Research Article

Multivalued Fixed Point Results for New Generalized F -Dominated Contractive Mappings on Dislocated Metric Space with Application

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The purpose of this paper is to find out fixed point results for semi- α_* -dominated multivalued mappings fulfilling a new generalized locally F -dominated multivalued contractive condition on a closed ball in complete dislocated metric space. Example and application both are given to show the novelty of our results. Our results merge, extend, and infer many results.

1. Introduction and Preliminaries

Fixed point theory has a foundational role in functional analysis. Banach [1] established the fundamental fixed point theorem, which has played a significant role in different fields of applied mathematics. Due to its significance, a large number of authors have proven many interesting multiplications of his result (see [1–33]). Many authors introduced fixed point theorems in complete dislocated metric space. The idea of dislocated topology has been applied in the field of logic programming semantics (see [14]).

Wardowski [33] introduced new type of contraction called F -contraction and showed a new generalized fixed point theorem. He generalized many previous fixed point results in different directions. Many other useful results on F -contractions can be seen in [3–5, 11, 12, 15, 18, 22, 24–27]. In this paper, we recalled the concept of F -contraction to obtain some common fixed point results for semi- α_* -dominated multivalued mappings on proximal sets satisfying a new type rational F -contraction in the context of complete dislocated metric spaces. We have also given an example in which the mapping is not α_* -admissible but it fulfills the condition of α_* -dominated.

Definition 1 (see [14]). Let $\dot{Z}$ be a nonempty set. A mapping $d_l : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$ is called a dislocated metric (or simply d_l -metric) if the following conditions hold, for any $\check{t}, \check{s}, \check{z} \in \dot{Z}$:

- (i) If $d_l(\check{t}, \check{s}) = 0$, then $\check{t} = \check{s}$.
- (ii) $d_l(\check{t}, \check{s}) = d_l(\check{s}, \check{t})$.
- (iii) $d_l(\check{t}, \check{s}) \leq d_l(\check{t}, \check{z}) + d_l(\check{z}, \check{s})$.

Then d_l is called a dislocated metric on $\dot{Z}$, and the pair $(\dot{Z}, d_l)$ is called dislocated metric space or d_l metric space. It is clear that if $d_l(\check{t}, \check{s}) = 0$, then, from (i), $\check{t} = \check{s}$. But if $\check{t} = \check{s}$, $d_l(\check{t}, \check{s})$ may not be 0. We use DMS instead of dislocated metric space.

Definition 2 (see [14]). Let $(\dot{Z}, d_l)$ be a DMS:

- (i) A sequence $\{c_{\check{n}}\}$ in $(\dot{Z}, d_l)$ is called a Cauchy sequence if given $\varepsilon > 0$; there corresponds $\check{n}_0 \in \mathbb{N}$ such that for all $\check{n}, m \geq \check{n}_0$ one has $d_l(c_{\check{n}}, c_{\check{m}}) < \varepsilon$ or $\lim_{\check{n}, m \rightarrow \infty} d_l(c_{\check{n}}, c_{\check{m}}) = 0$.
- (ii) A sequence $\{c_{\check{n}}\}$ is said to be dislocated—converges (for short d_l -converges) to l if $\lim_{\check{n} \rightarrow \infty} d_l(c_{\check{n}}, l) = 0$. In this case l is called a d_l -limit of $\{c_{\check{n}}\}$.

(iii) $(\dot{Z}, d_l)$ is called complete if every Cauchy sequence in $\dot{Z}$ converges to a point $l \in \dot{Z}$ such that $d_l(l, l) = 0$.

Definition 3 (see [29]). Let H be a nonempty subset of DMS $\dot{Z}$ and let $i \in \dot{Z}$. An element $v_0 \in H$ is called a best approximation in H if

$$d_l(i, H) = d_l(i, v_0), \quad \text{where } d_l(i, H) = \inf_{y \in H} d_l(i, y). \quad (1)$$

If each $i \in \dot{Z}$ has at least one best approximation in H , then H is called a proximal set. We denote $P(\dot{Z})$ as the set of all closed proximal subsets of $\dot{Z}$.

Definition 4 (see [29]). The function $H_{d_l} : P(\dot{Z}) \times P(\dot{Z}) \rightarrow \mathbb{R}^+$, defined by

$$H_{d_l}(N, M) = \max \left\{ \sup_{\tilde{n} \in N} d_l(\tilde{n}, M), \sup_{m \in M} d_l(N, m) \right\}, \quad (2)$$

is called dislocated Hausdorff metric on $P(\dot{Z})$.

Definition 5 (see [29]). Let $\tilde{S} : \dot{Z} \rightarrow P(\dot{Z})$ be a multivalued mapping and $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, +\infty)$. Let $K \subseteq \dot{Z}$; we say that $\tilde{S}$ is semi- α^* -admissible on K , whenever $\alpha(i, j) \geq 1$ implies that $\alpha^*(\tilde{S}i, \tilde{S}j) \geq 1$, for all $i, j \in K$, where $\alpha^*(\tilde{S}i, \tilde{S}j) = \inf\{\alpha(u, v) : u \in \tilde{S}i, v \in \tilde{S}j\}$. If $K = \dot{Z}$, then we say that $\tilde{S}$ is α^* -admissible on $\dot{Z}$.

Definition 6. Let $(\dot{Z}, d_l)$ be a DMS. Let $\tilde{S} : \dot{Z} \rightarrow P(\dot{Z})$ be multivalued mapping and $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, +\infty)$. Let $A \subseteq \dot{Z}$, and we say that $\tilde{S}$ is semi- α_* -dominated on H , whenever $\alpha_*(i, \tilde{S}i) \geq 1$ for all $i \in H$, where $\alpha_*(i, \tilde{S}i) = \inf\{\alpha(i, l) : l \in \tilde{S}i\}$. If $H = \dot{Z}$, then we say that $\tilde{S}$ is α_* -dominated on $\dot{Z}$. If $\tilde{S} : \dot{Z} \rightarrow \dot{Z}$ is a self-mapping, then $\tilde{S}$ is semi- α -dominated on H , whenever $\alpha(i, \tilde{S}i) \geq 1$ for all $i \in H$.

Definition 7 (see [33]). Let $(\dot{Z}, d)$ be a metric space. A mapping $H : \dot{Z} \rightarrow \dot{Z}$ is said to be an F -contraction if there exists $\tau > 0$ such that

$$\begin{aligned} d(Hj, Hk) &> 0 \implies \\ \tau + F(d(Hj, Hk)) &\leq F(d(j, k)) \end{aligned} \quad (3)$$

$$\forall j, k \in \dot{Z},$$

where $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing; that is, for all $j, k \in \mathbb{R}_+$, such that $j < k$, $F(j) < F(k)$.

(F2) For each sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$.

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

Lemma 8. Let $(\dot{Z}, d_l)$ be a DMS. Let $(P(\dot{Z}), H_{d_l})$ be a dislocated Hausdorff metric space on $P(\dot{Z})$. Then, for all $G, H \in$

$P(\dot{Z})$ and for each $g \in G$, there exists $h_g \in H$ satisfying $d_l(g, H) = d_l(g, h_g)$; then $H_{d_l}(G, H) \geq d_l(g, h_g)$.

Example 9 (see [14]). If $\dot{Z} = \mathbb{R}^+ \cup \{0\}$, then $d_l(j, k) = j + k$ defines a dislocated metric d_l on $\dot{Z}$.

Example 10 (see [33]). The family of F is not empty:

- (1) $F(\tilde{c}) = \ln(\tilde{c})$; $\tilde{c} > 0$.
- (2) $F(\tilde{c}) = \tilde{c} + \ln(\tilde{c})$; $\tilde{c} > 0$.
- (3) $F(\tilde{c}) = -1/\sqrt{\tilde{c}}$; $\tilde{c} > 0$.

