Research Article

Some Seminormed Difference Sequence Spaces over n -Normed Spaces Defined by a Musielak-Orlicz Function of Order (α, β)

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We first define the notion of lacunary statistical convergence of order (α, β) , and taking this notion into consideration, we introduce some seminormed difference sequence spaces over n -normed spaces with the help of Musielak-Orlicz function $\mathcal{M} = (M_k)$ of order (α, β) . We also examine some topological properties and prove inclusion relations between the resulting sequence spaces.

1. Introduction and Preliminaries

Gähler [1] extended the usual notion of normed spaces into 2-normed spaces, while the notion was again extended to n -normed spaces by Misiak [2]. Assume that X is a linear space over the field $\mathbb{K}$ of real or complex numbers of dimension $d \geq n \geq 2$, $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of natural numbers). A real valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the conditions,

- (N1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent in X ,
- (N2) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation,
- (N3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{K}$,
- (N4) $\|x_1 + x'_1, x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x'_1, x_2, \dots, x_n\|$,

is called a n -norm on X , and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called a n -normed space over $\mathbb{K}$.

A sequence $(x_k)_{k \in \mathbb{N}}$ in a n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be

- (i) convergent to $\xi \in X$ if $\lim_{k \rightarrow \infty} \|x_k - \xi, z_1, \dots, z_{n-1}\| = 0$ holds for every $z_1, \dots, z_{n-1} \in X$,
- (ii) Cauchy if $\lim_{k, i \rightarrow \infty} \|x_k - x_i, z_1, \dots, z_{n-1}\| = 0$ holds for every $z_1, \dots, z_{n-1} \in X$.

An n -Banach space $(X, \|\cdot, \dots, \cdot\|)$ is a complete n -normed space, where completeness means that, for every (x_k) in X with $\|x_k - x_i, z_1, \dots, z_{n-1}\| \rightarrow 0$ ($k, i \rightarrow \infty$), there exists $\xi \in X$ such that $\|x_k - \xi, z_1, \dots, z_{n-1}\| = 0$ ($k \rightarrow \infty$).

Kizmaz [3] was the first who introduced the idea of difference sequence spaces and studied $Z(\Delta) = \{x = (x_k) \in w : \Delta x \in Z\}$ ($Z = l_\infty, c, c_0$), where $\Delta x = x_k - x_{k+1}$ for all $k \in \mathbb{N}$, w is the space of all complex or real sequences, and the standard notations l_∞ , c , and c_0 denote bounded, convergent, and null sequences, respectively. Et and Çolak [4] presented a generalization of these difference sequence spaces and introduced the space $Z(\Delta^n)$; in this case, $\Delta^n x$ is given by $\Delta^n x = \Delta(\Delta^{n-1} x) = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$ for $n \geq 2$. Another generation is given by Tripathy and Esi [5] by considering Δ_m instead as Δ in Kizmaz spaces and is defined by $\Delta_m x = x_k - x_{k+m}$. In view of Δ^n and Δ_m , Tripathy et al. [6] introduced the difference sequence space as follows:

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}, \quad (1)$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta^0 x_k = x_k$, which is equivalent to the following binomial representation:

$$\Delta_m^n x_k = \sum_{v=0}^n (-1)^v \binom{n}{v} x_{k+mv}. \quad (2)$$

We remark that if we take $m = 1$, then difference sequence space $Z(\Delta_m^n)$ is reduced to $Z(\Delta^n)$ and for the choice of $n = m = 1$, we obtain the difference sequence space $Z(\Delta)$. For recent work related to various kinds of difference sequence spaces, we refer to [7–13] and references therein.

If X is a linear space and $g : X \rightarrow \mathbb{R}$ such that (i) $x = 0 \Rightarrow g(x) = 0$, (ii) $g(x + y) \leq g(x) + g(y)$, (iii) $g(-x) = g(x)$, and (iv) if $t_k \rightarrow t$ ($k \rightarrow \infty$) and $x_k \rightarrow x$ ($k \rightarrow \infty$) in the sense that $g(x_k - x) \rightarrow 0$ ($k \rightarrow \infty$) for scalars t_k, t and the vectors $x_k, x \in X$, then $t_k x_k \rightarrow tx$ ($k \rightarrow \infty$) in the sense that $g(t_k x_k - tx) \rightarrow 0$ ($k \rightarrow \infty$); then g is said to be a paranorm on X and the pair (X, g) is called a paranormed space.

