Research Article

A Result on the Existence and Uniqueness of Stationary Solutions for a Bioconvective Flow Model

Aníbal Coronel, Luis Friz, Ian Hess, and Alex Tello

GMA, Departamento de Ciencias Básicas, Facultad de Ciencias, Universidad del Bio-Bío, Campus Fernando May, Chillán, Chile

Correspondence should be addressed to Aníbal Coronel; acoronel@ubiobio.cl

Received 8 December 2017; Accepted 25 March 2018; Published 30 April 2018

1. Introduction

Bioconvection is an important process in the biological treatment and in the life of some microorganisms. In a broad sense, bioconvection originates from the concentration of upward swimming microorganisms in a culture fluid. It is well known that, under some physical assumptions, the process can be described by mathematical models which are called bioconvective flow models. The first model of this kind was derived by Moribe [1] and independently by Levandowsky et al. [2] (see also [3] for the mathematical analysis). In that model the unknowns are the velocity of the fluid, the pressure of the fluid, and the local concentration of microorganisms. More recently, Tuval et al. [4] have introduced a new bioconvective flow model considering an additional unknown variable, the oxygen concentration. Some advances in mathematical analysis and some numerical results for this new model are presented in [5] and [6], respectively.

In this paper, we are interested in the existence and uniqueness of solutions for the stationary problem associated with the bioconvective system given in [4] when the physical domain is a three-dimensional chamber [6] (a parallelepiped). Thus, the stationary bioconvective flow problem to be analyzed is formulated as follows. Given the external force \( F \), the source functions \( f_n, f_c \), and the dimensionless function \( r \), find the velocity of the fluid \( u = (u_1, u_2, u_3)^t \), the fluid pressure \( p \), the local concentration of bacteria \( n \), and the local concentration of oxygen \( c \) satisfying the boundary value problem:

\[
-\frac{\gamma}{\nu^2} \Delta u + (u \cdot \nabla) u + \frac{\gamma}{\nu^2} \nabla p = \gamma \frac{\gamma}{\nu^2} n g + F,
\]

in \( \Omega = \prod_{i=1}^{3} [0, L_i] \),

\[
\text{div}(u) = 0, \quad \text{in } \Omega,
\]

\[
-\Delta n + (u \cdot \nabla) n + \chi \text{div}(nr(c) \nabla c) = f_n, \quad \text{in } \Omega,
\]

\[
-\frac{\delta}{\nu^2} \Delta c + (u \cdot \nabla) c + \beta r(c) n = f_c, \quad \text{in } \Omega,
\]

\[
\nabla c \cdot v = \nabla n \cdot v = 0,
\]

\[
u = 0, \quad \text{on } \partial \Omega_L \ (x_3 = 0),
\]

\[
\chi nr(c) \nabla c \cdot v - \nabla n \cdot v = 0,
\]

\[
u = 0, \quad \text{on } \partial \Omega_U = \partial \Omega - \partial \Omega_L.
\]
Here $\nu$ is the unit external normal to $\partial \Omega$; $g = (0, 0, -g)$ is a gravitational field with constant acceleration $g$; and $S_c$, $\gamma$, $\alpha$, $\delta$, and $\beta$ are some physical parameters defined as follows:

\[
\begin{align*}
S_c &= \frac{\eta}{D_n \rho}, \\
\gamma &= \frac{V \eta n_r (\rho_h - \rho) L^3}{\eta D_n}, \\
\chi &= \frac{\kappa_{\text{air}}}{D_n},
\end{align*}
\]

(7)

with $\eta$ being the fluid viscosity, $D_n$ the diffusion constant for bacteria, $D_c$ the diffusion constant for oxygen, $\rho$ the fluid density, $\rho_h$ the bacterial density, $V_0 > 0$ the bacterial volume, $n_r$ a characteristic cell density, $L$ a characteristic length, $\chi$ the chemotactic sensitivity, $c_{\text{air}}$ the oxygen concentration above the fluid, and $k$ the oxygen consumption rate.

