Research Article

Approximation Property of the Stationary Stokes Equations with the Periodic Boundary Condition

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In this paper, we will consider the stationary Stokes equations with the periodic boundary condition and we will study approximation property of the solutions by using the properties of the Fourier series. Finally, we will discuss that our estimation for approximate solutions is optimal.

1. Introduction

The study of stability problems for various functional equations originated from a famous talk presented by Ulam in 1940. In this talk, he discussed a problem concerning the stability of homomorphisms. And Obłoza [1, 2] first investigated the Hyers-Ulam stability of the linear differential equations which have the form $y'(x) + g(x)y(x) = r(x)$. Thereafter, a number of mathematicians have dealt with this subject for different types of differential equations (see [3–8]).

For an open interval $I = (a, b)$ of $\mathbb{R}$ with $-\infty \leq a < b \leq +\infty$, we consider the linear differential equation of $n$th order

$$F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x) = 0$$

for all $x \in I$, where $y : I \to \mathbb{C}$ is an $n$ times continuously differentiable function.

We say that the differential equation (1) satisfies the Hyers-Ulam stability provided the following statement is true for any $\varepsilon > 0$: if an $n$ times continuously differentiable function $y : I \to \mathbb{C}$ satisfies the differential inequality

$$|F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x)| \leq \varepsilon$$

for all $x \in I$, then there exists a solution $y_0 : I \to \mathbb{C}$ to (1) such that

$$|y(x) - y_0(x)| \leq K(\varepsilon)$$

for all $x \in I$, where $K(\varepsilon)$ depends on $\varepsilon$ only and satisfies $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$.

Recently, several mathematicians investigated the Hyers-Ulam stability for the partial differential equations. One can refer to [9–14].

In this paper, we will investigate approximate properties of the solutions for the stationary Stokes equations with the periodic boundary condition. The stationary Stokes problem associated with the space periodicity condition is the following one: For a given $f$, find $u$ and $p$ such that

$$-\Delta u + \nabla p = f \quad \text{in } Q$$

$$\nabla \cdot u = 0 \quad \text{in } Q$$

$$u(x + Le_i) = u(x) \quad \forall x \in \mathbb{R}^n,$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, $L$ is the period in the $i$-th direction, and $Q = (0, L)^n$ is the cube of the period.

The advantage of the boundary condition (6) is that it leads to a simple functional setting, while many of the mathematical difficulties remain unchanged. In fact, in the next section we will introduce in detail the corresponding functional setting of the problem.

Finally, we will discuss that our estimation for approximate solutions is optimal.
2. Preliminary Results

In this section, we will introduce the useful functional settings and preliminary results for the solutions of the stationary Stokes equations with the periodic boundary condition. For the materials of this section, we totally refer to the book by Roger Temam [15]. So if the reader wants to understand more deeply, one can refer to this book.

For the functional spaces of the solutions, we will consider the Lebesgue space \( L^2(\mathbb{R}^n) \) with the periodic boundary condition. We set by \( H_p^m(\Omega) \) the Sobolev space of functions which are in \( L^2(\Omega) \), with all their derivatives of order \( \leq m \). Then, \( H_p^m(\Omega) \) is a Hilbert space with the inner product and the norm

\[
(u, v)_m = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v),
\]

\[
|u|_m = \left[ (u, u)_m \right]^{1/2},
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \alpha_i \in \mathbb{N}_0 \), \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), and

\[
D^n = D^{n_1} \cdots D^{n_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.
\]

We also set by \( H^m(\mathbb{R}^n) \), \( m \in \mathbb{N}_0 \), the space of functions which are periodic with period \( L \)

\[
u(x + Le) = \nu(x) \quad \forall i = 1, \ldots, n.
\]

For \( m = 0 \), \( H^0(\mathbb{R}^n) \) means simply \( L^2(\mathbb{R}^n) \). Then, for an arbitrary \( m \in \mathbb{N} \), \( H_p^m(\Omega) \) is a Hilbert space with the inner product

\[
(u, v)_m = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha u(x) D^\alpha v(x) \, dx
\]