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ as $x > 0$ and $M(x) \rightarrow \infty$ ($x \rightarrow \infty$). Clearly, if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda \in (0, 1)$. An Orlicz function M is said to satisfy Δ_2 -condition for all values of u , if there exists a constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of $L > 1$ (see Krasnoselskii and Rutitsky [14] and, more recent, Giannetti et al. [15]). Using the idea of Orlicz function, Lindenstrauss and Tzafriri [16] constructed the sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}, \quad (3)$$

which is called Orlicz sequence space and showed that ℓ_M is a Banach space with the following norm:

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}. \quad (4)$$

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \leq p < \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be a Musielak-Orlicz function [17]. A sequence $\mathcal{N} = (N_k)$ is defined by $N_k(v) = \sup\{|v|u - (M_k) : u \geq 0\}$ ($k = 1, 2, \dots$) which is said to be complementary function of $\mathcal{M}$. For a given $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined by

$$\begin{aligned} t_{\mathcal{M}} &= \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\}, \\ h_{\mathcal{M}} &= \{x \in w : I_{\mathcal{M}}(cx) < \infty \forall c > 0\}, \end{aligned} \quad (5)$$

where $I_{\mathcal{M}}$ denotes the convex modular and is defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} (M_k)(x_k) \quad (x = (x_k) \in t_{\mathcal{M}}). \quad (6)$$

It is noted that $t_{\mathcal{M}}$ can be considered equipped with the Luxemburg norm or equipped with the Orlicz norm, where Luxemburg and Orlicz norms are given by

$$\begin{aligned} \|x\| &= \inf \left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\}, \\ \|x\|^0 &= \inf \left\{ \frac{1}{k} (1 + I_{\mathcal{M}}(kx)) : k > 0 \right\}, \end{aligned} \quad (7)$$

respectively.

A sequence $x = (x_k) \in l_{\infty}$ is said to be almost convergent if all of its Banach limits coincide. The space $\widehat{c}$ of almost convergent sequences was defined by Lorentz [18] (also see [19]) as follows:

$$\widehat{c} = \left\{ x = (x_k) \in l_{\infty} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_{k+s} \text{ exists, uniformly in } s \right\}. \quad (8)$$

Recall that x is strongly almost convergent [20] to some number ξ if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |x_{k+s} - \xi| = 0 \quad \text{uniformly in } s. \quad (9)$$

The idea of statistical convergence first appeared, under the name of almost convergence, in the first edition of Zygmund [21]. Later, this idea was introduced by Fast [22] and Steinhaus [23], independently, and some of its basic properties were studied by Schoenberg [24], Šalát [25], and Fridy [26]. Some useful results, applications, and various developments on this topic have been presented by many authors; we refer to [27–35]. Çolak [36] extended this notion with the help of α -density (for $\alpha = 1$, α -density reduced to natural density) and called it statistical convergence of order α . In the recent past, Şenül [37] presented an interesting generalization of this notion by taking into account (α, β) instead of α and defined statistical convergence of order (α, β) as follows.

The sequence $x = (x_k)$ is said to be statistically convergent of order (α, β) , briefly S_{α}^{β} -convergence, to L if for each $\epsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha}} |\{k \leq n : |x_k - L| \geq \epsilon\}|^{\beta} = 0, \quad (10)$$

where $|\{k \leq n : k \in E\}|^{\beta}$ denotes the β th power of number of elements of E not exceeding n , and we write $S_{\alpha}^{\beta}\text{-}\lim x_k = L$.

2. Construction of Difference Sequence Spaces

Before defining some difference sequence spaces, let us first introduce the notion of lacunary statistical convergence of order (α, β) . Recall that the standard notation $\theta = (k_r)$ denotes the lacunary sequence, where (k_r) is a sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$, and $h_r := k_r - k_{r-1} \rightarrow \infty$ ($r \rightarrow \infty$), throughout the article, the intervals determined by the lacunary sequence θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $q_r = k_r/k_{r-1}$ ($r \neq 1$) by q_r (see [38]).

Suppose that $E \subset \mathbb{N}$ and $0 < \alpha \leq \beta \leq 1$. We define the $\theta(\alpha, \beta)$ -density of E by

$$\delta_{\theta}^{\alpha, \beta}(E) = \lim_n \frac{1}{h_r^{\alpha}} |\{k \in (k_{r-1}, k_r] : k \in E\}|^{\beta}, \quad (11)$$

in case this limit exists, where h_r^{α} denotes the α th power of h_r and $|\{k \in I_r : k \in E\}|^{\beta}$ denotes the β th power of number of elements of E in I_r .

