Research Article

Essential Norm of Difference of Composition Operators from Weighted Bergman Spaces to Bloch-Type Spaces

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We compute upper and lower bounds for essential norm of difference of composition operators acting from weighted Bergman spaces to Bloch-type spaces.

1. Introduction and Preliminaries

Let \( D \) be the open unit disk in the complex plane, \( H(D) \) the space of all holomorphic functions on \( D \), and \( S(D) \) the set of all holomorphic self-maps of \( D \). For \( \varphi \in S(D) \), the composition operator \( C_\varphi \) is a linear operator defined by

\[
C_\varphi f = f \circ \varphi, \quad f \in H(D).
\]

For a general reference on composition operator we refer to [1]. To understand the topological structures of spaces of composition operators, many people have studied difference of composition operators on different spaces of holomorphic functions; see, for example, [2–26] and references therein. Recently, Zhu and Yang [26] characterized boundedness and compactness of the differences of two composition operators from weighted Bergman spaces to Bloch spaces. Motivated by these results, in this paper we compute essential norm of difference of composition operators from weighted Bergman spaces to Bloch-type spaces.

For \( 0 < \alpha < \infty \) and \( 0 < p < \infty \), the weighted Bergman space \( \mathcal{A}_p^\alpha \) is the space of all functions \( f \in H(D) \) such that

\[
\|f\|_{\mathcal{A}_p^\alpha}^p = \int_D |f(z)|^p \left(1 - |z|^2\right)^\alpha \, dA(z) < \infty,
\]

where \( dA(z) \) is the normalized area measure on \( D \). For any \( z \in D \), the following point-evaluation estimate holds:

\[
|f(z)| \leq \frac{\|f\|_{\mathcal{A}_p^\alpha}}{(1 - |z|^2)^{(2\alpha)/p}}, \quad f \in \mathcal{A}_p^\alpha.
\]

The following equivalent norm in the weighted Bergman spaces is well-known. Let \( 0 < p < \infty \), \( n \in \mathbb{N} \), and \( f \in H(D) \). Then \( f \in \mathcal{A}_p^\alpha \) if and only if, for all \( n \in \mathbb{N} \), \( f^{(n)}(z)(1 - |z|^2)^n \in L^p((1 - |z|^2)^\alpha dA(z)) \)

\[
\|f\|_{\mathcal{A}_p^\alpha} = \left( \int_D \left| f^{(n)}(z) \right|^p \left(1 - |z|^2\right)^{\alpha n} dA(z) \right)^{1/p}.
\]

Thus, if \( f \in \mathcal{A}_p^\alpha \), then \( f^{(n)} \in \mathcal{A}_{pn\alpha} \) and \( \|f^{(n)}\|_{\mathcal{A}_{pn\alpha}} \leq C \|f\|_{\mathcal{A}_p^\alpha} \). Therefore, by (3) applied to the function \( f^{(n)} \), for every \( z \) in \( D \), we have
Moreover, by Lemma 2 in [26], for any $z, w \in \mathbb{D}$, there is a constant $C > 0$ such that
\begin{equation}
\left| f^{(n)}(z) \right| \leq \frac{\left\| f^{(n)} \right\|_{\mathcal{B}_\mu} (1 - |z|^2)^{(2p + 2p - 2)/p}}{(1 - |z|^2)^{(2p + 2p - 2)/p}} \leq \frac{\left\| f \right\|_{\mathcal{B}_\mu} (1 - |z|^2)^{(2p + 2p - 2)/p}}{(1 - |z|^2)^{(2p + 2p - 2)/p}} .
\end{equation}

(5)

Theorem 1. Let $\alpha \in (1, \infty)$, $\beta \in (0, \infty)$, $\varphi, \psi \in \mathcal{S}(\mathbb{D})$ such that $\varphi \equiv \psi \equiv 1$ and let $\mu$ be a weight function such that $\mathcal{C}_\varphi, \mathcal{C}_\psi : \mathcal{L}_\mu^2 \to \mathcal{B}_\mu$ is bounded. Then
\begin{equation}
\mathcal{C}_\varphi - \mathcal{C}_\psi \leq \lim sup_{\varphi(z) \to 1} \left\| \mathcal{D}_\mu^\varphi(\varphi(z)) \right\|_{\mathcal{B}_\mu} \rho(\varphi(z), \psi(z)) \n\end{equation}
\begin{equation}
+ \lim sup_{\varphi(z) \to 1} \left\| \mathcal{D}_\mu^\psi(\varphi(z)) \right\|_{\mathcal{B}_\mu} \rho(\varphi(z), \psi(z)) \n\end{equation}
\begin{equation}
+ \lim sup_{\varphi(z) \to 1} \left\| \mathcal{D}_\mu^\psi(\varphi(z)) - \mathcal{D}_\mu^\psi(\varphi(z)) \right\|_{\mathcal{B}_\mu} .
\end{equation}