Example 11. Let $\dot{Z} = \mathbb{R}$. Define the mapping $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$ by

$$\alpha(j, k) = \begin{cases} 1 & \text{if } j > k \\ \frac{1}{2} & \text{otherwise.} \end{cases} \quad (4)$$

Define the multivalued mappings $\tilde{S}, \tilde{T} : \dot{Z} \rightarrow P(\dot{Z})$ by

$$\begin{aligned} \tilde{S}j &= \{[j-4, j-3] \text{ if } j \in \dot{Z}\}, \\ \tilde{T}k &= \{[k-2, k-1] \text{ if } k \in \dot{Z}\}. \end{aligned} \quad (5)$$

Suppose that $j = 3$ and $k = 2.5$. As $3 > 2.5$, then $\alpha(3, 2.5) \geq 1$. Now, $\alpha_*(\tilde{S}3, \tilde{T}2.5) = \inf\{\alpha(a, b) : a \in \tilde{S}3, b \in \tilde{T}2.5\} = 1/2 \not\geq 1$, and this means that the pair $(\tilde{S}, \tilde{T})$ is not α_* -admissible. Also, $\alpha_*(\tilde{S}3, \tilde{S}2) \not\geq 1$ and $\alpha_*(\tilde{T}3, \tilde{T}2) \not\geq 1$. This implies that $\tilde{S}$ and $\tilde{T}$ are not α_* -admissible individually. Now, $\alpha_*(j, \tilde{S}j) = \inf\{\alpha(j, b) : b \in \tilde{S}j\} \geq 1$, for all $j \in \dot{Z}$. Hence $\tilde{S}$ is α_* -dominated mapping. Similarly, $\alpha_*(k, \tilde{T}k) = \inf\{\alpha(k, b) : b \in \tilde{T}k\} \geq 1$. Hence it is clear that $\tilde{S}$ and $\tilde{T}$ are α_* -dominated but not α_* -admissible.

2. Main Result

Let $(\dot{Z}, d_l)$ be a DMS, let $c_0 \in \dot{Z}$, and let $\tilde{S}, \tilde{T} : \dot{Z} \rightarrow P(\dot{Z})$ be the multifunctions on $\dot{Z}$. Let $c_1 \in \tilde{S}c_0$ be an element such that $d_l(c_0, \tilde{S}c_0) = d_l(c_0, c_1)$. Let $c_2 \in \tilde{T}c_1$ be such that $d_l(c_1, \tilde{T}c_1) = d_l(c_1, c_2)$. Let $c_3 \in \tilde{S}c_2$ be such that $d_l(c_2, \tilde{S}c_2) = d_l(c_2, c_3)$. Continuing this method, we get a sequence $c_{\tilde{n}}$ of points in $\dot{Z}$ such that $c_{2\tilde{n}+1} \in \tilde{S}c_{2\tilde{n}}$ and $c_{2\tilde{n}+2} \in \tilde{T}c_{2\tilde{n}+1}$, where $\tilde{n} = 0, 1, 2, \dots$. Also $d_l(c_{2\tilde{n}}, \tilde{S}c_{2\tilde{n}}) = d_l(c_{2\tilde{n}}, c_{2\tilde{n}+1})$, and $d_l(c_{2\tilde{n}+1}, \tilde{T}c_{2\tilde{n}+1}) = d_l(c_{2\tilde{n}+1}, c_{2\tilde{n}+2})$. We denote this iterative sequence by $\{\tilde{T}\tilde{S}(c_{\tilde{n}})\}$. We say that $\{\tilde{T}\tilde{S}(c_{\tilde{n}})\}$ is a sequence in $\dot{Z}$ generated by c_0 .

Theorem 12. Let $(\dot{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$. Let $\tilde{r} > 0$,

$c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq \dot{Z}$ and let $\check{S}, \check{T} : \dot{Z} \rightarrow P(\dot{Z})$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \left\{ \tau + \alpha_* (\check{e}, \check{S}\check{e}) F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & \quad + \alpha_* (\hat{y}, \check{T}\hat{y}) F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e})) \left. \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & \quad + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) \\ & \quad \left. + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (6)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_2, \eta_3, \eta_4 > 0$ and $\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r}, \quad (7)$$

where $\lambda = ((\eta_1 + \eta_2 + \eta_3)/(1 - \eta_3 - \eta_4))$ and $\eta_3 + \eta_4 \neq 1$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{\check{T}\check{S}(c_{\check{n}})\} \rightarrow \bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$. Also if inequality (6) holds for $\bar{u}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

Proof. Consider a sequence $\{\check{T}\check{S}(c_{\check{n}})\}$. From (7), we get

$$d_l(c_0, c_1) \leq d_l(c_0, \check{S}c_0) \leq \check{r}. \quad (8)$$

It follows that

$$c_1 \in \overline{B_{d_l}(c_0, \check{r})}. \quad (9)$$

Let $c_2, \dots, c_j \in \overline{B_{d_l}(c_0, \check{r})}$ for some $j \in N$. If $j = 2i + 1$, where $i = 1, 2, \dots, (j - 1)/2$, then since $\check{S}, \check{T} : \dot{Z} \rightarrow P(\dot{Z})$ are semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha_*(c_{2i}, \check{S}c_{2i}) \geq 1$ and $\alpha_*(c_{2i+1}, \check{T}c_{2i+1}) \geq 1$. As $\alpha_*(c_{2i}, \check{S}c_{2i}) \geq 1$, this implies that $\inf\{\alpha(c_{2i}, b) : b \in \check{S}c_{2i}\} \geq 1$. Also $c_{2i+1} \in \check{S}c_{2i}$, so $\alpha(c_{2i}, c_{2i+1}) \geq 1$. Now, by using Lemma 8, we have

$$\begin{aligned} & \tau + F(d_l(c_{2i+1}, c_{2i+2})) \leq \tau + F(H_{d_l}(\check{S}c_{2i}, \check{T}c_{2i+1})) \\ & \leq \max \left\{ \tau + \alpha_*(c_{2i}, \check{S}c_{2i}) F(H_{d_l}(\check{S}c_{2i}, \check{T}c_{2i+1})), \tau \right. \\ & \quad \left. + \alpha_*(c_{2i+1}, \check{T}c_{2i+1}) F(H_{d_l}(\check{T}c_{2i+1}, \check{S}c_{2i})) \right\} \\ & \leq F \left[\eta_1 d_l(c_{2i}, c_{2i+1}) + \eta_2 d_l(c_{2i}, \check{S}c_{2i}) \right. \\ & \quad + \eta_3 d_l(c_{2i}, \check{T}c_{2i+1}) \\ & \quad \left. + \eta_4 \frac{d_l^2(c_{2i}, \check{S}c_{2i}) \cdot d_l(c_{2i+1}, \check{T}c_{2i+1})}{1 + d_l^2(c_{2i}, c_{2i+1})} \right] \end{aligned}$$

$$\begin{aligned} & \leq F \left[\eta_1 d_l(c_{2i}, c_{2i+1}) + \eta_2 d_l(c_{2i}, c_{2i+1}) \right. \\ & \quad + \eta_3 d_l(c_{2i}, c_{2i+1}) + \eta_3 d_l(c_{2i+1}, c_{2i+2}) \\ & \quad \left. + \eta_4 \frac{d_l^2(c_{2i}, c_{2i+1}) \cdot d_l(c_{2i+1}, c_{2i+2})}{1 + d_l^2(c_{2i}, c_{2i+1})} \right] \\ & \leq F((\eta_1 + \eta_2 + \eta_3) d_l(c_{2i}, c_{2i+1}) + (\eta_3 \\ & \quad + \eta_4) d_l(c_{2i+1}, c_{2i+2})) - \tau; \end{aligned} \quad (10)$$

this implies that

$$\begin{aligned} F(d_l(c_{2i+1}, c_{2i+2})) & \leq F((\eta_1 + \eta_2 + \eta_3) d_l(c_{2i}, c_{2i+1}) \\ & \quad + (\eta_3 + \eta_4) d_l(c_{2i+1}, c_{2i+2})), \end{aligned} \quad (11)$$

for all $j \in N$. As F is strictly increasing, we have

$$\begin{aligned} d_l(c_{2i+1}, c_{2i+2}) & < (\eta_1 + \eta_2 + \eta_3) d_l(c_{2i}, c_{2i+1}) \\ & \quad + (\eta_3 + \eta_4) d_l(c_{2i+1}, c_{2i+2}), \end{aligned} \quad (12)$$

which implies that

$$\begin{aligned} & (1 - \eta_3 - \eta_4) d_l(c_{2i+1}, c_{2i+2}) \\ & < (\eta_1 + \eta_2 + \eta_3) d_l(c_{2i}, c_{2i+1}) \end{aligned} \quad (13)$$

$$d_l(c_{2i+1}, c_{2i+2}) < \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4} \right) d_l(c_{2i}, c_{2i+1}).$$