Suppose that $0 < \alpha \leq \beta \leq 1$ and let $\theta = (k_r)$ be a lacunary sequence. Then, we say that the sequence $x = (x_k)$ is lacunary statistically convergent of order (α, β) (shortly, $S_\alpha^\beta(\theta)$ -convergence) to the number L and we write $S_\alpha^\beta(\theta)\text{-}\lim x_k = L$ if for each $\epsilon > 0$, the set $A(\epsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has $\theta(\alpha, \beta)$ -density zero; that is,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} |\{k \in (k_{r-1}, k_r] : |x_k - L| \geq \epsilon\}|^\beta = 0. \quad (12)$$

Example 1. Consider a sequence $x = (x_k)$ defined by $x_k = k$ if $k_r - \lfloor \sqrt{h_r^\alpha} \rfloor + 1 \leq k \leq k_r$ ($r \in \mathbb{N}$) and $x_k = 0$ otherwise. Also, let $\beta = 1$. Then for any $\epsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} |\{k \in (k_{r-1}, k_r] : |x_k - 0| \geq \epsilon\}| \leq \frac{\lfloor \sqrt{h_r^\alpha} \rfloor}{h_r^\alpha} \rightarrow 0 \quad (r \rightarrow \infty). \quad (13)$$

We see that $x = (x_k)$ is $S_\alpha^\beta(\theta)$ -convergent to zero but not convergent.

Remark 2. For $\theta = (2^r)$, the notion of $S_\alpha^\beta(\theta)$ -convergence coincides with S_α^β -convergence in [37]. Also, if we take $\theta = (2^r)$ and $\beta = 1$, then $S_\alpha^\beta(\theta)$ -convergence reduces to the notion of statistical convergence of order α due to Çolak [36]. Further, the choice of $\theta = (2^r)$ and $\alpha = \beta = 1$ in the definition of lacunary statistically convergent of order (α, β) gives us the notion of statistical convergence due to Fast [22]. Moreover, if $\alpha = \beta = 1$ in the definition of $S_\alpha^\beta(\theta)$ -convergence, then we obtain lacunary statistical convergence introduced by Fridy and Orhan [39].

We denote by $w(n - X)$ the space of all sequences defined over $(X, \|\cdot, \dots, \cdot\|)$. Let X be a seminormed space, seminormed by $q = (q_k)$, and let $p = (p_k)$ be any bounded sequence of positive real numbers. We are now ready to define the following sequence spaces:

$$\begin{aligned} c_\theta^\beta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta &= \left\{ x = (x_k) \in w(n - X) : \lim_{r \rightarrow \infty} \right. \\ &\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right] \right]^\beta \\ &\quad \left. = 0 \text{ uniformly in } s, \text{ for some } L, \rho > 0 \right\}, \\ c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta &= \left\{ x = (x_k) \in w(n - X) : \lim_{r \rightarrow \infty} \right. \\ &\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right] \right]^\beta \\ &\quad \left. = 0 \text{ uniformly in } s, \text{ for some } \rho > 0 \right\}, \end{aligned}$$

$$\begin{aligned} c_\infty^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta &= \left\{ x = (x_k) \in w(n - X) : \sup_{r,s} \right. \\ &\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k=1}^n \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right] \right]^\beta \\ &\quad \left. < \infty \text{ uniformly in } s, \text{ for some } \rho > 0 \right\}. \end{aligned} \quad (14)$$

Note that if we consider $\mathcal{M}(x) = x$; then the above difference sequence spaces reduce to the following spaces:

$$\begin{aligned} c_\theta^\beta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta &= \left\{ x = (x_k) \in w(n - X) : \lim_{r \rightarrow \infty} \right. \\ &\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \\ &\quad \left. = 0 \text{ uniformly in } s, \text{ for some } L, \rho > 0 \right\}, \\ c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta &= \left\{ x = (x_k) \in w(n - X) : \lim_{r \rightarrow \infty} \right. \\ &\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \\ &\quad \left. = 0 \text{ uniformly in } s, \text{ for some } \rho > 0 \right\}, \\ c_\infty^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta &= \left\{ x = (x_k) \in w(n - X) : \sup_{r,s} \right. \\ &\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k=1}^n \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \\ &\quad \left. < \infty, \text{ uniformly in } s, \text{ for some } \rho > 0 \right\}. \end{aligned} \quad (15)$$