And the functions in \( H_p^m(\Omega) \) are characterized by their Fourier series expansion

\[
H_p^m(Q) = \left\{ u : u = \sum_{k \in \mathbb{Z}^n} c_k e^{2\pi i k \cdot x/L}, \ c_k \right\}
\]

\[
= c_{-k}, \sum_{k \in \mathbb{Z}^n} |k|^{2m} |c_k|^2 < \infty \right\}.
\]

We also denote

\[
H_p^m(Q) = \left\{ u \in H_p^m(\Omega) \right\} \text{ of type (11) : } c_0 = 0 \right\}.
\]

Then, for \( m \in \mathbb{N} \), \( H_p^m(\Omega) \) is a Hilbert space for the norm

\[
\sum_{k \in \mathbb{Z}^n} |k|^{2m} |c_k|^2 \text{, and } H_p^m(\Omega) \text{ and } H_p^m(Q) \text{ are in duality for all } m \in \mathbb{N}.
\]

Now, we introduce two important function spaces,

\[
\mathbb{V} = \left\{ u \in H^1_p(\Omega) : \nabla \cdot u = 0 \text{ in } \mathbb{R}^n \right\},
\]

\[
\mathbb{H} = \left\{ u \in H^0_p(\Omega) : \nabla \cdot u = 0 \text{ in } \mathbb{R}^n \right\}.
\]

where \( H^m_p(Q) = \{H^m_p(\Omega)\}^n \). We also introduce the inner product and the norm

\[
((u, v)) = \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right),
\]

\[
\|u\| = \left[ (u, u) \right]^{1/2}.
\]

One notes that \( \mathbb{V} \) is a Hilbert space with this norm. Also, the dual \( \mathbb{V}' \) of \( \mathbb{V} \) is

\[
\mathbb{V}' = \left\{ u \in H^{-1}_p(Q) = \left( H^1_p(Q) \right)' : \nabla \cdot u = 0 \text{ in } \mathbb{R}^n \right\};
\]

\[
\|\cdot\|_{\mathbb{V}'} \text{ will denote the dual norm of } \|\cdot\| \text{ on } \mathbb{V}'.
\]

For the boundary value, due to trace theorem we have that \( u \in \mathbb{V} \) if and only if its restriction \( u|_Q \) to \( Q \) belongs to

\[
\left\{ u \in H^1(Q) : \nabla \cdot u = 0, \ \nu|_{\Gamma_{j+n}} = 0 \right\}
\]

where we have numbered the faces \( \Gamma_1, \ldots, \Gamma_{2n} \) of \( Q \) as follows:

\[
\Gamma_j = \partial Q \cap \{ x_j = 0 \},
\]

\[
\Gamma_{j+n} = \partial Q \cap \{ x_j = L \},
\]

and \( \nu|_{\Gamma_j} \) is an improper notation for the trace of \( \nu \) on \( \Gamma_j \). And \( u \in \mathbb{H} \) if and only if \( u \) belongs to

\[
\left\{ u \in L^2(Q) : \nabla \cdot u = 0, \ \nu \cdot \gamma \mid_{\Gamma_{j+n}} = -\nu \cdot \gamma \mid_{\Gamma_j} \right\}
\]

Now, let us look at the stationary Stokes problem (4) with the periodic boundary condition (6); given \( f \in H^1_p(Q) \) or \( H^{-1}_p(Q) \), find \( u \in H^1_p(Q) \) and \( p \in L^2(Q) \) such that

\[
-\Delta u + \nabla p = f \quad \text{in } Q,
\]

\[
\nabla \cdot u = 0 \quad \text{in } Q.
\]

Here, to solve the above problem we use the Fourier series. Let us introduce the Fourier expansions of \( u, p, \) and \( f \);

\[
u = \sum_{k \in \mathbb{Z}^n} u_k e^{2\pi i k \cdot x/L},
\]

\[
p = \sum_{k \in \mathbb{Z}^n} p_k e^{2\pi i k \cdot x/L},
\]

\[
f = \sum_{k \in \mathbb{Z}^n} f_k e^{2\pi i k \cdot x/L}.
\]

