prod_{j=3}^r \frac{x_{i_j}}{1 + x_{i_j}} \left(\sum_{\substack{i_k \in \{3, \dots, n\} \\ i_k \neq i_j (j=3, \dots, n)}} \frac{x_{i_k}}{1 + x_{i_k}} \right), \end{aligned} \quad (29)$$

$$f'(x) = \frac{x^2 - 1}{x^2}, \quad (30)$$

$$f''(x) = 2x^{-3}, \quad (31)$$

$$\begin{aligned} \frac{\partial g(x, r)}{\partial x} &= E_{n-2}(v_3, v_4, \dots, v_n; r - 2) + f'(x) \\ &\quad \cdot E_{n-2}(v_3, v_4, \dots, v_n; r - 1) \\ &= \sum_{3 \leq i_3 < \dots < i_r \leq n} \prod_{j=3}^r \frac{x_{i_j}}{1 + x_{i_j}} \left(1 \right. \\ &\quad \left. + \frac{1}{r - 1} f'(x) \sum_{\substack{i_k \in \{3, \dots, n\} \\ i_k \neq i_j (j=3, \dots, n)}} \frac{x_{i_k}}{1 + x_{i_k}} \right), \end{aligned} \quad (32)$$

and

$$\frac{\partial^2 g(x, r)}{\partial x^2} = f''(x) E_{n-2}(v_3, v_4, \dots, v_n; r - 1). \quad (33)$$

Let $A = (n - r)/(n - 1)$; then $0 < A < 1$, from (30) we have

$$S = 1 + \frac{n - r}{r - 1} f'(A) \left(\frac{A}{1 + A} \right) = 0. \quad (34)$$

From (31), we have $f''(x) > 0$ for $x \in (0, +\infty)$; hence by (33), if $x_i \in (0, +\infty)$ ($i = 3, 4, \dots, n$), then $\partial^2 g(x, r)/\partial x^2 >$

0 for $x \in (0, +\infty)$; therefore, $\partial g(x, r)/\partial x$ is increasing on $(0, +\infty)$ with x ; further from (32) and (34), we obtain for $x, x_i \in (0, A)$ ($i = 3, 4, \dots, n$)

$$\frac{\partial g(x, r)}{\partial x} < \frac{\partial g(A, r)}{\partial x} = \sum_{3 \leq i_3 < \dots < i_r \leq n} \prod_{j=3}^r \frac{x_{i_j}}{1 + x_{i_j}} \left(1 + \frac{1}{r-1} f'(A) \sum_{\substack{i_k \in \{3, \dots, n\} \\ i_k \neq i_j (j=3, \dots, n)}} \frac{x_{i_k}}{1 + x_{i_k}} \right) \leq SM_{n-2}(A, \dots, A; r-2) = 0 \quad (35)$$

$$A, \dots, A; r-2) = 0$$

and have for $x, x_i \in (A, +\infty)$ ($i = 3, 4, \dots, n$)

$$\frac{\partial g(x, r)}{\partial x} > \frac{\partial g(A, r)}{\partial x} = \sum_{3 \leq i_3 < \dots < i_r \leq n} \prod_{j=3}^r \frac{x_{i_j}}{1 + x_{i_j}} \left(1 + \frac{1}{r-1} f'(A) \sum_{\substack{i_k \in \{3, \dots, n\} \\ i_k \neq i_j (j=3, \dots, n)}} \frac{x_{i_k}}{1 + x_{i_k}} \right) \geq SM_{n-2}(A, \dots, A; r-2) = 0. \quad (36)$$

$$A, \dots, A; r-2) = 0.$$

Hence by (35) and (36), respectively, if $x_i \in (0, A)$ ($i = 3, 4, \dots, n$), then $g(x, r)$ is decreasing on $(0, A)$ with x , and if $x_i \in (A, +\infty)$ ($i = 3, 4, \dots, n$), then $g(x, r)$ is increasing on $(A, +\infty)$ with x ; further from (28) we derive that $Q_n(x, r) \geq 0$ for $x \in (0, A)^n$, and $Q_n(x, r) \leq 0$ for $x \in (A, +\infty)^n$. Therefore, from Lemma 8, $M_n(x, r)$ is Schur multiplicatively convex in $(0, (n-r)/(n-1))^n$ and Schur multiplicatively concave in $((n-r)/(n-1), +\infty)^n$.

The proof of Theorem 3 is completed. $\square$

Remark 10. It is easy to see that the key point of our proof of Theorem 3 is to construct one monotone function $g(x, r)$ and prove that $g(x, r)$ is decreasing on $(0, A)$, and increasing on $(A, +\infty)$ with x , which is different from the proof of Theorem 3.3 in [15].

4. Applications

In this section, we establish some inequalities by use of Theorems 1–3 and the theory of majorization.

Theorem 11. Let $n \geq 2$, $x = (x_1, x_2, \dots, x_n)$, $A_n(x) = (1/n) \sum_{i=1}^n x_i$, and $r \in \{1, 2, \dots, n\}$.