It is to further notice that if we take $p_k = 1$ for all $k \in \mathbb{N}$, then the spaces $c_\theta^\beta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$, $c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$, and $c_\infty^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$ reduce to the following spaces:

$$\begin{aligned} c_\theta^\beta[\mathcal{M}, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta &= \left\{ x = (x_k) \in w(n - X) : \lim_{r \rightarrow \infty} \right. \\ &\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right] \right]^\beta \\ &\quad \left. = 0, \text{ uniformly in } s, \text{ for some } L, \rho > 0 \right\}, \end{aligned}$$

$$\begin{aligned}
c_0^\theta[\mathcal{M}, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta &= \left\{ x = (x_k) \in w(n-X) : \lim_{r \rightarrow \infty} \right. \\
&\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^\beta \right. \\
&\quad \left. = 0, \text{ uniformly in } s, \text{ for some } \rho > 0 \right\}, \\
c_\infty^\theta[\mathcal{M}, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta &= \left\{ x = (x_k) \in w(n-X) : \sup_{r,s} \right. \\
&\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k=1}^n \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^\beta \right. \\
&\quad \left. < \infty, \text{ uniformly in } s, \text{ for some } \rho > 0 \right\}.
\end{aligned} \tag{16}$$

Remark 3. If $\alpha = \beta = 1$, $n = 1$, and $q_k(x) = x \forall k$ in our difference sequence spaces defined above, then these spaces reduce to the difference sequence spaces defined by Raj et al. [40]. Moreover, if we take $\alpha = \beta = 1$ in our difference sequence spaces and take $u_k = 1$ for all k in the difference sequence spaces defined by Raj and Sharma [41], then we observe that difference sequence spaces of both the manuscripts coincide.

The following inequality will be used to prove some of our results in the next section. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$, then

$$\begin{aligned}
&\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m (\alpha x_{k+s} + \beta y_{k+s})}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \\
&\leq D \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \frac{1}{2^{p_k}} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m (x_{k+s})}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \\
&\quad + D \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \frac{1}{2^{p_k}} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m (y_{k+s})}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \\
&\leq D \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m (x_{k+s})}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \\
&\quad + D \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m (y_{k+s})}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \rightarrow 0 \quad \text{as } r \rightarrow \infty, \text{ uniformly in } s.
\end{aligned} \tag{19}$$

$$|a_k + b_k|^{p_k} \leq D \{ |a_k|^{p_k} + |b_k|^{p_k} \}, \tag{17}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

3. Main Results

In this section, we are going to study some topological properties and to obtain some inclusion relations between the difference sequence spaces defined in the previous section.

Theorem 4. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and let $p = (p_k)$ be a bounded sequence of positive real numbers. Then the spaces $c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$, $c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$ and $c_\infty^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$ are linear over the field of complex numbers $\mathbb{C}$.

Proof. Let $x = (x_k), y = (y_k) \in c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\begin{aligned}
&\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \\
&= 0 \quad \text{uniformly in } s, \\
&\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \\
&= 0 \quad \text{uniformly in } s.
\end{aligned} \tag{18}$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is nondecreasing convex function, by (17), we obtain

Thus, we have $(\alpha x + \beta y) \in c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Hence, $c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$ is a linear space. In a similar way, one can prove that $c^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$ and $c_\infty^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$ are linear spaces. $\square$

Theorem 5. For any Musielak-Orlicz function $\mathcal{M} = (M_k)$ and a bounded sequence of positive real numbers $p = (p_k)$, the space $c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$ is a topological linear space paranormed by

$$g(x) = \inf \left\{ \rho^{p_r/K} : \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right]^{1/K} \leq 1, r, s \in \mathbb{N} \right\}, \quad (20)$$

where $K = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_k) \in c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Since $M_k(0) = 0$, we obtain $g(0) = 0$. Again, if $g(x) = 0$, then we have

$$\inf \left\{ \rho^{p_r/K} : \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right]^{1/K} \leq 1, r, s \in \mathbb{N} \right\} = 0. \quad (21)$$

This implies that, for a given $\epsilon > 0$, there exists some ρ_ϵ ($0 < \rho_\epsilon < \epsilon$) such that

$$\begin{aligned} & \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m (x_{k+s} + y_{k+s})}{\rho} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \right]^\beta \right]^{1/K} \\ & \leq \left(\left[\sum_{k \in I_r} \left[\frac{\rho_1}{\rho_1 + \rho_2} M_k \left(q_k \left(\left\| \frac{\Delta_n^m (x_{k+s})}{\rho_1} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \right]^\beta + \frac{\rho_2}{\rho_1 + \rho_2} M_k \left(q_k \left(\left\| \frac{\Delta_n^m (y_{k+s})}{\rho_2} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \right]^\beta \right]^{1/K} \\ & \leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m (x_{k+s})}{\rho_1} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \right]^\beta \right]^{1/K} \\ & \quad + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m (y_{k+s})}{\rho_2} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \right]^\beta \right]^{1/K} \leq 1. \end{aligned} \quad (26)$$

$$\left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho_\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right]^{1/K} \leq 1. \quad (22)$$