(13)

Proof. Let $(z_j)_{j \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$ and
\begin{equation}
\lim_{j \to \infty} \left\| \mathcal{D}_\mu^\varphi(\varphi(z_j)) \right\|_{\mathcal{B}_\mu} \rho(\varphi(z_j), \psi(z_j)) \n\end{equation}
\begin{equation}
= \lim sup_{\varphi(z) \to 1} \left\| \mathcal{D}_\mu^\varphi(\varphi(z)) \right\|_{\mathcal{B}_\mu} \rho(\varphi(z), \psi(z)) .
\end{equation}

(14)

For each $k \in \mathbb{N}$, let $f_{j,k,\varphi}$ be defined as
\begin{equation}
f_{j,k,\varphi}(z) = \frac{1 - |\varphi(z_j)|^2}{(1 - \varphi(z_j) z)^{k + (2\alpha + 2)/p}} .
\end{equation}

(15)

Now consider the following functions:
\begin{equation}
g_{j,\varphi}(z) = \frac{\varphi - (2 + \alpha + \beta)}{2 \alpha + 2 \beta} f_{j,1,\varphi}(z),
\end{equation}
\begin{equation}
h_{j,\varphi}(z) = \frac{\varphi - (2 + \alpha + 2 \beta)}{2 \alpha + 2 \beta} f_{j,1,\varphi}(z) - \frac{\varphi^2}{\varphi(z_j)} f_{j,1,\varphi}(z) .
\end{equation}

(16)

Then it is easy to see that $g_{j,\varphi}$ and $h_{j,\varphi}$ are norm bounded sequences in $\mathcal{L}_\mu^2$. Moreover, both sequences $(g_{j,\varphi})$ and $(h_{j,\varphi})$ converge to $0$ uniformly on compact subsets of $\mathbb{D}$. Let $K : \mathcal{L}_\mu^2 \to \mathcal{B}_\mu$ be any compact operator. Then
\begin{equation}
\left\| \mathcal{C}_\varphi - \mathcal{C}_\psi \right\|_{\mathcal{B}_\mu} \n\end{equation}
\begin{equation}
\geq C \lim sup_{j \to \infty} \left\| \left( \mathcal{C}_\varphi - \mathcal{C}_\psi - K \right) g_{j,\varphi} \right\|_{\mathcal{B}_\mu} \n\end{equation}
\begin{equation}
\geq C \lim sup_{j \to \infty} \left\| \left( \mathcal{C}_\varphi - \mathcal{C}_\psi \right) g_{j,\varphi} \right\|_{\mathcal{B}_\mu} \n\end{equation}
\begin{equation}
- \lim sup_{j \to \infty} \left\| K g_{j,\varphi} \right\|_{\mathcal{B}_\mu} \n\end{equation}
\begin{equation}
\geq C \lim sup_{j \to \infty} \left\| \left( \mathcal{C}_\varphi - \mathcal{C}_\psi - K \right) h_{j,\varphi} \right\|_{\mathcal{B}_\mu} .
\end{equation}

(17)
\[
\begin{align*}
\geq & \ C \lim_{j \to \infty} \| (C_{\varphi} - C_{\psi}) h_{j,\varphi} \|_{\mathcal{B}_{\mu}} \\
& - \lim_{j \to \infty} \| K h_{j,\varphi} \|_{\mathcal{B}_{\mu}}.
\end{align*}
\]

Again
\[
\begin{align*}
\| (C_{\varphi} - C_{\psi}) g_{j,\varphi} \|_{\mathcal{B}_{\mu}} &= \sup_{z \in D} \mu(z) \left| (C_{\varphi} - C_{\psi}) g_{j,\varphi} \right| (z) \\
& \geq \left| \left( \mu(z_j) \left( 1 - |\varphi(z_j)|^2 \right) \varphi'(z_j) \right) \right| \\
& \cdot \left( 1 - \left| \varphi(z_j) \right|^2 \right) \left( 1 - \left| \psi(z_j) \right|^2 \right) \frac{1}{(1 - \varphi(z_j) \psi(z_j))^2} \\
& \cdot \left( 1 - \left| \varphi(z_j) \right|^2 \right)^{1+(2+\alpha)/p} \\
& \cdot \left| \varphi(z_j) - \psi(z_j) \right| \\
& \cdot \left| (1 - \varphi(z_j)) \left( 1 - \varphi(z_j) \right)^{1+(2+\alpha)/p} \right| \\
& \cdot \left| 1 - \varphi(z_j) \psi(z_j) \right| \\
& \cdot \rho(\varphi(z_j), \psi(z_j)).
\end{align*}
\]