Equation (20) reduces for every \( k \neq 0 \) to

\[
4\pi^2 |k|^2 u_k + \frac{2\pi i k}{L} p_k = f_k
\]

and

\[
k \cdot u_k = 0.
\]
Taking the scalar product of (22) with \( k \) and using (23) we find the \( p_k \)'s:

\[
p_k = \frac{Lk \cdot f_k}{2\pi i |k|^2} \quad \text{for} \quad k \in \mathbb{Z}^n, \quad k \neq 0
\]

(24)

then (22) provided the \( u_k \)'s;

\[
u_k = \frac{L^2}{4\pi^2 |k|^2} \left( f_k - \frac{(k \cdot f_k) k}{|k|^2} \right)
\]

(25)

for \( k \in \mathbb{Z}^n, \quad k \neq 0 \).

By definition (11) of \( H^m_p(\Omega) \), if \( f \in \dot{H}^0_p(\Omega) \) then \( u \in \dot{H}^2_p(\Omega) \) and \( p \in \dot{H}^1_p(\Omega) \); if \( f \in \dot{H}^1_p(\Omega) \) then \( u \in \dot{H}^3_p(\Omega) \) and \( p \in \dot{H}^2_p(\Omega) \).

Now if \( f \) belongs to \( H \), then \( k \cdot f_k = 0 \) for every \( k \) so that \( p = 0 \) and \( u_k = f_k L^2 / 4\pi^2 |k|^2 \).

3. Approximate Properties for the Solutions

In this section, we will discuss approximate properties for the solutions of the stationary Stokes equations with the periodic boundary condition. In this paper, we will prove theorems for \( n = 2 \) while one can extend our result to \( \mathbb{R}^n \).

**Theorem 1.** Let the functions \( v \in \dot{H}^2_p(\Omega) \) and \( q \in \dot{H}^1_p(\Omega) \) satisfy the equations

\[
\begin{align*}
-\Delta v + \nabla q - f &= g \quad \text{in} \quad Q, \\
\nabla \cdot v &= 0 \quad \text{in} \quad Q,
\end{align*}
\]

(26)

where \( \| g \|_{L^2} \leq \varepsilon \) and \( f, g \in \dot{H}^0_p(\Omega) \). Then there exist \( u \in \dot{H}^2_p(\Omega) \) and \( p \in \dot{H}^1_p(\Omega) \) satisfying

\[
\begin{align*}
-\Delta u + \nabla p - f &= 0 \quad \text{in} \quad Q, \\
\nabla \cdot u &= 0 \quad \text{in} \quad Q
\end{align*}
\]

(27)

such that

\[
\begin{align*}
\| u - v \|_{\dot{H}^2} &\leq K_{j_i} \| g \|_{L^2} \leq K_{j_i} \varepsilon \quad \text{for} \quad i = 0, 1, 2, \\
\| p - q \|_{\dot{H}^1} &\leq M_i \| g \|_{L^2} \leq M_i \varepsilon \quad \text{for} \quad i = 0, 1
\end{align*}
\]

(28) (29)

for some constants \( K_{j_i} \) and \( M_i \).

**Proof.** For existence of the solutions \( u \in \dot{H}^2_p(\Omega) \) and \( p \in \dot{H}^1_p(\Omega) \), one can prove by (24) and (25). Next, to obtain (28) and (29) we denote the Fourier expansions of \( v, q, \) and \( g \) as the following:

\[
\begin{align*}
v &= \sum_{k \in \mathbb{Z}^n} v_k e^{2\pi i k \cdot x / L}, \\
q &= \sum_{k \in \mathbb{Z}^n} q_k e^{2\pi i k \cdot x / L}, \\
g &= \sum_{k \in \mathbb{Z}^n} g_k e^{2\pi i k \cdot x / L}.
\end{align*}
\]

(30)