Thus,

$$\begin{aligned} & \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\epsilon} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \right]^\beta \right]^{1/K} \\ & \leq \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho_\epsilon}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right]^{1/K} \leq 1, \end{aligned} \quad (23)$$

for each r and s . Assume that $x_k \neq 0$ for each $k \in \mathbb{N}$. This implies that $\Delta_n^m x_{k+s} \neq 0$ for each $k, s \in \mathbb{N}$. Let $\epsilon \rightarrow 0$; then $q_k(\|\Delta_n^m x_{k+s}/\epsilon, z_1, \dots, z_{n-1}\|) \rightarrow \infty$. It follows that

$$\left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\epsilon} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \right]^\beta \right]^{1/K} \rightarrow \infty, \quad (24)$$

which is a contradiction. Consequently, we have $\Delta_n^m x_{k+s} = 0$ for each k, s and so $x_k = 0$ for each $k \in \mathbb{N}$. Suppose that $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\begin{aligned} & \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho_1} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \right]^\beta \right]^{1/K} \\ & \leq 1, \\ & \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho_2} \right\|, z_1, \dots, z_{n-1} \right) \right)^{p_k} \right]^\beta \right]^{1/K} \\ & \leq 1, \end{aligned} \quad (25)$$

for each r and s . Assume that $\rho = \rho_1 + \rho_2$. Then, by Minkowski's inequality, we have

Since ρ 's are nonnegative, one obtains

$$\begin{aligned} g(x+y) &= \inf \left\{ \rho^{p_r/K} : \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m(x_{k+s} + y_{k+s})}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right)^{1/K} \leq 1, r, s \in \mathbb{N} \right\} \\ &\leq \inf \left\{ \rho_1^{p_r/K} : \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m(x_{k+s})}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right)^{1/K} \leq 1, r, s \in \mathbb{N} \right\} \\ &\quad + \inf \left\{ \rho_2^{p_r/K} : \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m(y_{k+s})}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right)^{1/K} \leq 1, r, s \in \mathbb{N} \right\}. \end{aligned} \quad (27)$$

Consequently, $g(x+y) \leq g(x) + g(y)$. It is only left to show that the scalar multiplication is continuous. Let us consider a complex number λ . Then, by definition, we obtain

$$\begin{aligned} g(\lambda x) &= \inf \left\{ \rho^{p_r/K} : \right. \\ &\quad \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m \lambda x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right)^{1/K} \\ &\quad \left. \leq 1, r, s \in \mathbb{N} \right\}. \end{aligned} \quad (28)$$

Then

$$\begin{aligned} g(\lambda x) &= \inf \left\{ (|\lambda|t)^{p_r/K} : \right. \\ &\quad \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{t}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right)^{1/K} \\ &\quad \left. \leq 1, r, s \in \mathbb{N} \right\}, \end{aligned} \quad (29)$$

where $t = \rho/|\lambda|$. Since $|\lambda|^{p_r} \leq \max(1, |\lambda|^{\sup p_r})$, we have

$$\begin{aligned} g(\lambda x) &\leq \max(1, |\lambda|^{\sup p_r}) \inf \left\{ t^{p_r/K} : \right. \\ &\quad \left(\frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{t}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right)^{1/K} \\ &\quad \left. \leq 1, r, s \in \mathbb{N} \right\}. \end{aligned} \quad (30)$$

So, the fact that scalar multiplication is continuous follows from the above inequality. $\square$

Theorem 6. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then $c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta \subset c_\infty^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$.

Proof. Let $x = (x_k) \in c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$. There exists some positive ρ_1 such that

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right] \\ = 0 \quad \text{uniformly in } s. \end{aligned} \quad (31)$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is nondecreasing and convex, by using (17), we obtain

$$\begin{aligned} &\sup_{r,s} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right] \\ &\leq D \sup_{r,s} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[\frac{1}{2^{p_k}} \right. \right. \\ &\quad \cdot M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \left. \left. \right]^\beta \right] + D \sup_{r,s} \\ &\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[\frac{1}{2^{p_k}} M_k \left(q_k \left(\left\| \frac{L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right] \\ &\leq D \sup_{r,s} \\ &\quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right] \\ &\quad + D \sup_{r,s} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{L}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right], \end{aligned} \quad (32)$$

which gives

$$\sup_{r,s} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right] \right]^\beta < \infty. \quad (33)$$

Hence, $x = (x_k) \in c_\infty^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$. $\square$

Theorem 7. Let $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ and let $\mathcal{M} = (M_k)$ and $\mathcal{M}' = (M'_k)$ be two Musielak-Orlicz functions satisfying Δ_2 -condition. Then

- (i) $c_0^\theta[\mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta \subset c_0^\theta[\mathcal{M} \circ \mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$,
- (ii) $c^\theta[\mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta \subset c^\theta[\mathcal{M} \circ \mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$,
- (iii) $c_\infty^\theta[\mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta \subset c_\infty^\theta[\mathcal{M} \circ \mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$.