Here $\lambda = ((\eta_1 + \eta_2 + \eta_3)/(1 - \eta_3 - \eta_4)) < 1$. Hence

$$\begin{aligned} d_l(c_{2i+1}, c_{2i+2}) & < \lambda d_l(c_{2i}, c_{2i+1}) < \lambda^2 d_l(c_{2i-1}, c_{2i}) < \dots \\ & < \lambda^j d_l(c_0, c_1). \end{aligned} \quad (14)$$

Now,

$$\begin{aligned} d_l(c_0, c_{j+1}) & \leq d_l(c_0, c_1) + d_l(c_1, c_2) + \dots + d_l(c_j, c_{j+1}) \\ & \leq d_l(c_0, c_1) [1 + \lambda + \dots + \lambda^j] \\ & \leq (1 - \lambda) \check{r} \frac{(1 - \lambda^{j+1})}{(1 - \lambda)} < \check{r}. \end{aligned} \quad (15)$$

Thus, $c_{j+1} \in \overline{B_{d_l}(c_0, \tilde{r})}$. Hence $c_{\tilde{n}} \in \overline{B_{d_l}(c_0, \tilde{r})}$, for all $\tilde{n} \in N$. Continuing this process, we get

$$\begin{aligned}
 \tau + F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) &\leq \tau + F(H_{d_l}(\check{S}c_{\tilde{n}-1}, \check{T}c_{\tilde{n}})) \\
 &\leq \max \left\{ \tau + \alpha_*(c_{\tilde{n}-1}, \check{S}c_{\tilde{n}-1}) F(H_{d_l}(\check{S}c_{\tilde{n}-1}, \check{T}c_{\tilde{n}})), \tau \right. \\
 &\quad \left. + \alpha_*(c_{\tilde{n}}, \check{T}c_{\tilde{n}}) F(H_{d_l}(\check{T}c_{\tilde{n}}, \check{S}c_{\tilde{n}-1})) \right\} \\
 &\leq F \left[\eta_1 d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) + \eta_2 d_l(c_{\tilde{n}-1}, \check{S}c_{\tilde{n}-1}) \right. \\
 &\quad \left. + \eta_3 d_l(c_{\tilde{n}-1}, \check{T}c_{\tilde{n}}) \right. \\
 &\quad \left. + \eta_4 \frac{d_l^2(c_{\tilde{n}-1}, \check{S}c_{\tilde{n}-1}) \cdot d_l(c_{\tilde{n}}, \check{T}c_{\tilde{n}})}{1 + d_l^2(c_{\tilde{n}-1}, c_{\tilde{n}})} \right] \\
 &\leq F \left[\eta_1 d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) + \eta_2 d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) + \eta_3 d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) \right. \\
 &\quad \left. + \eta_3 d_l(c_{\tilde{n}}, c_{\tilde{n}+1}) + \eta_4 \frac{d_l^2(c_{\tilde{n}-1}, c_{\tilde{n}}) \cdot d_l(c_{\tilde{n}}, c_{\tilde{n}+1})}{1 + d_l^2(c_{\tilde{n}-1}, c_{\tilde{n}})} \right] \\
 &\leq F[(\eta_1 + \eta_2 + \eta_3) d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) + (\eta_3 \\
 &\quad + \eta_4) d_l(c_{\tilde{n}}, c_{\tilde{n}+1})] - \tau;
 \end{aligned} \tag{16}$$

this implies that

$$\begin{aligned}
 F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) &\leq F[(\eta_1 + \eta_2 + \eta_3) d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) \\
 &\quad + (\eta_3 + \eta_4) d_l(c_{\tilde{n}}, c_{\tilde{n}+1})],
 \end{aligned} \tag{17}$$

for all $\tilde{n} \in N$. As F is strictly increasing, we have

$$\begin{aligned}
 d_l(c_{\tilde{n}}, c_{\tilde{n}+1}) &< (\eta_1 + \eta_2 + \eta_3) d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) \\
 &\quad + (\eta_3 + \eta_4) d_l(c_{\tilde{n}}, c_{\tilde{n}+1}),
 \end{aligned} \tag{18}$$

which implies that

$$\begin{aligned}
 (1 - \eta_3 - \eta_4) d_l(c_{\tilde{n}}, c_{\tilde{n}+1}) &< (\eta_1 + \eta_2 + \eta_3) d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) \\
 d_l(c_{\tilde{n}}, c_{\tilde{n}+1}) &< \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4} \right) d_l(c_{\tilde{n}-1}, c_{\tilde{n}}).
 \end{aligned} \tag{19}$$

Here $\lambda = ((\eta_1 + \eta_2 + \eta_3)/(1 - \eta_3 - \eta_4)) < 1$. Hence

$$d_l(c_{\tilde{n}}, c_{\tilde{n}+1}) < \lambda d_l(c_{\tilde{n}-1}, c_{\tilde{n}}) < d_l(c_{\tilde{n}-1}, c_{\tilde{n}}). \tag{20}$$

Consequently,

$$\tau + F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) \leq F(d_l(c_{\tilde{n}-1}, c_{\tilde{n}})), \tag{21}$$

which implies that

$$\begin{aligned}
 F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) &\leq F(d_l(c_{\tilde{n}-1}, c_{\tilde{n}})) - \tau \\
 &\vdots \\
 &\leq F(d_l(c_0, c_1)) - \tilde{n}\tau.
 \end{aligned} \tag{22}$$

This implies that

$$F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) \leq F(d_l(c_0, c_1)) - \tilde{n}\tau. \tag{23}$$

And so $\lim_{\tilde{n} \rightarrow \infty} F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) = -\infty$. By (F_2) , we find that

$$\lim_{\tilde{n} \rightarrow \infty} F(d_l(c_{\tilde{n}}, c_{\tilde{n}+1})) = 0. \tag{24}$$

We shall prove that $\{\check{T}\check{S}(c_{\tilde{n}})\}$ is Cauchy in $(\dot{Z}, d_l)$. So it suffices to show that $\lim_{\tilde{n} \rightarrow \infty} d_l(c_{\tilde{n}}, c_m) = 0$. We argue by contradiction. Suppose that there exist $\epsilon > 0$ and sequences $(\check{n}(q))$ and $(m(q))$ of natural numbers such that

$$\begin{aligned}
 d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}) &\geq \epsilon, \\
 d_l(c_{\check{n}(q)+1}, c_{m(q)}) &< \epsilon \\
 m(q) &> \check{n}(q) > q, \quad \forall q \in N.
 \end{aligned} \tag{25}$$

By triangular inequality, we have

$$\begin{aligned}
 d_l(c_{\check{n}(q)}, c_{m(q)}) &\leq d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}) + d_l(c_{\check{n}(q)+1}, c_{m(q)}) \\
 &\leq \epsilon + d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}) \\
 &\leq \epsilon + d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}).
 \end{aligned} \tag{26}$$

From (24), there exist $q_1 \in N$ such that, for all $q \geq q_1$,

$$d_l(c_{\check{n}(q)}, \check{T}c_{\check{n}(q)}) < \epsilon. \tag{27}$$

Combining (26) with (27) yields

$$d_l(c_{\check{n}(q)}, c_{m(q)}) < 2\epsilon, \quad \forall q \geq q_1. \tag{28}$$