Then, by (24), (25), and (26) we obtain

\[
\begin{align*}
u_k &= \frac{Lk \cdot (f_k + g_k)}{2\pi i |k|^2} \quad \text{for} \quad k \in \mathbb{Z}^2, \quad k \neq 0 \\
p_k &= \frac{Lk \cdot f_k}{2\pi i |k|^2} \quad \text{for} \quad k \in \mathbb{Z}^2, \quad k \neq 0
\end{align*}
\]

(31) (32)

Also, by (24), (25), and (27) we obtain

\[
\begin{align*}
u_k &= \frac{L^2}{4\pi^2 |k|^2} \left( f_k - \frac{(k \cdot f_k) k}{|k|^2} \right) \\
p_k &= \frac{L^2}{4\pi^2 |k|^2} \left( f_k - \frac{(k \cdot f_k) k}{|k|^2} \right)
\end{align*}
\]

(33) (34)

Then, for \( |k| \geq 2 \), from (35) we have

\[
\| u_k - v_k \| \leq \frac{L^2}{8\pi^2} |g_k|
\]

(36)

and for \( |k| = 1 \) and \( g_k = (g_k^1, g_k^2), \) we have

\[
\begin{align*}
g_k &= \frac{(k \cdot g_k) k}{|k|^2} = (0, g_k^2), \quad \text{for} \quad k = (1, 0), \\
g_k &= \frac{(k \cdot g_k) k}{|k|^2} = (0, g_k^1), \quad \text{for} \quad k = (-1, 0), \\
g_k &= \frac{(k \cdot g_k) k}{|k|^2} = (g_k^1, 0), \quad \text{for} \quad k = (0, 1), \\
g_k &= \frac{(k \cdot g_k) k}{|k|^2} = (g_k^1, 0), \quad \text{for} \quad k = (0, -1)
\end{align*}
\]

(37)

which implies

\[
\| u_k - v_k \| \leq \frac{L^2}{4\pi^2} |g_k|.
\]

(38)

Hence, we have

\[
\| u - v \|_{L^2} \leq \left[ \sum_{k \in \mathbb{Z}^n} |u_k - v_k|^2 \right]^{1/2} \leq \frac{L^2}{4\pi^2} \| g \|_{L^2} \leq K_{j_i} \varepsilon
\]

(39)
Hence, our estimation for
\[ \|p - q\|_{L^2} = \left( \sum_{k \in Z^n} |p_k - q_k|^2 \right)^{1/2} \leq \frac{L}{2\pi} \|g\|_{L^2} \leq M_1 \varepsilon. \] (40)

Similarly, for \( H^1 \)-norm of \( u - v \) and \( p - q \), we obtain
\[ \|u - v\|_{H^1} = \left( \sum_{k \in Z^n} |k|^2 |u_k - v_k|^2 \right)^{1/2} \leq \frac{L^2}{4\pi^2} \|g\|_{L^2} \leq K_2 \varepsilon \] (41)
and
\[ \|p - q\|_{H^1} = \left( \sum_{k \in Z^n} |k|^4 |p_k - q_k|^2 \right)^{1/2} \leq \frac{L^2}{2\pi} \|g\|_{L^2} \leq M_2 \varepsilon. \] (42)

Also, for \( H^2 \)-norm of \( u - v \), we have
\[ \|u - v\|_{H^2} = \left( \sum_{k \in Z^n} |k|^4 |u_k - v_k|^2 \right)^{1/2} \leq \frac{L^2}{4\pi^2} \|g\|_{L^2} \leq K_3 \varepsilon. \] (43)

Therefore, by (39)–(43), we complete the proof. \( \square \)

**Remark 2.** We consider the function \( g \) as \( g_k = (0, 0) \) for \( k \neq (1, 0) \) and \( g_k = (0, 1) \) for \( k = (1, 0) \). Then, we have
\[ \|u - v\|_{L^2} = \left( \sum_{k \in Z^n} |u_k - v_k|^2 \right)^{1/2} = L^2 \|g\|_{L^2}, \]
\[ \|u - v\|_{H^1} = \left( \sum_{k \in Z^n} |k|^2 |u_k - v_k|^2 \right)^{1/2} = L^2 \|g\|_{L^2}, \] (44)
\[ \|u - v\|_{H^2} = \left( \sum_{k \in Z^n} |k|^4 |u_k - v_k|^2 \right)^{1/2} = L^2 \|g\|_{L^2}. \]