Proof. Let $x = (x_k) \in c^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta$. Then, we have

$$\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M'_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right] \right]^\beta = 0, \quad (34)$$

uniformly in s and for some $L \in \mathbb{C}$. Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Let

$$y_{k,s} = M'_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \quad (35)$$

$\forall k, s \in \mathbb{N}$.

We can write

$$\begin{aligned} & \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} [M_k(y_{k,s})]^{p_k} \right]^\beta \\ &= \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} \right]^\beta \\ & \quad + \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r, y_{k,s} > \delta} [M_k(y_{k,s})]^{p_k} \right]^\beta. \end{aligned} \quad (36)$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2 -condition, we have

$$\begin{aligned} & \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} \right]^\beta \\ & \leq [M_k(1)]^H \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r, y_{k,s} \leq \delta} [M_k(y_{k,s})]^{p_k} \right]^\beta. \end{aligned} \quad (37)$$

For $y_{k,s} > \delta$, we obtain

$$y_{k,s} < \frac{y_{k,s}}{\delta} < 1 + \frac{y_{k,s}}{\delta}. \quad (38)$$

It follows from the fact that $\mathcal{M} = (M_k)$ is nondecreasing and convex that

$$\begin{aligned} M_k(y_{k,s}) &< M_k \left(1 + \frac{y_{k,s}}{\delta} \right) \\ &< \frac{1}{2} M_k(2) + \frac{1}{2} M_k \left(\frac{2y_{k,s}}{\delta} \right). \end{aligned} \quad (39)$$

Since (M_k) satisfies Δ_2 -condition, we can write

$$\begin{aligned} M_k(y_{k,s}) &< \frac{1}{2} T \frac{y_{k,s}}{\delta} M_k(2) + \frac{1}{2} T \frac{y_{k,s}}{\delta} M_k(2) \\ &= T \frac{y_{k,s}}{\delta} M_k(2). \end{aligned} \quad (40)$$

Hence,

$$\begin{aligned} & \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r, y_{k,s} > \delta} [M_k(y_{k,s})]^{p_k} \right]^\beta \\ & \leq \max \left(1, \left(\frac{T M_k(2)}{\delta} \right)^H \right) \\ & \quad \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r, y_{k,s} > \delta} [(y_{k,s})]^{p_k} \right]^\beta. \end{aligned} \quad (41)$$

Equation (37) together with (41) gives

$$x = (x_k) \in c_0^\theta[\mathcal{M} \circ \mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta. \quad (42)$$

This completes the proof of (i). In a similar way, using Δ_2 -condition for (M_k) , we can prove that

$$\begin{aligned} & c^\theta[\mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta \\ & \subset c^\theta[\mathcal{M} \circ \mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta, \\ & c_\infty^\theta[\mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta \\ & \subset c_\infty^\theta[\mathcal{M} \circ \mathcal{M}', p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta. \end{aligned} \quad (43)$$

$\square$

Corollary 8. Let $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ and let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function satisfying Δ_2 -condition. Then

$$\begin{aligned} & c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta \\ & \subset c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta, \\ & c_\infty^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta \\ & \subset c_\infty^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|]_\alpha^\beta. \end{aligned} \quad (44)$$

Proof. The proof follows by taking $\mathcal{M}'(x) = x$ in the above Theorem 7. $\square$

Theorem 9. If $\mathcal{M} = (M_k)$ be the Musielak-Orlicz function, then the following statements are equivalent:

- (i) $c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \subset c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$,
- (ii) $c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \subset c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$,
- (iii) $\sup_r (1/h_r^\alpha) [\sum_{k \in I_r} [M_k(t/\rho)]^{p_k}]^\beta < \infty$ ($t, \rho > 0$).

Proof.