(1) Suppose that $x \in (0, 1)^n$. If r is even integer, then

$$\sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r \frac{1}{x_{i_j} - 1} \geq \frac{n!}{r! (n-r)!} \left[\frac{1}{A_n(x-1)} \right]^r, \quad (37)$$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r (x_{i_j} - 1) \leq \frac{n!}{r! (n-r)!} [A_n(x-1)]^r, \quad (38)$$

while if r is odd integer, then the inequalities (37) and (38) are reversed

(2) If $x \in (1, +\infty)^n$, the inequalities (37) and (38) hold.

Proof. It is easy to see that

$$(A_n(x), A_n(x), \dots, A_n(x)) < (x_1, x_2, \dots, x_n). \quad (39)$$

From Theorems 1 and 2, respectively, and applying (39), we get that Theorem 11 holds. $\square$

Theorem 12. Let $n \geq 2$, $x = (x_1, x_2, \dots, x_n)$, $H_n(x) = n / \sum_{i=1}^n (1/x_i)$, and $r \in \{1, 2, \dots, n\}$.

(1) If $x \in (0, 1)^n$, then

$$\sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r \frac{x_{i_j}}{1 - x_{i_j}} \geq \frac{n!}{(n-r)! r!} \left[\frac{H_n(x)}{H_n(x) - 1} \right]^r, \quad (40)$$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r \frac{1 - x_{i_j}}{x_{i_j}} \leq \frac{n!}{(n-r)! r!} \left[\frac{1 - H_n(x)}{H_n(x)} \right]^r; \quad (41)$$

(2) Suppose that $x \in (1, +\infty)^n$. If r is even integer, then the inequalities (40) and (41) hold, while if r is odd integer, the inequalities (40) and (41) are reversed.

Proof. It is easy to see that

$$\left(\frac{1}{H_n(x)}, \frac{1}{H_n(x)}, \dots, \frac{1}{H_n(x)} \right) < \left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n} \right). \quad (42)$$

From Theorems 1 and 2, respectively, and applying (42), we obtain that Theorem 12 holds. $\square$

Theorem 13. Let $n \geq 2$, $x = (x_1, x_2, \dots, x_n)$, $G_n(x) = (\prod_{i=1}^n x_i)^{1/n}$, and $r \in \{1, 2, \dots, n\}$. If $x \in (0, (n-r)/(n-1))^n$, then

$$\sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r \frac{x_{i_j}}{1 + x_{i_j}} \geq \frac{n!}{(n-r)! r!} \left[\frac{G_n(x)}{1 + G_n(x)} \right]^r, \quad (43)$$

if $x \in [(n-r)/(n-1), +\infty)^n$, the inequality (43) is reversed.

Proof. It is easy to see that

$$\log(G_n(x), G_n(x), \dots, G_n(x)) < \log(x_1, x_2, \dots, x_n). \quad (44)$$

From Theorems 3 and Remark 4, and applying (44), we obtain that Theorem 13 holds. $\square$

Remark 14. It is easy to see that Theorem 13 is a generalization of the Theorem 4.5 in [15].

Theorem 15. Let $\mathcal{A} = A_1 A_2 \dots A_{n+1}$ be an n -dimensional simplex in R^n and P be an arbitrary point in the interior of $\mathcal{A}$. If B_i is the intersection point of straight $A_i P$ and hyperplane

$\sum_i = A_1 A_2 \cdots A_{i-1} A_{i+1} \cdots A_{n+1}$ ($i = 1, 2, \dots, n+1$). Then for $r \in \{1, 2, \dots, n+1\}$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n+1} \prod_{j=1}^r \frac{A_{i_j} B_{i_j}}{P A_{i_j}} \geq \frac{(n+1)!}{(n+1-r)! r!} \left(\frac{n+1}{n} \right)^r, \quad (45)$$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n+1} \prod_{j=1}^r \frac{P A_{i_j}}{A_{i_j} B_{i_j}} \leq \frac{(n+1)!}{(n+1-r)! r!} \left(\frac{n}{n+1} \right)^r, \quad (46)$$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n+1} \prod_{j=1}^r \frac{A_{i_j} B_{i_j}}{P B_{i_j}} \geq \frac{(n+1)!}{(n+1-r)! r!} (n+1)^r, \quad (47)$$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n+1} \prod_{j=1}^r \frac{P B_{i_j}}{A_{i_j} B_{i_j}} \leq \frac{(n+1)!}{(n+1-r)! r!} \left(\frac{1}{n+1} \right)^r, \quad (48)$$

Proof. It is easy to see that $\sum_{i=1}^{n+1} (P B_i / A_i B_i) = 1$ and $P A_i / A_i B_i = 1 - P B_i / A_i B_i$, $i = 1, 2, \dots, n+1$; these imply that

$$\left(\frac{1}{n+1}, \frac{1}{n+1}, \dots, \frac{1}{n+1} \right) < \left(\frac{P B_1}{A_1 B_1}, \frac{P B_2}{A_2 B_2}, \dots, \frac{P B_{n+1}}{A_{n+1} B_{n+1}} \right), \quad (49)$$

$$\left(\frac{n}{n+1}, \frac{n}{n+1}, \dots, \frac{n}{n+1} \right) < \left(\frac{P A_1}{A_1 B_1}, \frac{P A_2}{A_2 B_2}, \dots, \frac{P A_{n+1}}{A_{n+1} B_{n+1}} \right). \quad (50)$$

Therefore, (45)-(48) follow from Theorems 1(2) and 2(2), respectively, together with (49) and (50). $\square$

Remark 16. The inequalities (45)-(48) can be found in [33, p.473-480]; here we give a different proof of them.