Multiplying (18) by \( \rho(\varphi(z_j), \psi(z_j)) \) and then adding it to (19), we get
\[
\begin{align*}
\| (C_{\varphi} - C_{\psi}) g_{j,\varphi} \|_{\mathcal{B}_{\mu}} + \| (C_{\varphi} - C_{\psi}) h_{j,\varphi} \|_{\mathcal{B}_{\mu}} \\
& \geq \| (C_{\varphi} - C_{\psi}) h_{j,\varphi} \|_{\mathcal{B}_{\mu}} \\
& \quad + \| (C_{\varphi} - C_{\psi}) g_{j,\varphi} \|_{\mathcal{B}_{\mu}} \rho(\varphi(z_j), \psi(z_j)) \\
& \geq \| \mathfrak{D}_{\mu}^\varphi(z_j) \| \rho(\varphi(z_j), \psi(z_j)).
\end{align*}
\]

Similarly, by considering a sequence \( (z_j)_{j \in \mathbb{N}} \) in \( D \) such that \( |\psi(z_j)| \to 1 \) as \( j \to \infty \) and
\[
\begin{align*}
\lim_{j \to \infty} \| \mathfrak{D}_{\mu}^\varphi(z_j) \| \rho(\varphi(z_j), \psi(z_j)) \\
& = \lim_{|\psi(z)| \to 1} \| \mathfrak{D}_{\mu}^\varphi(z) \| \rho(\varphi(z), \psi(z)),
\end{align*}
\]
we can prove that
\[
\begin{align*}
\| (C_{\varphi} - C_{\psi}) g_{j,\varphi} \|_{\mathcal{B}_{\mu}} + \| (C_{\varphi} - C_{\psi}) h_{j,\varphi} \|_{\mathcal{B}_{\mu}} \\
& \geq \| \mathfrak{D}_{\mu}^\varphi(z_j) \| \rho(\varphi(z_j), \psi(z_j)).
\end{align*}
\]

Combining (17) and (20) and using the fact that \( \| K g_{j,\varphi} \|_{\mathcal{B}_{\mu}} \to 0 \) and \( \| K h_{j,\varphi} \|_{\mathcal{B}_{\mu}} \to 0 \) as \( j \to \infty \), we have
\[
\begin{align*}
\| (C_{\varphi} - C_{\psi}) g_{j,\varphi} \|_{\mathcal{B}_{\mu}} + \| (C_{\varphi} - C_{\psi}) h_{j,\varphi} \|_{\mathcal{B}_{\mu}} \\
& \geq \| \mathfrak{D}_{\mu}^\varphi(z_j) \| \rho(\varphi(z_j), \psi(z_j)).
\end{align*}
\]

Combining (17) and (22) and using the fact that \( \| K g_{j,\varphi} \|_{\mathcal{B}_{\mu}} \to 0 \) and \( \| K h_{j,\varphi} \|_{\mathcal{B}_{\mu}} \to 0 \) as \( j \to \infty \), we have
\[
\begin{align*}
\| (C_{\varphi} - C_{\psi}) g_{j,\varphi} \|_{\mathcal{B}_{\mu}} + \| (C_{\varphi} - C_{\psi}) h_{j,\varphi} \|_{\mathcal{B}_{\mu}} \\
& \geq \| \mathfrak{D}_{\mu}^\varphi(z_j) \| \rho(\varphi(z_j), \psi(z_j)).
\end{align*}
\]

Again, let \( (z_j)_{j \in \mathbb{N}} \) be a sequence in \( D \) such that \( |\varphi(z_j)| \wedge |\psi(z_j)| \to 1 \) as \( j \to \infty \) and
\[
\begin{align*}
\lim_{j \to \infty} \| \mathfrak{D}_{\mu}^\varphi(z_j) \| \rho(\varphi(z_j), \psi(z_j)) \\
& = \lim_{|\psi(z)| \to 1} \| \mathfrak{D}_{\mu}^\varphi(z) \| \rho(\varphi(z), \psi(z)).
\end{align*}
\]