We consider the standard notation of the Lebesgue and Sobolev spaces which are used in the analysis of Navier-Stokes and related equations of fluid mechanics; see [7–11] for details and specific definitions. In particular, we use the following rather common spaces notation:

\[
\begin{align*}
H^m(\Omega) &= W^{m,2}(\Omega), \\
H^1(\Omega) &= \left\{ f \in H^1(\Omega) : \int_{\Omega} f \, dx = 0 \right\}, \\
H^1_0(\Omega) &= \left\{ v \in C_0^\infty(\Omega) : \text{div} (v) = 0 \right\}, \\
C_0^\infty(\Omega) &= \{ v \in C_0^\infty(\Omega) : \text{div} (v) = 0 \}, \\
V &= \left\{ \frac{\nabla \eta}{c_{\text{air}}D_n} \right\},
\end{align*}
\]

where $A_{1,h}$ denotes the completion of $A$ in $B$. Also, we consider the notation for the applications $a_0 : V \times V \to \mathbb{R}$, $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $b_0 : V \times V \times V \to \mathbb{R}$, and $b : V \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, which are defined as

\[
\begin{align*}
a_0 (u, v) &= (\nabla u, \nabla v), \\
a (u, v) &= (\nabla \cdot u, \nabla \cdot v), \\
b_0 (u, v, w) &= ((u \cdot \nabla) v, w), \\
b (u, v, \psi) &= (u \cdot \nabla \psi),
\end{align*}
\]

(9)

where $(\cdot, \cdot)$ is the standard inner product in $L^2(\Omega)$ or $L^2(\Omega)$.

It is well known that $a_0$ and $a$ are bilinear coercive forms and $b_0$ and $b$ are well defined trilinear forms with the following properties:

\[
\begin{align*}
b_0 (u, v, w) &= -b_0 (u, w, v), \\
b (u, \phi, \psi) &= b (u, \psi, \phi), \\
b_0 (u, v, v) &= 0, \\
b (u, \phi, \psi) &= 0,
\end{align*}
\]

(10)

for all $u, v, w \in V$. Moreover, we need to introduce some notation related to these useful Sobolev embeddings and estimates for $b$ and $b_0$. There exist $C_{\text{poi}} > 0$, $C_{\text{tr}} > 0$, and $C_1$ depending only on $\Omega$ such that

\[
\begin{align*}
\| u \|_{L^2(\Omega)} &\leq C_{\text{poi}} \| u \|_V, \\
\| c \|_{L^2(\Omega)} &\leq C_{\text{poi}} \| c \|_{\tilde{H}^1(\Omega)}, \\
\| \phi \|_{L^2(\Omega)} &\leq C_{\text{tr}} \| \phi \|_{W^{1,2}(\Omega)},
\end{align*}
\]

(11)

\[
\begin{align*}
\| b_0 (u, v, w) \| &\leq C_1 \| u \|_V \| v \|_V \| w \|_V, \\
\| b (u, c, n) \| &\leq C_1 \| u \|_V \| c \|_{H^1(\Omega)} \| n \|_{\tilde{H}^1(\Omega)},
\end{align*}
\]

(12)

for all $u, v, w \in V$, $c, n \in H^1(\Omega)$, and $\phi \in W^{1,1}(\Omega)$. For details on Poincaré and trace inequalities, we refer to [8] and for the estimates of $b_0$ and $b$ consult [11].