As $\check{S}, \check{T} : \dot{Z} \rightarrow P(\dot{Z})$ are semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \tilde{r})}$, $\alpha_*(c_{\check{n}(q)}, \check{S}c_{\check{n}(q)}) \geq 1$ and $\alpha_*(c_{m(q)}, \check{T}c_{m(q)}) \geq 1$, for all $m, \check{n} \in N$. Now, by using Lemma 8 and condition (6), we get

$$\begin{aligned}
 F(\epsilon) &\leq F(H_{d_l}(\check{S}c_{\check{n}(q)}, \check{T}c_{m(q)})) \leq \max \left\{ \tau \right. \\
 &\quad \left. + \alpha_*(c_{\check{n}(q)}, \check{S}c_{\check{n}(q)}) F(H_{d_l}(\check{S}c_{\check{n}(q)}, \check{T}c_{m(q)})), \tau \right. \\
 &\quad \left. + \alpha_*(c_{m(q)}, \check{T}c_{m(q)}) F(H_{d_l}(\check{T}c_{m(q)}, \check{S}c_{\check{n}(q)})) \right\}
 \end{aligned}$$

$$\begin{aligned}
&\leq F \left[\eta_1 d_l(c_{\tilde{n}(q)}, c_{m(q)}) + \eta_2 d_l(c_{\tilde{n}(q)}, \check{S}c_{\tilde{n}(q)}) \right. \\
&\quad + \eta_3 d_l(c_{\tilde{n}(q)}, \check{T}c_{m(q)}) \\
&\quad \left. + \eta_4 \frac{d_l^2(c_{\tilde{n}(q)}, \check{S}c_{\tilde{n}(q)}) \cdot d_l(c_{m(q)}, \check{T}c_{m(q)})}{1 + d_l^2(c_{\tilde{n}(q)}, c_{m(q)})} \right] - \tau, \\
F(\epsilon) &\leq F \left[\eta_1 d_l(c_{\tilde{n}(q)}, c_{m(q)}) + \eta_2 d_l(c_{\tilde{n}(q)}, c_{\tilde{n}(q)+1}) \right. \\
&\quad + \eta_3 d_l(c_{\tilde{n}(q)}, c_{m(q)+1}) \\
&\quad \left. + \eta_4 \frac{d_l^2(c_{\tilde{n}(q)}, c_{\tilde{n}(q)+1}) \cdot d_l(c_{m(q)}, c_{m(q)+1})}{1 + d_l^2(c_{\tilde{n}(q)}, c_{m(q)})} \right] - \tau, \\
F(\epsilon) &\leq F \left[\eta_1 d_l(c_{\tilde{n}(q)}, c_{m(q)}) + \eta_2 d_l(c_{\tilde{n}(q)}, c_{\tilde{n}(q)+1}) \right. \\
&\quad + \eta_3 d_l(c_{\tilde{n}(q)}, c_{m(q)}) + \eta_3 d_l(c_{m(q)}, c_{m(q)+1}) \\
&\quad \left. + \eta_4 \frac{d_l^2(c_{\tilde{n}(q)}, c_{\tilde{n}(q)+1}) \cdot d_l(c_{m(q)}, c_{m(q)+1})}{1 + d_l^2(c_{\tilde{n}(q)}, c_{m(q)})} \right] - \tau,
\end{aligned} \tag{29}$$

which implies that

$$F(\epsilon) \leq F[2\eta_1 \epsilon + \eta_2 \epsilon + 3\eta_3 \epsilon + \eta_4 \epsilon] - \tau. \tag{30}$$

As $2\eta_1 + \eta_2 + 3\eta_3 + \eta_4 < 1$, we get

$$2\eta_1 \epsilon + \eta_2 \epsilon + 3\eta_3 \epsilon + \eta_4 \epsilon < \epsilon, \tag{31}$$

we deduce that

$$F(\epsilon) < F(\epsilon); \tag{32}$$

which is a contradiction. Thus, $\{\check{T}\check{S}(c_{\tilde{n}})\}$ is a Cauchy sequence in $(\overline{B_{d_l}(c_0, \check{r})}, d_l)$. Since $(\overline{B_{d_l}(c_0, \check{r})}, d_l)$ is a complete metric space, so there exist $\bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$ such that $\{\check{T}\check{S}(c_{\tilde{n}})\} \rightarrow \bar{u}$ as $\tilde{n} \rightarrow \infty$; then

$$\lim_{\tilde{n} \rightarrow \infty} d_l(c_{\tilde{n}}, \bar{u}) = 0. \tag{33}$$

Since $\alpha_*(\bar{u}, \check{T}\bar{u}) \geq 1$ and $\alpha_*(c_{2\tilde{n}}, \check{S}c_{2\tilde{n}}) \geq 1$, by using Lemma 8 and inequality (6), we have

$$\begin{aligned}
F(d_l(c_{2\tilde{n}+1}, \check{T}\bar{u})) &\leq F(H_{d_l}(\check{S}c_{2\tilde{n}}, \check{T}\bar{u})) \leq \max \{ \tau \\
&\quad + \alpha_*(c_{2\tilde{n}}, \check{S}c_{2\tilde{n}}) F(H_{d_l}(\check{S}c_{2\tilde{n}}, \check{T}\bar{u})), \tau \\
&\quad + \alpha_*(\bar{u}, \check{T}\bar{u}) F(H_{d_l}(\check{T}\bar{u}, \check{S}c_{2\tilde{n}})) \} \\
&\leq F \left[\eta_1 d_l(c_{2\tilde{n}}, \bar{u}) + \eta_2 d_l(c_{2\tilde{n}}, \check{S}c_{2\tilde{n}}) + \eta_3 d_l(c_{2\tilde{n}}, \check{T}\bar{u}) \right. \\
&\quad \left. + \eta_4 \frac{d_l^2(c_{2\tilde{n}}, \check{S}c_{2\tilde{n}}) \cdot d_l(\bar{u}, \check{T}\bar{u})}{1 + d_l^2(c_{2\tilde{n}}, \bar{u})} \right] - \tau \\
&\leq F \left[\eta_1 d_l(c_{2\tilde{n}}, \bar{u}) + \eta_2 d_l(c_{2\tilde{n}}, \check{S}c_{2\tilde{n}}) + \eta_3 d_l(c_{2\tilde{n}}, \bar{u}) \right. \\
&\quad \left. + \eta_3 (\bar{u}, \check{T}\bar{u}) + \eta_4 \frac{d_l^2(c_{2\tilde{n}}, \check{S}c_{2\tilde{n}}) \cdot d_l(\bar{u}, \check{T}\bar{u})}{1 + d_l^2(c_{2\tilde{n}}, \bar{u})} \right] - \tau.
\end{aligned} \tag{34}$$

By using (33), we get

$$F(d_l(\bar{u}, \check{T}\bar{u})) \leq F[\eta_3 d_l(\bar{u}, \check{T}\bar{u})] - \tau, \tag{35}$$

which implies that

$$d_l(\bar{u}, \check{T}\bar{u}) < \eta_3 d_l(\bar{u}, \check{T}\bar{u}) < d_l(\bar{u}, \check{T}\bar{u}), \tag{36}$$

which is a contradiction; hence $d_l(\bar{u}, \check{T}\bar{u}) = 0$ or $\bar{u} \in \check{T}\bar{u}$. Similarly, by using Lemma 8, inequality (6), and the inequality

$$d_l(\bar{u}, \check{S}\bar{u}) \leq d_l(\bar{u}, c_{2\tilde{n}+2}) + d_l(c_{2\tilde{n}+2}, \check{S}\bar{u}), \tag{37}$$

we can show that $d_l(\bar{u}, \check{S}\bar{u}) = 0$. $\bar{u} \in \check{S}\bar{u}$. Hence, $\check{S}$ and $\check{T}$ have a common fixed point $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$. Now,

$$d_l(\bar{u}, \bar{u}) \leq d_l(\bar{u}, \check{T}\bar{u}) + d_l(\check{T}\bar{u}, \bar{u}) \leq 0. \tag{38}$$