(i) $\Rightarrow$ (ii). It is obvious in view of the following fact that

$$c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \subset c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta. \quad (45)$$

(ii) $\Rightarrow$ (iii). Let $c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \subset c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Assume that (iii) does not hold. Then, for some $t, \rho > 0$, we obtain

$$\sup_r \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(\frac{t}{\rho} \right) \right]^{p_k} \right]^\beta = \infty, \quad (46)$$

and so one can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(j)}^\alpha} \left[\sum_{k \in I_{r(j)}} \left[M_k \left(\frac{j^{-1}}{\rho} \right) \right]^{p_k} \right]^\beta > j, \quad j = 1, 2. \quad (47)$$

Let us define the sequence $x = (x_k)$ by

$$\Delta_n^m x_{k+s} = \begin{cases} j^{-1}, & \text{if } k \in I_{r(j)}, \\ 0, & \text{if } k \notin I_{r(j)}, \end{cases} \quad \forall s \in \mathbb{N}. \quad (48)$$

Then $x = (x_k) \in c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$ but, by (47), $x = (x_k) \notin c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$ which contradicts (ii). Hence, the condition (iii) must hold.

(iii) $\Rightarrow$ (i). Suppose that (iii) holds and $x = (x_k) \in c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Also assume that $x = (x_k) \notin c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Then

$$\sup_{r,s} \frac{1}{h_{r,s}^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right]^\beta = \infty. \quad (49)$$

Let $t = q_k(\|\Delta_n^m x_{k+s}, z_1, \dots, z_{n-1}\|)$ for each k and fixed s . It follows from (49) that

$$\sup_r \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(\frac{t}{\rho} \right) \right]^{p_k} \right]^\beta = \infty, \quad (50)$$

which contradicts (iii). Hence, the condition (i) holds. $\square$

Theorem 10. Let $1 \leq p_k \leq \sup p_k < \infty$ and let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:

- (i) $c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \subset c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$,
- (ii) $c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \subset c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$,
- (iii) $\inf_r (1/h_r^\alpha) [\sum_{k \in I_r} [M_k(t/\rho)]^{p_k}]^\beta > 0$ ($t, \rho > 0$).

Proof. It is trivial to obtain that (i) implies (ii). We are now proving that (ii) implies (iii). Let the condition (ii) hold. Assume that (iii) does not hold. Therefore,

$$\inf_r \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(\frac{t}{\rho} \right) \right]^{p_k} \right]^\beta = 0 \quad (t, \rho > 0) \quad (51)$$

and so we can find a subinterval $I_{r(j)}$ ($j = 1, 2$) of I_r such that

$$\frac{1}{h_{r(j)}^\alpha} \left[\sum_{k \in I_{r(j)}} \left[M_k \left(\frac{j}{\rho} \right) \right]^{p_k} \right]^\beta < j^{-1}. \quad (52)$$

For all $s \in \mathbb{N}$, define $\Delta_n^m x_{k+s} = j$ if $k \in I_{r(j)}$ and $\Delta_n^m x_{k+s} = 0$ if $k \notin I_{r(j)}$. It follows from (52) that $x = (x_k) \in c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$ but $x = (x_k) \notin c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$ which contradicts (ii). Thus, the condition (iii) holds. It is left to prove that (iii) implies (i). Suppose that (iii) holds and also assume that $x = (x_k) \in c_0^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Therefore,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta = 0 \quad \text{uniformly in } s, \text{ for some } \rho > 0. \quad (53)$$

Again, assume that $x = (x_k) \notin c_0^\theta[p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Then, for some number $\epsilon > 0$ and a subinterval $I_{r(j)}$ of I_r , one obtains $\|\Delta_n^m x_{k+s}, z_1, \dots, z_{n-1}\| \geq \epsilon$ for all $k \in \mathbb{N}$ and for some $s \geq s_0$. Therefore, in view of the properties of the Orlicz function, one writes

$$\begin{aligned} & \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \\ & \geq \left[M_k \left(\frac{\epsilon}{\rho} \right) \right]^{p_k}. \end{aligned} \quad (54)$$

By taking into account (53), we obtain

$$\lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(\frac{\epsilon}{\rho} \right) \right]^{p_k} \right]^\beta = 0, \quad (55)$$

which contradicts (iii). Hence, (i) must hold. $\square$

Theorem 11. Let $0 < p_k \leq r_k$ for all $k \in \mathbb{N}$ and let (r_k/p_k) be bounded. Then

$$\begin{aligned} & c^\theta[\mathcal{M}, r, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \\ & \subset c^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta. \end{aligned} \quad (56)$$