The main result of the paper is the existence and uniqueness of weak solutions for (1)–(6). Indeed, let us introduce some appropriate notation:

\[
\begin{align*}
\Theta_1 &= \frac{1 - C_{\text{tr}}}{1 - C_{\text{tr}} - 2 \chi \| f \|_{L^2(\Omega)} C_{\text{poi}} C_{\text{tr}}}, \\
\Theta_2 &= \frac{1}{1 - C_{\text{tr}} - C_{\text{tr}} C_{\text{poi}}}, \\
\Gamma_0 &= \frac{\| \Omega \| \Theta_{C_{\text{poi}}} C_{\text{poi}}}{\| \Omega \| - \chi \beta \| \Omega \| C_{\text{poi}} C_{\text{poi}} \Theta_2 \left[ \frac{\| \Omega \| \Theta_{C_{\text{poi}}} C_{\text{poi}}}{\| \Omega \|} \right]}, \\
\Gamma_1 &= \frac{1}{S - C_{\text{tr}} C_{\text{poi}} (\gamma g t_0 + \| F \|_{L^2(\Omega)})}, \\
\Gamma_2 &= \frac{1}{1 - C_{\text{tr}}}, \\
\Gamma_3 &= \frac{1}{\delta (1 - C_{\text{tr}} - C_{\text{tr}} C_{\text{poi}}) - (C_1^3)^{1/2} \| n \|_{L^2(\Omega)}}
\end{align*}
\]

(13)

(14)

(15)

such that the result is proved as follows.

**Theorem 1.** Let us consider that $f_0, f_1 \in L^2(\Omega)$, $F \in L^2(\Omega)$ and $\bar{n}$, the average of $n$ on $\Omega$, are given. Also consider notations (12)–(15). If we assume that the following assumptions,

\[
\begin{align*}
r \in L^{\infty} (\mathbb{R}) \cap L^1 (\mathbb{R}), \\
1 - C_{\text{tr}} > C_{\text{tr}} C_{\text{poi}} \max \{ 2 \chi \| f \|_{L^2(\Omega)} , 1 \}, \\
1 > \chi \beta \| f \|_{L^2(\Omega)}^2 C_{\text{poi}}^2 \Theta_1 \Theta_2,
\end{align*}
\]

(16)
are satisfied, there is \((u, p, n, c) \in V \times H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)\) satisfying (1)–(6). Moreover, if we consider that additionally \(r \in Lip(\mathbb{R})\) and the following inequalities,

\[
S_c - C_1 C_{po} \left( \gamma g \Gamma_0 + \|f\|_{L^2(\Omega)} \right) > 0, \\
\delta \left( 1 - C_{tr} - C_{tr} C_{po} \right) - (C_1)^3 \|r\|_{L^1(\Omega)} \Gamma_0 > 0, \\
C_1 \|r\|_{Lip(\mathbb{R})} \Gamma_0 < 1, \\
\Pi = \Gamma_1 \Gamma_2 - C_1 \Gamma_0 + \frac{\|r\|_{L^1(\Omega)}}{(1 - C_1 \|r\|_{Lip(\mathbb{R})}) \Gamma_0 \left( \beta C_{po} \|r\|_{L^1(\Omega)} \right) + \|f_2\|_{L^2(\Omega)}} < 1,
\]

are satisfied, the weak solution is unique.

It should be noted that existence and uniqueness results are derived in [12, 13] for the bioconvection problem, when the concentration of oxygen is assumed to be constant. In the case of [12], the proof is based on the application of the Galerkin approximation and in [13] on the application of the Gossez theorem. Moreover, other related results are given in [3, 5]. In particular, in [5], a well detailed discussion of some particular models derived from (1)–(6) is given.

### 2. Proof of Theorem 1

#### 2.1. Variational Formulation.

By standard arguments, the variational formulation of (1)–(6) is given by

Find \((u, n, c) \in V \times H^1(\Omega) \times H^1(\Omega)\) such that

\[
S_c a_0 (u, v) + b_0 (u, u, v) = \gamma S_0 (ng, v) + (F, v), \\
\forall v \in V, \\
a(n, \phi) + b(u, u, \phi) = \chi (nr (c) \nabla c, \nabla \phi) + (f_1, \phi), \\
\forall \phi \in H^1(\Omega), \\
\delta a(c, \phi) + b(u, c, \phi) = -\beta (r (c) n, \phi) + \delta \int_{\partial \Omega_1} \nabla c \cdot \nabla \phi dS + (f_2, \phi), \\
\forall \phi \in H^1(\Omega).
\]