This implies that $d_l(\bar{u}, \bar{u}) = 0$. $\square$

Example 13. Let $\dot{Z} = Q^+ \cup \{0\}$ and let $d_l : \dot{Z} \times \dot{Z} \rightarrow \dot{Z}$ be the complete DMS on $\dot{Z}$ defined by

$$d_l(i, j) = i + j \quad \forall i, j \in \dot{Z}. \tag{39}$$

Define the multivalued mapping, $\check{S}, \check{T} : \dot{Z} \times \dot{Z} \rightarrow P(\dot{Z})$, by

$$\begin{aligned}
\check{S}z &= \begin{cases} \left[\frac{\dot{z}}{3}, \frac{2}{3}\dot{z} \right] & \text{if } \dot{z} \in [0, 7] \cap \dot{Z} \\ [\dot{z}, \dot{z} + 1] & \text{if } \dot{z} \in (7, \infty) \cap \dot{Z}, \end{cases} \\
\check{T}z &= \begin{cases} \left[\frac{\dot{z}}{4}, \frac{3}{4}\dot{z} \right] & \text{if } \dot{z} \in [0, 7] \cap \dot{Z} \\ [\dot{z} + 1, \dot{z} + 3] & \text{if } \dot{z} \in (7, \infty) \cap \dot{Z}. \end{cases}
\end{aligned} \tag{40}$$

Considering that $x_0 = 1$, $\tilde{r} = 8$, $\overline{B_{d_l}(x_0, \tilde{r})} = [0, 7] \cap \dot{Z}$. So we obtain a sequence $\{\tilde{T}\tilde{S}(x_{\tilde{n}})\} = \{1, 1/3, 1/12, 1/36, \dots\}$ in $\dot{Z}$ generated by x_0 . Also, $\overline{B_{d_l}(x_0, \tilde{r})} \cap \{\tilde{T}\tilde{S}(x_{\tilde{n}})\} = \{1, 1/3, 1/12, 1/36, \dots\}$. Let

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 7] \cap \dot{Z} \\ \frac{3}{2} & \text{otherwise,} \end{cases} \quad (41)$$

$\tau \in (0, 12/95]$, $\eta_1 = 1/5$, $\eta_2 = 1/10$, $\eta_3 = 1/15$, $\eta_4 = 1/30$, and $\lambda = 4/9$.

$$d_l(x_0, \check{S}x_0) = \frac{4}{3} < \left(1 - \frac{4}{9}\right) 8 = (1 - \lambda) \tilde{r}. \quad (42)$$

Now, if $x, y \in \overline{B_{d_l}(x_0, \tilde{r})} \cap \{\tilde{T}\tilde{S}(x_{\tilde{n}})\}$, then we have the following cases.

Case 1. If $\max\{\tau + \alpha_*(x, \check{S}x)F(H_{d_l}(\check{S}x, \check{T}y)), \tau + \alpha_*(y, \check{T}y)F(H_{d_l}(\check{T}y, \check{S}x))\} = \tau + \alpha_*(x, \check{S}x)F(H_{d_l}(\check{S}x, \check{T}y))$, then we consider only

$$\begin{aligned} H_{d_l}(\check{S}x, \check{T}y) &= \max \left\{ \sup_{a \in \check{S}x} d_l(a, \check{T}y), \sup_{b \in \check{T}y} d_l(\check{S}x, b) \right\} \\ &= \max \left\{ \sup_{a \in \check{S}x} d_l\left(a, \left[\frac{y}{4}, \frac{3y}{4}\right]\right), \right. \\ &\quad \left. \sup_{b \in \check{T}y} d_l\left(\left[\frac{x}{3}, \frac{2x}{3}\right], b\right) \right\} \\ &= \max \left\{ d_l\left(\frac{2x}{3}, \left[\frac{y}{4}, \frac{3y}{4}\right]\right), d_l\left(\left[\frac{x}{3}, \frac{2x}{3}\right], \frac{3y}{4}\right) \right\} \\ &= \max \left\{ d_l\left(\frac{2x}{3}, \frac{y}{4}\right), d_l\left(\frac{x}{3}, \frac{3y}{4}\right) \right\} = \max \left\{ \frac{2x}{3} \right. \\ &\quad \left. + \frac{y}{4}, \frac{x}{3} + \frac{3y}{4} \right\} < \frac{1}{5} d_l(x, y) + \frac{1}{10} d_l\left(x, \left[\frac{x}{3}, \frac{2x}{3}\right]\right) \\ &\quad + \frac{1}{15} d_l\left(x, \left[\frac{y}{4}, \frac{3y}{4}\right]\right) + \frac{1}{30} \\ &\quad \cdot \frac{d_l^2(x, [x/3, (2/3)x]) \cdot d_l(y, [y/4, (3/4)y])}{1 + d_l^2(x, y)} \\ &= \frac{1}{5} (x + y) + \frac{2x}{15} + \frac{4x + y}{60} + \frac{5x^2 y^2}{54 \{1 + (x + y)^2\}}. \end{aligned} \quad (43)$$

Thus,

$$\begin{aligned} H_{d_l}(\check{S}x, \check{T}y) &< \eta_1 d_l(x, y) + \eta_2 d_l(x, \check{S}x) \\ &\quad + \eta_3 d_l(x, \check{T}y) \\ &\quad + \eta_4 \frac{d_l^2(x, \check{S}x) \cdot d_l(y, \check{T}y)}{1 + d_l^2(x, y)}, \end{aligned} \quad (44)$$

which implies that

$$\begin{aligned} \tau + \ln(H_{d_l}(\check{S}x, \check{T}y)) &\leq \ln \left(\eta_1 d_l(x, y) \right. \\ &\quad \left. + \eta_2 d_l(x, \check{S}x) + \eta_3 d_l(x, \check{T}y) \right. \\ &\quad \left. + \eta_4 \frac{d_l^2(x, \check{S}x) \cdot d_l(y, \check{T}y)}{1 + d_l^2(x, y)} \right). \end{aligned} \quad (45)$$

That is, if $F(\check{c}) = \ln(\check{c})$, then

$$\begin{aligned} \tau + \alpha_*(x, \check{S}x) F(H_{d_l}(\check{S}x, \check{T}y)) &\leq F \left(\eta_1 d_l(x, y) \right. \\ &\quad \left. + \eta_2 d_l(x, \check{S}x) + \eta_3 d_l(x, \check{T}y) \right. \\ &\quad \left. + \eta_4 \frac{d_l^2(x, \check{S}x) \cdot d_l(y, \check{T}y)}{1 + d_l^2(x, y)} \right). \end{aligned} \quad (46)$$

Case 2. If $\max\{\tau + \alpha_*(x, \check{S}x)F(H_{d_l}(\check{S}x, \check{T}y)), \tau + \alpha_*(y, \check{T}y)F(H_{d_l}(\check{T}y, \check{S}x))\} = \tau + \alpha_*(y, \check{T}y)F(H_{d_l}(\check{T}y, \check{S}x))$, then by using the similar arguments of Case 1 we can get the same results. Now, if $x = 8$, $y = 9 \in (7, \infty) \cap \dot{Z}$, then

$$\begin{aligned} \tau + \alpha_*(y, \check{T}y) F(H_{d_l}(\check{S}x, \check{T}y)) &> F \left(\eta_1 d_l(x, y) \right. \\ &\quad \left. + \eta_2 d_l(x, \check{S}x) + \eta_3 d_l(x, \check{T}y) \right. \\ &\quad \left. + \eta_4 \frac{d_l^2(x, \check{S}x) \cdot d_l(y, \check{T}y)}{1 + d_l^2(x, y)} \right). \end{aligned} \quad (47)$$

And consequently condition (6) does not hold on $\dot{Z}$. Thus the mappings $\check{S}$ and $\check{T}$ satisfy all the conditions of Theorem 12 on closed ball rather than on whole space.

If we take $\check{S} = \check{T}$ in Theorem 12, then we are left with the result.