Proof. Suppose that $x \in c^\theta[\mathcal{M}, r, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Write

$$t_k = \left[\left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{r_k} \right]^\beta, \quad (57)$$

and $\mu_k = p_k/r_k$ for all $k \in \mathbb{N}$. Then $0 < \mu_k \leq 1$ for all $k \in \mathbb{N}$. Take $0 < \mu < \mu_k$ for $k \in \mathbb{N}$. Consider two sequences (u_k) and (v_k) defined by $u_k = t_k$ and $v_k = 0$ if $t_k \geq 1$ and $u_k = 0$ and $v_k = t_k$ for $t_k < 1$. Clearly

$$\begin{aligned} t_k &= u_k + v_k, \\ t_k^{\mu_k} &= u_k^{\mu_k} + v_k^{\mu_k}, \\ &\forall k \in \mathbb{N}. \end{aligned} \quad (58)$$

It follows that $u_k^{\mu_k} \leq u_k \leq t_k$ and $v_k^{\mu_k} \leq v_k^{\mu}$. Therefore,

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} t_k^{\mu_k} = \frac{1}{h_r^\alpha} \sum_{k \in I_r} (u_k^{\mu_k} + v_k^{\mu_k}) \leq \frac{1}{h_r^\alpha} \sum_{k \in I_r} t_k + \frac{1}{h_r^\alpha} \sum_{k \in I_r} v_k^{\mu}. \quad (59)$$

Now, for each k , we obtain

$$\begin{aligned} \frac{1}{h_r^\alpha} \sum_{k \in I_r} v_k^{\mu} &= \sum_{k \in I_r} \left(\frac{1}{h_r^\alpha} v_k \right)^\mu \left(\frac{1}{h_r^\alpha} \right)^{1-\mu} \\ &\leq \left(\sum_{k \in I_r} \left[\left(\frac{1}{h_r^\alpha} v_k \right)^\mu \right]^{1/\mu} \right)^\mu \\ &\cdot \left(\sum_{k \in I_r} \left[\left(\frac{1}{h_r^\alpha} \right)^{1-\mu} \right]^{1/(1-\mu)} \right)^{1-\mu} = \left(\frac{1}{h_r^\alpha} \sum_{k \in I_r} v_k \right)^\mu. \end{aligned} \quad (60)$$

By taking the above two inequalities into account, we get

$$\frac{1}{h_r^\alpha} \sum_{k \in I_r} t_k^{\mu_k} \leq \frac{1}{h_r^\alpha} \sum_{k \in I_r} t_k + \left(\frac{1}{h_r^\alpha} \sum_{k \in I_r} v_k \right)^\mu. \quad (61)$$

Hence, we conclude that $x \in c^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. $\square$

Theorem 12.

(i) If $0 < \inf p_k \leq p_k \leq 1$ for all $k \in \mathbb{N}$, then

$$\begin{aligned} c^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \\ \subset c^\theta[\mathcal{M}, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta. \end{aligned} \quad (62)$$

(ii) If $1 \leq p_k \leq \sup p_k < \infty$ for all $k \in \mathbb{N}$, then

$$\begin{aligned} c^\theta[\mathcal{M}, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \\ \subset c^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta. \end{aligned} \quad (63)$$

Proof. (i) Assume that $x \in c^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Then

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \\ = 0. \end{aligned} \quad (64)$$

Since $0 < \inf p_k \leq p_k \leq 1$, we have

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^\beta \right. \\ \left. \cdot \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \right]^\beta. \end{aligned} \quad (65)$$

Thus, we have

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^\beta \right]^\beta \\ = 0, \end{aligned} \quad (66)$$

which gives $x \in c^\theta[\mathcal{M}, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Therefore, we conclude that

$$\begin{aligned} c^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta \\ \subset c^\theta[\mathcal{M}, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta. \end{aligned} \quad (67)$$

(ii) Suppose that $p_k \geq 1$ for each k and $\sup p_k < \infty$. Let $x \in c^\theta[\mathcal{M}, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. Then, for each $\epsilon > 0$ there exists a positive integer N such that

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^{p_k} \right]^\beta \\ = 0 < 1. \end{aligned} \quad (68)$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right)^{p_k} \right]^\beta \right. \\ \left. \leq \lim_{r \rightarrow \infty} \frac{1}{h_r^\alpha} \left[\sum_{k \in I_r} \left[M_k \left(q_k \left(\left\| \frac{\Delta_n^m x_{k+s} - L}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right) \right]^\beta \right] = 0 < 1. \end{aligned} \quad (69)$$

Hence, $x \in c^\theta[\mathcal{M}, p, \Delta_n^m, q, \|\cdot, \dots, \cdot\|_\alpha]^\beta$. $\square$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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