Theorem 17. Let $\mathcal{A} = A_1 A_2 \cdots A_{n+1}$ be an n -dimensional simplex in R^n and r be the inradius of $\mathcal{A}$. Let h_i be the altitude of $\mathcal{A}$ from the vertex A_i , i.e., the distance from A_i to hyperplane $\sum_i = A_1 A_2 \cdots A_{i-1} A_{i+1} \cdots A_{n+1}$ ($i = 1, 2, \dots, n+1$), and r_i be the radius of i -th escribed hypersphere of $\mathcal{A}$. Then for $k \in \{1, 2, \dots, n+1\}$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+1} \prod_{j=1}^k \frac{r_{i_j}}{r_{i_j} - r} \geq \frac{(n+1)!}{(n+1-k)! k!} \left(\frac{n+1}{2} \right)^k, \quad (51)$$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+1} \prod_{j=1}^k \frac{r_{i_j} - r}{r_{i_j}} \leq \frac{(n+1)!}{(n+1-k)! k!} \left(\frac{2}{n+1} \right)^k, \quad (52)$$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+1} \prod_{j=1}^k \frac{h_{i_j}}{h_{i_j} - r} \geq \frac{(n+1)!}{(n+1-k)! k!} \left(\frac{n+1}{n} \right)^k, \quad (53)$$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+1} \prod_{j=1}^k \frac{h_{i_j} - r}{h_{i_j}} \geq \frac{(n+1)!}{(n+1-k)! k!} \left(\frac{n}{n+1} \right)^k, \quad (54)$$

Proof. Let V denotes the volume of $\mathcal{A}$, F_i ($i = 1, 2, \dots, n+1$) denote the $(n-1)$ -dimensional volume of the side face of

$\mathcal{A}$ opposite to the vertex A_i ($i=1,2,\dots,n+1$). Let $F = \sum_{i=1}^{n+1} F_i$; noting the well-known fact (see [33, p.463])

$$r = \frac{nV}{F},$$

$$h_i = \frac{nV}{F_i}, \quad (55)$$

$$r_i = \frac{nV}{F - 2F_i},$$

we obtain

$$\frac{r}{r_i} = \frac{F - 2F_i}{F},$$

$$\frac{r}{h_i} = \frac{F_i}{F}. \quad (56)$$

Thus we have

$$\sum_{i=1}^{n+1} \frac{r}{r_i} = n - 1, \quad (57)$$

$$\sum_{i=1}^{n+1} \frac{r}{h_i} = 1,$$

and further we obtain

$$\left(\frac{n-1}{n+1}, \frac{n-1}{n+1}, \dots, \frac{n-1}{n+1} \right) < \left(\frac{r}{r_1}, \frac{r}{r_2}, \dots, \frac{r}{r_{n+1}} \right), \quad (58)$$

$$\left(\frac{1}{n+1}, \frac{1}{n+1}, \dots, \frac{1}{n+1} \right) < \left(\frac{r}{h_1}, \frac{r}{h_2}, \dots, \frac{r}{h_{n+1}} \right). \quad (59)$$

Therefore, (51)-(54) follow from Theorems 1(2) and 2(2), respectively, together with (58) and (59). $\square$

Remark 18. The inequality (51) can be found in the Proposition 4 from [19]; here we give a different proof of it. Mitrinović, Pečarić, and Volenec (see [33, p.463-473]) established a series of inequalities for r , h_i , and r_i ; obviously, our inequalities (52)-(54) in Theorem 17 are different from theirs.

5. Conclusions

In this paper, we investigate the Schur harmonic convexity for two classes of symmetric functions defined by Guan (2007), Xia and Chu (2009), and the Schur multiplicative convexity for a class of symmetric functions defined by Xia et al. (2010) by using a new method and generalize the main results of Xia et al. (2010). As applications, we establish some inequalities by use of the theory of majorization; in particular, we give some new geometric inequalities in the n -dimensional space. It is easy to see that

(1) Our Theorem 13 is a generalization of the Theorem 4.5 in Xia et al. (2010).

(2) The inequalities (45)-(48) can be found in Mitrinović (1989); here we give a different proof of them.

(3) The inequality (51) can be found in the Proposition 4 from Wu (2005); here we give a different proof of it.

Data Availability

No data of any type is used in this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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