Corollary 14. Let $(\dot{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$. Let $\tilde{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \tilde{r})} \subseteq \dot{Z}$, and let $\check{S} : \dot{Z} \rightarrow P(\dot{Z})$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \tilde{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} \max \{ \tau + \alpha_*(\check{e}, \check{S}\check{e}) F(H_{d_l}(\check{S}\check{e}, \check{S}\hat{y})), \tau \} \\ + \alpha_*(\hat{y}, \check{S}\hat{y}) F(H_{d_l}(\check{S}\hat{y}, \check{S}\check{e})) \} &\leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ &\quad + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{S}\hat{y}) \\ &\quad \left. + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{S}\hat{y})}{1 + d_l^2(x, \hat{y})} \right) \end{aligned} \quad (48)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{S}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_2, \eta_3, \eta_4 > 0$ and $\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r}, \quad (49)$$

where $\lambda = ((\eta_1 + \eta_2 + \eta_3)/(1 - \eta_3 - \eta_4))$ and $\eta_3 + \eta_4 \neq 1$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{\check{S}\check{S}(c_{\check{n}})\} \rightarrow \bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$. Also if inequality (48) holds for $\check{e}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

If we take $\eta_2 = 0$ in Theorem 12, then we are left with the result.

Corollary 15. Let $(\check{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \check{Z} \times \check{Z} \rightarrow [0, \infty)$. Let $\check{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \check{Z}$, and let $\check{S}, \check{T} : \check{Z} \rightarrow P(\check{Z})$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \left\{ \tau + \alpha_*(\check{e}, \check{S}\check{e}) F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & \left. + \alpha_*(\hat{y}, \check{T}\hat{y}) F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e})) \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & \left. + \eta_3 d_l(\check{e}, \check{T}\hat{y}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (50)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_3, \eta_4 > 0$ and $\eta_1 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r}, \quad (51)$$

where $\lambda = ((\eta_1 + \eta_3)/(1 - \eta_3 - \eta_4))$ and $\eta_3 + \eta_4 \neq 1$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{\check{T}\check{S}(c_{\check{n}})\} \rightarrow \bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$. Also if inequality (50) holds for $\bar{u}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

If we take $\eta_3 = 0$ in Theorem 12, then we are left with the result.

Corollary 16. Let $(\check{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \check{Z} \times \check{Z} \rightarrow [0, \infty)$. Let $\check{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \check{Z}$, and let $\check{S}, \check{T} : \check{Z} \rightarrow P(\check{Z})$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \left\{ \tau + \alpha_*(\check{e}, \check{S}\check{e}) F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & \left. + \alpha_*(\hat{y}, \check{T}\hat{y}) F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e})) \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & \left. + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (52)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_2, \eta_4 > 0$ and $\eta_1 + \eta_2 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r}, \quad (53)$$

where $\lambda = ((\eta_1 + \eta_2)/(1 - \eta_4))$ and $1 - \eta_4 \neq 0$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{\check{T}\check{S}(c_{\check{n}})\} \rightarrow \bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$. Also if inequality (52) holds for $\bar{u}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

If we take $\eta_4 = 0$ in Theorem 12, then we are left only with the result.

Corollary 17. Let $(\check{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \check{Z} \times \check{Z} \rightarrow [0, \infty)$. Let $\check{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \check{Z}$, and let $\check{S}, \check{T} : \check{Z} \rightarrow P(\check{Z})$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \left\{ \tau + \alpha_*(\check{e}, \check{S}\check{e}) F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & \left. + \alpha_*(\hat{y}, \check{T}\hat{y}) F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e})) \right\} \leq F(\eta_1 d_l(\check{e}, \hat{y}) \\ & + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y})) \end{aligned} \quad (54)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_2, \eta_3 > 0$ and $\eta_1 + \eta_2 + 2\eta_3 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r}, \quad (55)$$

where $\lambda = ((\eta_1 + \eta_2 + \eta_3)/(1 - \eta_3))$ and $1 - \eta_3 \neq 0$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{\check{T}\check{S}(c_{\check{n}})\} \rightarrow \bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$. Also if inequality (54) holds for $\bar{u}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

3. Fixed Point Results for Graphic Contractions

In this section, we present an application of Theorem 12 in graph theory. Jachymski [21], proved the result concerning for contraction mappings on metric space with a graph. Hussain et al. [16] introduced the fixed points theorem for ψ_b -graphic contraction and gave an application to system of integral equations. A graph $\check{G}$ is a connected graph if there must exist a path among any two different vertices (for details, see [13, 32]).

Definition 18. Let $\check{Z}$ be a nonempty set and let $\check{G} = (V(\check{G}), E(\check{G}))$ be a graph such that $V(\check{G}) = \check{Z}$, and $\check{S} : \check{Z} \rightarrow P(\check{Z})$ is said to be multigraph-dominated if $(\check{z}, \check{u}) \in E(\check{G})$, for all $\check{u} \in \check{S}\check{z}$.

Theorem 19. Let $(\dot{Z}, d_l)$ be a complete DMS endowed with a graph $\check{G}$. Suppose that there exists a function $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$. Let $\check{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \check{r})}$, $\check{S}, \check{T} : \dot{Z} \rightarrow P(\dot{Z})$, and assume a sequence $\{\check{T}\check{S}(c_{\check{n}})\}$ in $\dot{Z}$ generated by c_0 , with $(c_0, c_1) \in E(\check{G})$. Assume that the following hold:

(i) $\check{S}$ and $\check{T}$ are multigraph-dominated for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$.

(ii) There exist some $\eta_1, \eta_2, \eta_3, \eta_4, \tau > 0$, and

$$\begin{aligned} & \max \left\{ \tau + \alpha_* (\check{e}, \check{S}\check{e}) F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & + \alpha_* (\hat{y}, \check{T}\hat{y}) F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e})) \left. \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) \\ & \left. + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (56)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ and $(\check{e}, \hat{y}) \in E(\check{G})$ or $(\hat{y}, \check{e}) \in E(\check{G})$.

(iii) $\sum_{i=0}^{\check{n}} \lambda^i (d_l(c_0, \check{S}c_0)) \leq \check{r}$ for all $\check{n} \in N \cup \{0\}$.

Then, $\{\check{T}\check{S}(c_{\check{n}})\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $(c_{\check{n}}, c_{\check{n}+1}) \in E(\check{G})$, and $\{\check{T}\check{S}(c_{\check{n}})\} \rightarrow m^*$. Also, if inequality (56) holds for m^* and $(c_{\check{n}}, m^*) \in E(\check{G})$ or $(m^*, c_{\check{n}}) \in E(\check{G})$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point m^* in $\overline{B_{d_l}(c_0, \check{r})}$ and $d_l(m^*, m^*) = 0$.

Proof. Define $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$ by

$$\begin{aligned} & \alpha(\check{e}, \hat{y}) \\ & = \begin{cases} 1, & \text{if } \check{e} \in \overline{B_{d_l}(c_0, \check{r})}, (\check{e}, \hat{y}) \in E(\check{G}) \text{ or } (\hat{y}, \check{e}) \in E(\check{G}) \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (57)$$

As $\check{S}$ and $\check{T}$ are semigraph-dominated on $\overline{B_{d_l}(c_0, \check{r})}$, then, for $\check{e} \in \overline{B_{d_l}(c_0, \check{r})}$, $(\check{e}, \hat{y}) \in E(\check{G})$ for all $\hat{y} \in \check{S}\check{e}$ and $(\check{e}, \hat{y}) \in E(\check{G})$ for all $\hat{y} \in \check{T}\check{e}$. So, $\alpha(\check{e}, \hat{y}) = 1$ for all $\hat{y} \in \check{S}\check{e}$ and $\alpha(\check{e}, \hat{y}) = 1$ for all $\hat{y} \in \check{T}\check{e}$. This implies that $\inf\{\alpha(\check{e}, \hat{y}) : \hat{y} \in \check{S}\check{e}\} = 1$ and $\inf\{\alpha(\check{e}, \hat{y}) : \hat{y} \in \check{T}\check{e}\} = 1$. Hence $\alpha_*(\check{e}, \check{S}\check{e}) = 1$, $\alpha_*(\check{e}, \check{T}\check{e}) = 1$ for all $\check{e} \in \overline{B_{d_l}(c_0, \check{r})}$. So, $\check{S}, \check{T} : \dot{Z} \rightarrow P(\dot{Z})$ are the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$. Moreover, inequality (56) can be written as