Then from (18) we have
\[
\begin{align*}
\| (C_{\varphi} - C_{\psi}) g_{j,\varphi} \|_{\mathcal{B}_{\mu}} = \sup_{z \in D} \mu(z) \\
& \cdot \left| \left( \mu(z) \left( 1 - |\varphi(z)|^2 \right) \varphi'(z) \right) \right| \\
& \cdot \left( 1 - \left| \varphi(z) \right|^2 \right) \left( 1 - \left| \psi(z) \right|^2 \right) \frac{1}{(1 - \varphi(z) \psi(z))^2} \\
& \cdot \left( 1 - \left| \varphi(z) \right|^2 \right)^{1+(2+\alpha)/p} \\
& \cdot \left| 1 - \varphi(z) \psi(z) \right| \\
& \cdot \rho(\varphi(z), \psi(z)).
\end{align*}
\]
\[
- \frac{\mu(z) \left(1 - |\varphi(z)|^2\right) \psi'(z)}{(1 - |\varphi(z)|^2)^{(2+2\alpha)/p}} \geq \|\Phi^\nu_m(z)\|
\]

\[
- \Phi^\nu_m(z) - |\Phi^\nu_m(z)| \times 1
\]

\[
\geq \left| \Phi^\nu_m(z) - \Phi^\nu_m(z) \right| - \left| \Phi^\nu_m(z) \right|
\]

Using (20) and (26), we have

\[
\sup_{z \in \mathcal{D}} \mu(z) \left(\left| (C_{\varphi} - C_{\psi}) g_{j,\varphi} \right| + \left| (C_{\varphi} - C_{\psi}) h_{j,\varphi} \right| \right)
\geq \|\Phi^\nu_m(z) - \Phi^\nu_m(z)\|.
\]

Combining (17) and (27) and using the fact that \(Kg_j\|\|_{\mathcal{B}_\mu} \to 0\) and \(Kh_j\|\|_{\mathcal{B}_\mu} \to 0\) as \(j \to \infty\), we have

\[
\|C_{\varphi} - C_{\psi}\|_{c,\mathcal{A}_\mu^p - \mathcal{B}_\mu}
\geq \lim_{|\varphi(z)\cap |\psi(z)| \to 1} \|\Phi^\nu_m(z) - \Phi^\nu_m(z)\|.
\]

Combining (23), (24), and (28), we get the lower bound as

\[
\sup_{z \in \mathcal{D}} \mu(z) \left(\left| (C_{\varphi} - C_{\psi}) (I - L_k) f \right| \right)
\geq \lim sup_{|\varphi(z)| = 1} \left| \Phi^\nu_m(z) - \Phi^\nu_m(z) \right|
\]

\[
+ \lim sup_{|\varphi(z)| = 1} \left| \Phi^\nu_m(z) - \Phi^\nu_m(z) \right|
\]

Let \(r_k(z) = (k/(k+1))z\). Then \(r_k\|_{\infty} < 1\). Let

\[
L_k f(z) = C_{r_k} f(z) = f \left( \frac{k}{k+1} z \right).
\]
Similarly, we can show that

\[
\left| f'(w) - f' \left( \frac{k}{k+1} w \right) \right| \leq \frac{|w|}{k+1} \sup_{\zeta \in D(0, r)} |f''(\zeta)| \leq \frac{|w|}{k+1} \sup_{\zeta \in D(0, r)} \left\| f \right\|_{A^p_k} \left( 1 - |\zeta|^2 \right)^{2+(2+\alpha)/p}.
\]

Using (35) with \( \omega = \phi(0) \) and \( \omega = \psi(0) \), we have

\[
\left| (C_{\phi} - C_{\psi}) (I - L_k) f(0) \right| = \left| f(\phi(0)) - f \left( \frac{k}{k+1} \phi(0) \right) \right| + \left| f(\psi(0)) - f \left( \frac{k}{k+1} \psi(0) \right) \right| \leq \frac{|\psi(0)| + |\phi(0)|}{k+1} \left( 1 - r^2 \right)^{1+(2+\alpha)/p} \rightarrow 0
\]
as \( k \rightarrow \infty \).

Combining (38) and (39), we have

\[
\sup_{1 \leq r \leq 1} \sup_{|\phi(z)| \leq r} \left| (C_{\phi} - C_{\psi}) (I - L_k) f(z) \right| \rightarrow 0
\]
as \( k \rightarrow \infty \). Finally, we have

\[
\mu(z) \left| (C_{\phi} - C_{\psi}) (I - L_k) f(z) \right| = \mu(z) \left| \phi'(z) \right| \left| f'(z) - f' \left( \frac{k}{k+1} \phi(z) \right) \right| - \mu(z) \left| \psi'(z) \right| \left| f'(z) - f' \left( \frac{k}{k+1} \psi(z) \right) \right|
\]