We notice that if \(f_2 = f_1 = 0\) and \(u_0\) is a solution of (1)–(2) with \(n = 0\), we have that \((u_0, 0, 0)\) is a solution of (19). However, \((u_0, 0, 0)\) does not describe the bioconvective flow problem and we need to study the variational problem when the total local concentration of bacteria and the total local concentration of oxygen are some given strictly positive constants, that is, \(\int_\Omega n_a d\mathbf{x} = \alpha_1 > 0\) and \(\int_\Omega c_a d\mathbf{x} = \alpha_2 > 0\).

Thus, by considering the change of variable \(\tilde{n}_a = n_a - \alpha_1 |\Omega|^{-1}\) and \(\tilde{c}_a = c_a - \alpha_2 |\Omega|^{-1}\), we can rewrite (19) as follows:

Given \(\alpha = (\alpha_1, \alpha_2) \in [0, 1] \times [0, 1]\) find \((u_a, \tilde{n}_a, \tilde{c}_a)\)

\[
eq \gamma S_0 (\tilde{n}_a, g) + (F, v), \\
a(\tilde{n}_a, \phi) + b(u_a, \tilde{n}_a, \phi) = \chi (\tilde{n}_a + \frac{\alpha_1}{|\Omega|} r (\tilde{c}_a + \frac{\alpha_2}{|\Omega|}) \nabla \tilde{c}_a, \nabla \phi), \\
+ (f_2, \phi), \\
\delta a(\tilde{c}_a, \phi) + b(u_a, \tilde{c}_a, \phi) = -\beta (r (\tilde{c}_a + \frac{\alpha_2}{|\Omega|}) \tilde{n}_a + \frac{\alpha_1}{|\Omega|}) \nabla \tilde{c}_a, \nabla \phi, \\
+ \delta \int_{\partial \Omega_1} \nabla \tilde{c}_a \cdot \nabla \phi dS + (f_2, \phi), \\
\forall (v, \phi, \varphi) \in V \times H^1(\Omega) \times H^1(\Omega).
\]

#### 2.2. Some A Priori Estimates for \(u_a, \tilde{n}_a,\) and \(\tilde{c}_a\)

**Proposition 2.** Consider that the assumptions for the existence result of Theorem 1 are satisfied. If we assume that \((u_a, \tilde{n}_a, \tilde{c}_a)\) is a solution of (20)–(24), then \(\|u_a\|_{L^2(\Omega)} \leq T_0\) with \(T_0\) defined on (13). Furthermore, the following estimates are valid:

\[
\|u_a\|_V \leq C_{po} (\gamma g \Gamma_0 + \|f\|_{L^2(\Omega)}), \\
\|\tilde{n}_a\|_{F(\Omega)} \leq \frac{\Theta C_{po}}{\delta} \left( \beta C_{po} \|r\|_{L^1(\Omega)} \Gamma_0 + \|f_2\|_{L^2(\Omega)} \right), \\
\|\tilde{c}_a\|_{F(\Omega)} \leq \frac{\Theta C_{po}}{\delta} \left( \beta C_{po} \|r\|_{L^1(\Omega)} \Gamma_0 + \|f_2\|_{L^2(\Omega)} \right).
\]

**Proof.** In order to prove the estimates, we select the test functions \((v, \phi, \varphi) = (u_a, \tilde{n}_a, \tilde{c}_a)\) in (21)–23. From (21) and (10), we deduce that

\[
\|u_a\|_V \leq \gamma g C_{po}^2 \|\tilde{n}_a\|_{F(\Omega)} + (S_c)^{-1} C_{po} \|f_2\|_{L^2(\Omega)}.
\]