$$\begin{aligned} & \max \left\{ \tau + \alpha_* (\check{e}, \check{S}\check{e}) F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & + \alpha_* (\hat{y}, \check{T}\hat{y}) F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e})) \left. \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) \\ & \left. + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (58)$$

for all elements $\check{e}, \hat{y}$ in $\overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$. Also, (iii) holds. Then, by Theorem 12, we have that $\{\check{T}\check{S}(c_{\check{n}})\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$ and $\{\check{T}\check{S}(c_{\check{n}})\} \rightarrow m^* \in \overline{B_{d_l}(c_0, \check{r})}$. Now, $c_{\check{n}}, m^* \in \overline{B_{d_l}(c_0, \check{r})}$ and either $(c_{\check{n}}, m^*) \in E(\check{G})$ or $(m^*, c_{\check{n}}) \in E(\check{G})$ implies that either $\alpha(c_{\check{n}}, m^*) \geq 1$ or $\alpha(m^*, c_{\check{n}}) \geq 1$. So, all the conditions of Theorem 12 are satisfied. Hence, by Theorem 12, $\check{S}$ and $\check{T}$ have a common fixed point m^* in $\overline{B_{d_l}(c_0, \check{r})}$ and $d_l(m^*, m^*) = 0$. $\square$

4. Fixed Point Results for Single Valued Mapping

In this section, we discuss some new fixed point results for single valued mapping in complete DMS.

Theorem 20. Let $(\dot{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$. Let $\check{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \dot{Z}$, and let $\check{S}, \check{T} : \dot{Z} \rightarrow \dot{Z}$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \left\{ \tau + \alpha_* (\check{e}, \check{S}\check{e}) F(d_l(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & + \alpha_* (\hat{y}, \check{T}\hat{y}) F(d_l(\check{T}\hat{y}, \check{S}\check{e})) \left. \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) \\ & \left. + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (59)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{c_{\check{n}}\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_2, \eta_3, \eta_4 > 0$ and $\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda) \check{r}, \quad (60)$$

where $\lambda = ((\eta_1 + \eta_2 + \eta_3)/(1 - \eta_3 - \eta_4))$ and $\eta_3 + \eta_4 \neq 1$. Then $\{c_{\check{n}}\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{c_{\check{n}}\} \rightarrow v \in \overline{B_{d_l}(c_0, \check{r})}$. Also if inequality (59) holds for v and either $\alpha(c_{\check{n}}, v) \geq 1$ or $\alpha(v, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point v in $\overline{B_{d_l}(c_0, \check{r})}$.

Proof. The proof of the above Theorem is the same as Theorem 12. $\square$

If we take $\check{S} = \check{T}$ in Theorem 20, then we have the following result.

Corollary 21. Let $(\dot{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$. Let $\check{r} > 0$, let

$c_0 \in \overline{B_{d_l}(c_0, \tilde{r})} \subseteq \dot{Z}$, and let $\check{S} : \dot{Z} \rightarrow \dot{Z}$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \tilde{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \left\{ \tau + \alpha_* (\check{e}, \check{S}\check{e}) F(d_l(\check{S}\check{e}, \check{S}\hat{y})), \tau \right. \\ & \quad + \alpha_* (\hat{y}, \check{S}\hat{y}) F(d_l(\check{S}\hat{y}, \check{S}\check{e})) \left. \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & \quad + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{S}\hat{y}) \\ & \quad \left. + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{S}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (61)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \tilde{r})} \cap \{c_{\check{n}}\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_2, \eta_3, \eta_4 > 0$ and $\eta_1 + \eta_2 + \eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda) \tilde{r}, \quad (62)$$

where $\lambda = ((\eta_1 + \eta_2)/(1 - \eta_3 - \eta_4))$ and $\eta_3 + \eta_4 \neq 1$. Then $\{c_{\check{n}}\}$ is a sequence in $\overline{B_{d_l}(c_0, \tilde{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{c_{\check{n}}\} \rightarrow v \in \overline{B_{d_l}(c_0, \tilde{r})}$. Also if inequality (61) holds for v and either $\alpha(c_{\check{n}}, v) \geq 1$ or $\alpha(v, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point v in $\overline{B_{d_l}(c_0, \tilde{r})}$.

If we take $\eta_2 = 0$ in Theorem 20, then we are left with the result.

Corollary 22. Let $(\dot{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$. Let $\tilde{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \tilde{r})} \subseteq \dot{Z}$, and let $\check{S}, \check{T} : \dot{Z} \rightarrow \dot{Z}$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \tilde{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \left\{ \tau + \alpha_* (\check{e}, \check{S}\check{e}) F(d_l(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & \quad + \alpha_* (\hat{y}, \check{T}\hat{y}) F(d_l(\check{T}\hat{y}, \check{S}\check{e})) \left. \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & \quad \left. + \eta_3 d_l(\check{e}, \check{T}\hat{y}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (63)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \tilde{r})} \cap \{c_{\check{n}}\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_3, \eta_4 > 0$ and $\eta_1 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda) \tilde{r}, \quad (64)$$

where $\lambda = ((\eta_1 + \eta_3)/(1 - \eta_3 - \eta_4))$ and $\eta_3 + \eta_4 \neq 1$. Then $\{c_{\check{n}}\}$ is a sequence in $\overline{B_{d_l}(c_0, \tilde{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{c_{\check{n}}\} \rightarrow v \in \overline{B_{d_l}(c_0, \tilde{r})}$. Also if inequality (63) holds for v and either $\alpha(c_{\check{n}}, v) \geq 1$ or $\alpha(v, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point v in $\overline{B_{d_l}(c_0, \tilde{r})}$.

If we take $\eta_3 = 0$ in Theorem 20, then we are left with the result.

Corollary 23. Let $(\dot{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$. Let $\tilde{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \tilde{r})} \subseteq \dot{Z}$, and let $\check{S}, \check{T} : \dot{Z} \rightarrow \dot{Z}$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \tilde{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \left\{ \tau + \alpha_* (\check{e}, \check{S}\check{e}) F(d_l(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & \quad + \alpha_* (\hat{y}, \check{T}\hat{y}) F(d_l(\check{T}\hat{y}, \check{S}\check{e})) \left. \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & \quad \left. + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (65)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \tilde{r})} \cap \{c_{\check{n}}\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_2, \eta_4 > 0$ and $\eta_1 + \eta_2 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda) \tilde{r}, \quad (66)$$

where $\lambda = ((\eta_1 + \eta_2)/(1 - \eta_4))$ and $1 - \eta_4 \neq 0$. Then $\{c_{\check{n}}\}$ is a sequence in $\overline{B_{d_l}(c_0, \tilde{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{c_{\check{n}}\} \rightarrow v \in \overline{B_{d_l}(c_0, \tilde{r})}$. Also if inequality (65) holds for v and either $\alpha(c_{\check{n}}, v) \geq 1$ or $\alpha(v, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point v in $\overline{B_{d_l}(c_0, \tilde{r})}$.

If we take $\eta_4 = 0$ in Theorem 20, then we are left with the result.

Corollary 24. Let $(\dot{Z}, d_l)$ be a complete DMS. Suppose that there exists a function $\alpha : \dot{Z} \times \dot{Z} \rightarrow [0, \infty)$. Let $\tilde{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \tilde{r})} \subseteq \dot{Z}$, and let $\check{S}, \check{T} : \dot{Z} \rightarrow \dot{Z}$ be the semi- α_* -dominated mappings on $\overline{B_{d_l}(c_0, \tilde{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \left\{ \tau + \alpha_* (\check{e}, \check{S}\check{e}) F(d_l(\check{S}\check{e}, \check{T}\hat{y})), \tau \right. \\ & \quad + \alpha_* (\hat{y}, \check{T}\hat{y}) F(d_l(\check{T}\hat{y}, \check{S}\check{e})) \left. \right\} \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) \right. \\ & \quad \left. + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) \right) \end{aligned} \quad (67)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \tilde{r})} \cap \{c_{\check{n}}\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$, where $\eta_1, \eta_2, \eta_3 > 0$ and $\eta_1 + \eta_2 + 2\eta_3 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda) \tilde{r}, \quad (68)$$

where $\lambda = ((\eta_1 + \eta_2 + \eta_3)/(1 - \eta_3))$ and $1 - \eta_3 \neq 0$. Then $\{c_{\check{n}}\}$ is a sequence in $\overline{B_{d_l}(c_0, \tilde{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, and $\{c_{\check{n}}\} \rightarrow v \in \overline{B_{d_l}(c_0, \tilde{r})}$. Also if inequality (67) holds for v and either $\alpha(c_{\check{n}}, v) \geq 1$ or $\alpha(v, c_{\check{n}}) \geq 1$ for all $\check{n} \in N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have common fixed point v in $\overline{B_{d_l}(c_0, \tilde{r})}$.