\[
\leq \left| \mathcal{D}_\mu^p (z) \right| \left( 1 - |\phi(z)|^2 \right)^{1+(2+\alpha)/p} + \left| \mathcal{D}_\mu^p (z) \right| \left( 1 - |\psi(z)|^2 \right)^{1+(2+\alpha)/p} \cdot \left| f'(z) - f' \left( \frac{k}{k+1} \phi(z) \right) \right|
\]

\[
\leq \frac{M_1}{k+1} \left( 1 - r^2 \right)^{2+(2+\alpha)/p} + \frac{1}{k+1} \left( 1 - r^2 \right)^{2+(2+\alpha)/p}
\]

\[

\rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\]
and by Banach-Steinhaus theorem; it converges to zero uniformly on compact subsets of \( \mathbb{D} \), so we have

\[
\lim_{k \to \infty} \sup_{|z| \leq r} \left| f'(z) - \frac{k}{k+1} f' \left( \frac{k}{k+1} z \right) \right| = 0. \tag{43}
\]

Also, for the boundedness of \( C_{\varphi}, C_{\psi} : \mathcal{A}_\mu \to \mathcal{B}_\mu \), we have the facts that \( \sup_{z \in \mathbb{D}} |\varphi(z)| < \infty \) and \( \sup_{z \in \mathbb{D}} |\psi(z)| < \infty \). Using these facts and (41), we see that for each \( r \in (0, 1) \) and \( |\psi(z)| \leq r \) the right hand side of (41) is dominated by a constant multiple of

\[
\sup_{|\varphi(z)| > r} \left| \mathcal{D}_\mu^\varphi(z) \rho(\varphi(z), \psi(z)) \right|. \tag{44}
\]

If \( |\varphi(z)| > r \), then we see that the right hand side of (41) is dominated by a constant multiple of

\[
\sup_{|\varphi(z)| > r} \left| \mathcal{D}_\mu^\varphi(z) \right| \rho(\varphi(z), \psi(z)) + \sup_{|\varphi(z)| \leq r} \left| \mathcal{D}_\mu^\varphi(z) - \mathcal{D}_\mu^\psi(z) \right|. \tag{45}
\]

Thus

\[
\lim_{r \to 1^-} \lim_{k \to \infty} \sup_{|\varphi(z)| > r} \mu(z)
\cdot \left( (C_{\varphi} - C_{\psi}) (I - L_k) f \right)'(z)
\leq \lim_{|\varphi(z)| \to 1^-} \sup_{|\varphi(z)| > r} \left| \mathcal{D}_\mu^\varphi(z) \right|
\cdot \rho(\varphi(z), \psi(z))
+ \lim_{|\varphi(z)| \to 1^-} \sup_{|\varphi(z)| \leq r} \left| \mathcal{D}_\mu^\varphi(z) - \mathcal{D}_\mu^\psi(z) \right|. \tag{46}
\]

Similarly, we can show that

\[
\lim_{r \to 1^-} \lim_{k \to \infty} \sup_{|\varphi(z)| > r} \mu(z)
\cdot \left( (C_{\varphi} - C_{\psi}) (I - L_k) f \right)'(z)
\leq \lim_{|\varphi(z)| \to 1^-} \sup_{|\varphi(z)| > r} \left| \mathcal{D}_\mu^\varphi(z) \right|
\cdot \rho(\varphi(z), \psi(z))
+ \lim_{|\varphi(z)| \to 1^-} \sup_{|\varphi(z)| \leq r} \left| \mathcal{D}_\mu^\varphi(z) - \mathcal{D}_\mu^\psi(z) \right|. \tag{47}
\]

Combining (31), (32), (40), (46), and (47), we have the fact that

\[
\| C_{\varphi} - C_{\psi} \|_{\mathcal{A}, \mathcal{A}_\mu} \leq \lim_{|\varphi(z)| \to 1^-} \sup_{|\varphi(z)| > r} \left| \mathcal{D}_\mu^\varphi(z) - \mathcal{D}_\mu^\psi(z) \right|
+ \lim_{|\varphi(z)| \to 1^-} \sup_{|\varphi(z)| > r} \left| \mathcal{D}_\mu^\varphi(z) \right| \rho(\varphi(z), \psi(z))
+ \lim_{|\varphi(z)| \to 1^-} \sup_{|\varphi(z)| \leq r} \left| \mathcal{D}_\mu^\varphi(z) - \mathcal{D}_\mu^\psi(z) \right| \rho(\varphi(z), \psi(z)). \tag{48}
\]

Combining (29) and (48), we get the desired result. 

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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