Now, by the trace inequality and integration by parts, we have that

\[
\int_{\partial \Omega} |\nabla \tilde{n}_a \cdot \nu \tilde{n}_a| dS \leq C_{tr} \|\tilde{n}_a\|_{F(\Omega)} \|\nabla \tilde{n}_a\|_{W^{1,\infty}(\Omega)}
\]

\[
\leq C_{tr} C_{po} \|\tilde{n}_a\|_{F(\Omega)} \Gamma_0 + C_{tr} \int_{\partial \Omega} |\nabla \tilde{n}_a \cdot \nu \tilde{n}_a| dS,
\]

which implies that

\[
\int_{\partial \Omega} |\nabla \tilde{n}_a \cdot \nu \tilde{n}_a| dS \leq \frac{C_{tr} C_{po}}{1 - C_{tr}} \|\tilde{n}_a\|_{F(\Omega)}^2.
\]
Here, we have used the fact that $1 - C_{tr} > 0$, as a consequence of the assumption (16). Then, by integration by parts we get the bound
\[
\left( \tilde{n}_a r \left( z_a + \frac{\alpha_2}{|\Omega|} \right) \nabla \tilde{n}_a, \nabla \tilde{n}_a \right) = \left( \nabla \left[ \int_0^{z_a} r \left( m + \frac{\alpha_2}{|\Omega|} \right) \, dm \right], \nabla \left( \frac{\tilde{n}_a^2}{2} \right) \right) = \left( \int_0^{z_a} r \left( m + \frac{\alpha_2}{|\Omega|} \right) \, dm, \Delta \left( \frac{\tilde{n}_a^2}{2} \right) \right) + \int_{\partial\Omega} \left[ \int_0^{z_a} r \left( m + \frac{\alpha_2}{|\Omega|} \right) \, dm \right] \nabla \left( \frac{\tilde{n}_a^2}{2} \right) \cdot v dS \\
\leq 2 \| r \|_{L^1(\Omega)} \int_{\partial\Omega} |\tilde{n}_a| \nabla \tilde{n}_a \cdot v \, dS \leq \frac{2}{1 - C_{tr}} \| \tilde{n}_a \|_{F^1(\Omega)},
\]
with $\Theta_1$ being defined in (12). Similarly, from (23) and (28) with $\tilde{c}_a$ instead of $\tilde{n}_a$, we deduce that
\[
\| \tilde{c}_a \|_{F^1(\Omega)} \leq \Theta_1 \left[ \frac{\chi \alpha_1}{|\Omega|} \| r \|_{L^\infty(\Omega)} \| \tilde{c}_a \|_{F^1(\Omega)} + C_{pol} \| f_n \|_{L^2(\Omega)} \right],
\]
where $\Theta_2$ is given in (12). Now, replacing the estimate (32) (31) and (31) by (16), we deduce the existence of $\Gamma_0$ defined in (13) such that $\| \tilde{n}_a \|_{F^1(\Omega)} \leq \Gamma_0$. We notice that the second and third relation in (16) imply that $\Theta > 1$, $i = 1, 2$, and $\alpha > \chi \beta \alpha_1 \| r \|_{L^\infty(\Omega)} C_{pol} \Theta_1 \Theta_2$, respectively, that is, $\Gamma > 0$ under (16). Moreover, from (26) and (31), we deduce the estimates given in (25) for $\| u_n \|_V$ and $\| c_n \|_{F^1(\Omega)}$, concluding the proof of the Proposition.