5. Application to the Systems of Integral Equations

Theorem 25. Let $(\dot{Z}, d_l)$ be a complete DMS. Let $\tilde{r} > 0$, let $c_0 \in \overline{B_{d_l}(c_0, \tilde{r})} \subseteq \dot{Z}$, and let $\check{S}, \check{T} : \dot{Z} \rightarrow \dot{Z}$ be the dominated mappings on $\overline{B_{d_l}(c_0, \tilde{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} & \max \{ \tau + F(d_l(\check{S}\check{e}, \check{T}\hat{y})), \tau + F(d_l(\check{T}\hat{y}, \check{S}\check{e})) \} \\ & \leq F \left(\eta_1 d_l(\check{e}, \hat{y}) + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) \right. \\ & \quad \left. + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}) \cdot d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})} \right) \end{aligned} \quad (69)$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \tilde{r})} \cap \{c_n\}$ with $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, where $\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\tilde{r}, \quad (70)$$

where $\lambda = ((\eta_1 + \eta_2 + \eta_3)/(1 - \eta_3 - \eta_4))$ and $\eta_3 + \eta_4 \neq 1$. Then $\{c_n\}$ is a sequence in $\overline{B_{d_l}(c_0, \tilde{r})}$, for all $\check{n} \in N \cup \{0\}$ and $\{c_n\} \rightarrow v \in \overline{B_{d_l}(c_0, \tilde{r})}$. Then $\check{S}$ and $\check{T}$ have common fixed point v in $\overline{B_{d_l}(c_0, \tilde{r})}$.

Proof. The proof of the above theorem is the same as the theorem proven in the previous section. In this section, we discuss the application of fixed point Theorem 25 in form of Volterra type integral equations.

$$\begin{aligned} \check{u}(k) &= \int_0^k H_1(k, h, \check{u}(h)) dh, \\ \check{c}(k) &= \int_0^k H_2(k, h, \check{c}(h)) dh \end{aligned} \quad (71)$$

for all $k \in [0, 1]$. We find the solution of (71). Let $\dot{Z} = \dot{C}([0, 1], \mathbb{R}_+)$ be the set of all continuous functions on $[0, 1]$, endowed with the complete dislocated metric. For $\check{u} \in \dot{C}([0, 1], \mathbb{R}_+)$, define supremum norm as $\|\check{u}\|_\tau = \sup_{k \in [0, 1]} \{|\check{u}(k)|e^{-\tau k}\}$, where $\tau > 0$ is taken arbitrarily. Then define

$$d_\tau(\check{u}, \check{c}) = \sup_{k \in [0, 1]} \{|\check{u}(k) + \check{c}(k)|e^{-\tau k}\} = \|\check{u} + \check{c}\|_\tau \quad (72)$$

for all $\check{u}, \check{c} \in \dot{C}([0, 1], \mathbb{R}_+)$, and, with these settings, $(\dot{C}([0, 1], \mathbb{R}_+), d_\tau)$ becomes a complete DMS. $\square$

Now we prove the following theorem to ensure the existence of solution of integral equations.

Theorem 26. Assume that the following conditions are satisfied:

- (i) $H_1, H_2 : [0, 1] \times [0, 1] \times \dot{C}([0, 1], \mathbb{R}_+) \rightarrow \mathbb{R}$.

(ii) Define

$$\begin{aligned} \check{S}\check{u}(k) &= \int_0^k H_1(k, h, \check{u}(h)) dh, \\ \check{T}\check{c}(k) &= \int_0^k H_2(k, h, \check{c}(h)) dh. \end{aligned} \quad (73)$$

Suppose that there exists $\tau > 0$, such that

$$|H_1(k, h, \check{u}) + H_2(k, h, \check{c})| \leq \frac{\tau N(\check{u}, \check{c})}{(\tau \sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1)^2} \quad (74)$$

for all $k, h \in [0, 1]$ and $\check{u}, \check{c} \in \dot{C}([0, 1], \mathbb{R})$, where

$$\begin{aligned} N(\check{u}, \check{c}) &= \eta_1 [|\check{u}(k) + \check{c}(k)|] + \eta_2 [|\check{u}(k) + \check{S}\check{u}(k)|] \\ & \quad + \eta_3 [|\check{u}(k) + \check{T}\check{c}(k)|] \\ & \quad + \eta_4 \frac{[|\check{u}(k) + \check{S}\check{u}(k)|]^2 \cdot [|\check{c}(k) + \check{T}\check{c}(k)|]}{1 + [|\check{u}(k) + \check{c}(k)|]^2}, \end{aligned} \quad (75)$$

where $\eta_1, \eta_2, \eta_3, \eta_4 \geq 0$ and $\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$. Then integral equations (71) have a solution.

Proof. By assumption (ii),

$$\begin{aligned} & |\check{S}\check{u}(k) + \check{T}\check{c}(k)| \\ &= \int_0^k |H_1(k, h, \check{u}(h)) + H_2(k, h, \check{c}(h))| dh, \\ &\leq \int_0^k \frac{\tau}{(\tau \sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1)^2} ([N(\check{u}, \check{c})] e^{-\tau h}) e^{\tau h} dh, \\ &\leq \int_0^k \frac{\tau}{(\tau \sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1)^2} \|N(\check{u}, \check{c})\|_\tau e^{\tau h} dh, \\ &\leq \frac{\tau \|N(\check{u}, \check{c})\|_\tau}{(\tau \sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1)^2} \int_0^k e^{\tau h} dh, \\ &\leq \frac{\|N(\check{u}, \check{c})\|_\tau}{(\tau \sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1)^2} e^{\tau k}. \end{aligned} \quad (76)$$

This implies that

$$\begin{aligned} |\check{S}\check{u}(k) + \check{T}\check{c}(k)| e^{-\tau k} &\leq \frac{\|N(\check{u}, \check{c})\|_\tau}{(\tau \sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1)^2}, \\ \|\check{S}\check{u}(k) + \check{T}\check{c}(k)\|_\tau &\leq \frac{\|N(\check{u}, \check{c})\|_\tau}{(\tau \sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1)^2}, \\ \frac{\tau \sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1}{\sqrt{\|N(\check{u}, \check{c})\|_\tau}} &\leq \frac{1}{\sqrt{\|\check{S}\check{u}(k) + \check{T}\check{c}(k)\|_\tau}}, \\ \tau + \frac{1}{\sqrt{\|N(\check{u}, \check{c})\|_\tau}} &\leq \frac{1}{\sqrt{\|\check{S}\check{u}(k) + \check{T}\check{c}(k)\|_\tau}}, \end{aligned} \quad (77)$$

which further implies that

$$\tau - \frac{1}{\sqrt{\|\tilde{S}\tilde{u}(k) + \tilde{T}\tilde{c}(k)\|_\tau}} \leq \frac{-1}{\sqrt{\|N(\tilde{u}, \tilde{c})\|_\tau}}. \quad (78)$$

So all the conditions of Theorem 25 are satisfied for $(\tilde{c}) = -1/\sqrt{\tilde{c}}; \tilde{c} > 0$; and $d_\tau(\tilde{u}, \tilde{c}) = \|\tilde{u} + \tilde{c}\|_\tau$. Hence integral equations given in (71) have a unique common solution. $\square$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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