And we consider the function \( g \) as \( g_k = (0, 0) \) for \( k \neq (1, 0) \) and \( g_k = (1, 0) \) for \( k = (1, 0) \). Then we have
\[ \|p - q\|_{L^2} = \left( \sum_{k \in Z^n} |p_k - q_k|^2 \right)^{1/2} = \frac{L}{2\pi} \|g\|_{L^2}, \] (45)
\[ \|p - q\|_{H^1} = \left( \sum_{k \in Z^n} |k|^4 |p_k - q_k|^2 \right)^{1/2} = \frac{L}{2\pi} \|g\|_{L^2}. \]

Hence, our estimation for \( K_i \) and \( M_i \) is optimal.

**Corollary 3.** Let the functions \( v \in H^1_p(Q) \) and \( q \in H^1_p(Q) \) satisfy the equations
\[ -\Delta v + \nabla q - f = g \quad \text{in } Q, \]
\[ \nabla \cdot v = 0 \quad \text{in } Q, \] (46)
where \( f \in H^1 \) and \( g \in H^1 \) with \( \|g\|_{L^2} \leq \varepsilon \). Then there exist \( u \in H^1_p(Q) \) and \( p \in H^1_p(Q) \) satisfying
\[ -\Delta u + \nabla p - f = 0 \quad \text{in } Q, \]
\[ \nabla \cdot u = 0 \quad \text{in } Q, \] (47)
and
\[ \|u - v\|_{L^2} \leq K_i \|g\|_{L^2} \leq K_i \varepsilon \quad \text{for } i = 0, 1, 2, \]
\[ \|p - q\|_{L^2} \leq M_i \|g\|_{L^2} \leq M_i \varepsilon \quad \text{for } i = 0, 1. \] (48)

**Corollary 4.** Let the functions \( v \in H^1_p(Q) \) and \( q \in H^1_p(Q) \) satisfy the equations
\[ -\Delta v + \nabla q - f = g \quad \text{in } Q, \]
\[ \nabla \cdot v = 0 \quad \text{in } Q, \] (49)
where \( f, g \in H \) with \( \|g\|_{L^2} \leq \varepsilon \). Then there exist \( u \in H^1_p(Q) \) and \( p \in H^1_p(Q) \) satisfying
\[ -\Delta u + \nabla p - f = 0 \quad \text{in } Q, \]
\[ \nabla \cdot u = 0 \quad \text{in } Q, \] (50)
and
\[ \|u - v\|_{L^2} \leq K_i \|g\|_{L^2} \leq K_i \varepsilon \quad \text{for } i = 0, 1, 2, \]
\[ \|p - q\|_{L^2} = 0 \quad \text{for } i = 0, 1. \] (51)

**Corollary 5.** Let the functions \( v \in H^1_p(Q) \) and \( q \in H^1_p(Q) \) satisfy the equations
\[ -\Delta v + \nabla q - f = g \quad \text{in } Q, \]
\[ \nabla \cdot v = 0 \quad \text{in } Q, \] (52)
where \( f \in H^1_p(Q) \) and \( g \in H \) with \( \|g\|_{L^2} \leq \varepsilon \). Then there exist \( u \in H^1_p(Q) \) and \( p \in H^1_p(Q) \) satisfying
\[ -\Delta u + \nabla p - f = 0 \quad \text{in } Q, \]
\[ \nabla \cdot u = 0 \quad \text{in } Q, \] (53)
and
\[ \|u - v\|_{L^2} \leq K_i \|g\|_{L^2} \leq K_i \varepsilon \quad \text{for } i = 0, 1, 2, \]
\[ \|p - q\|_{L^2} = 0 \quad \text{for } i = 0, 1. \] (54)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
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