\subsection{Proof of Theorem 1.}
To prove the existence, we can apply the Gosszé theorem [9, 14]. Let us first define the mapping $G : V \times \tilde{H}^1(\Omega) \times \tilde{H}^1(\Omega) \to (V \times \tilde{H}^1(\Omega) \times \tilde{H}^1(\Omega))^\ast$ by the relation
\[
\langle G(u, n, c), (v, \phi, \psi) \rangle = \lambda_1 \langle S_c, u \rangle + b_0(u, u, v) - \gamma S_c(n, v) - (F, v) + \lambda_2 \left\{ a(n, \phi) + b(u, n, \phi) \right\} \\
- \chi \left( \left( n + \frac{\alpha_1}{|\Omega|} \right) r \left( c + \frac{\alpha_2}{|\Omega|} \right) \nabla c, \nabla \phi \right) - (f, \phi) + \lambda_3 \delta a(c, \phi) + b(u, c, \phi) + \beta \left( r \left( c + \frac{\alpha_2}{|\Omega|} \right) \left( n + \frac{\alpha_1}{|\Omega|} \right), \phi \right) - \delta \int_{\partial\Omega} \nabla c \cdot \nabla \phi - (f, \phi) \right\},
\]
with $\langle \cdot, \cdot \rangle$ denoting the duality pairing between $V \times \tilde{H}^1(\Omega)$ and $(V \times \tilde{H}^1(\Omega) \times \tilde{H}^1(\Omega))^\ast$, and $\lambda_1, \lambda_2, \lambda_3$ are positive fixed constants. From (10), (12), and (29), we then have that
\[
\langle G(u, n, c), (u, n, c) \rangle \geq \left\{ \lambda_1 S_c \| u \|^2_V \\
- \lambda_1 \gamma S_c \left( C_{pol} \right)^2 \| n \|_{F^1(\Omega)} \| u \|_V + \frac{\lambda_2}{3 \Theta_1} \| r \|_{F^1(\Omega)} \right\} \\
+ \frac{\lambda_1 \gamma S_c \left( C_{pol} \right)^2}{3 \Theta_1} \| n \|_{F^1(\Omega)} \| u \|_V + \frac{\lambda_2}{2 \Theta_1} \| r \|_{F^1(\Omega)} \right\} \\
+ \lambda_3 \delta \left( C_{pol} \right)^2 \| r \|_{L^\infty(\Omega)} \| c \|_{F^1(\Omega)} \| n \|_{F^1(\Omega)} \right\} \\
+ \lambda_3 \delta \left( C_{pol} \right)^2 \| r \|_{L^\infty(\Omega)} \| c \|_{F^1(\Omega)} \| n \|_{F^1(\Omega)} \right\} \\
+ \frac{\lambda_2}{2 \Theta_1} \| r \|_{F^1(\Omega)} \| n \|_{F^1(\Omega)} + \lambda_3 \| f_n \|_{L^2(\Omega)} \| u \|_V \right\} \\
:= Y_1 + Y_2 - Y_3 + Y_4,
\]
Now, selecting $\lambda_1, \lambda_2, \lambda_3$ and $r$ such that
\[
\lambda_1 < \frac{4 \lambda_2}{3 \Theta_1 \gamma^2 S_c \left( C_{pol} \right)^2},
\]
\[
\lambda_2 < \frac{6 \lambda_3 |\Omega|^2 \lambda_3}{\Theta_1 \Theta_2 \left( \chi \alpha_1 \| r \|_{L^\infty(\Omega)} \right)^2},
\]
\begin{equation}
\lambda_3 < \frac{48 \lambda_2}{6 \Theta_1 \Theta_2 \left( \beta \left( C_{\text{pol}} \right)^2 \| r \|_{L^{\infty}(\Omega)} \right)^2}
\end{equation}

\begin{equation}
r < \frac{Y_1 + Y_2}{C_{\text{pol}} \left( \lambda_1 \| F \|_{L^2(\Omega)} + \lambda_2 \| f_n \|_{L^2(\Omega)} + \lambda_3 \| f_c \|_{L^2(\Omega)} \right)}.
\end{equation}

we can prove that $\langle G(\mathbf{u}, n, c), (\mathbf{u}, n, c) \rangle$ is positive for all $(\mathbf{u}, n, c) \in \mathbf{V} \times H^1(\Omega) \times H^1(\Omega)$ such that $\| (\mathbf{u}, n, c) \|_{\mathbf{V} \times H^1(\Omega) \times H^1(\Omega)} = r$. Moreover, we notice that it is straightforward to deduce that $G$ is continuous between the weak topologies of $\mathbf{V} \times H^1(\Omega) \times H^1(\Omega)$ and $(\mathbf{V} \times H^1(\Omega) \times H^1(\Omega))'$. Then, we deduce that $\langle G(\mathbf{u}, n, c), (\mathbf{u}, n, c) \rangle = 0$, concluding the proof of existence.

To prove the uniqueness we consider that there are two solutions $(\mathbf{u}^i, n^i, c^i), i = 1, 2$, satisfying (21)–(23). Then, subtracting the test functions $(\mathbf{v}, \phi, \psi) = (\mathbf{u}^i - \mathbf{u}^2, n^1 - n^2, c^1 - c^2)$, using (10), (16), (17), and applying Proposition 2, we get

\begin{equation}
\| \mathbf{u}^1 - \mathbf{u}^2 \|_\mathbf{V} \leq \Gamma_1 \| n^1 - n^2 \|_{H^1(\Omega)},
\end{equation}

\begin{equation}
\| n^1 - n^2 \|_{H^2(\Omega)} \leq \Gamma_2 \left[ C_1 \| \mathbf{u}^1 - \mathbf{u}^2 \|_\mathbf{V} \right] \left[ \| \mathbf{v} \|_{L^2(\Omega)} \| \mathbf{V} \|_{L^2(\Omega)} \right]
+ \| \mathbf{r} \|_{L^2(\Omega)} \left[ \| \mathbf{e} \|_{L^2(\Omega)} + \mathbf{C} \right],
\end{equation}

\begin{equation}
\| c^1 - c^2 \|_{H^2(\Omega)} \leq \Gamma_3 \left[ \| \mathbf{u}^1 - \mathbf{u}^2 \|_\mathbf{V} \right] \left[ \| \mathbf{v} \|_{L^2(\Omega)} \| \mathbf{V} \|_{L^2(\Omega)} \right]
+ \left( C_1 \right) \left[ \| \mathbf{r} \|_{L^2(\Omega)} \| \mathbf{V} \|_{L^2(\Omega)} \right],
\end{equation}

with $\Gamma_i$ being defined in (13)–(15). From (38), Proposition 2, and the first inequality in (18), we have that

\begin{equation}
\| c^1 - c^2 \|_{H^2(\Omega)} \leq \frac{C_1 \Gamma_3 \Theta_2 C_{\text{pol}}}{\delta (1 - \left( C_1 \right)^2) \| \mathbf{r} \|_{L^2(\Omega)} \| \mathbf{V} \|_{L^2(\Omega)} \| \mathbf{V} \|_{L^2(\Omega)} \| \mathbf{V} \|_{L^2(\Omega)} \} \left[ \beta C_{\text{pol}} \| r \|_{L^{\infty}(\Omega)} \right],
\end{equation}

Then, replacing (39) in (37), using Proposition 2 to estimate $\| n^1 \|_{H^2(\Omega)}$, we obtain the bound $\| n^1 - n^2 \|_{H^2(\Omega)} \leq \Pi (\Gamma_1)^{-1} \| \mathbf{u}^1 - \mathbf{u}^2 \|_\mathbf{V}$ with $\Pi$ being defined in (18). Now, using this estimate in (36), we get that $\| \mathbf{u}^1 - \mathbf{u}^2 \|_\mathbf{V} \leq \Pi \| \mathbf{u}^1 - \mathbf{u}^2 \|_\mathbf{V}$. Thus using the fact that $\Pi \leq 1$ we deduce that $\mathbf{u}^1 = \mathbf{u}^2$ on $\mathbf{V}$, which also implies that $n^1 = n^2$ and $c^1 = c^2$ on $H^1(\Omega)$, concluding the uniqueness proof.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References
