Research Article

Fixed Points of $\mathcal{L}$ -Fuzzy Mappings in Ordered b -Metric Spaces

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We prove common fixed point theorems for weakly commuting and occasionally coincidentally idempotent $\mathcal{L}$ -fuzzy mappings in ordered b -metric spaces. We also obtain common fixed point for pair of mapping satisfying (JCLR) property. An application to integral type and usual contractive condition is given.

1. Introduction

In 1967, Goguen [1] introduced the notion of $\mathcal{L}$ -fuzzy sets as a generalization of fuzzy sets [2]. Afterward, Heilpern [3] gave the concept of fuzzy mappings and proved fixed point theorems for fuzzy contractive mappings in metric linear spaces as a generalization of Nadler [4] contraction principle. Subsequently, several authors studied the generalizations/extensions and applications of these results; in many papers sufficient conditions for the existence of fixed point for fuzzy and $\mathcal{L}$ -fuzzy contractive mappings in metric spaces and b -metric spaces are obtained (see [5–8]). Rashid et al. [9] proved common $\mathcal{L}$ -fuzzy fixed point theorem in complete metric spaces. They extended the results of Heilpern [3] into $\mathcal{L}$ -fuzzy mappings in metric spaces. In this paper, we prove some common fixed point theorems for $\mathcal{L}$ -fuzzy mappings in ordered b -metric spaces. An application to integral type and usual contractive condition is also given, we will generalize and extend results [10, 11].

2. Preliminaries

Definition 1 (see [12]). Let X be a nonempty set. A mapping $d : X \times X \rightarrow [0, \infty)$ is called b -metric if there exists a real number $b \geq 1$ such that, for every $x, y, z \in X$, we have

$$(d_1) \quad d(x, y) = 0 \Leftrightarrow x = y,$$

$$(d_2) \quad d(x, y) = d(y, x),$$

$$(d_3) \quad d(x, z) \leq b\{d(x, y) + d(y, z)\}.$$

In this case, the pair (X, d) is called a b -metric space.

Definition 2 (see [13]). Let (X, d) be a b -metric space. A sequence $\{x_n\}$ in X is called

(i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$,

(ii) Cauchy if and only if $d(x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \infty$.

A b -metric space is said to be complete if and only if each Cauchy sequence in this space is convergent.

Let (X, d) be a b -metric space, let $CL(X)$ denote the collection of all nonempty closed subsets of X , let $W(X)$ be the set of all fuzzy sets of X , where its α -level sets are nonempty compact subsets of X , and let $\mathbb{R}^+$ be the set of nonnegative real numbers; define the metrics $d : X \times CL(X) \rightarrow \mathbb{R}^+$ and $H : CL(X) \times CL(X) \rightarrow \mathbb{R}^+$ as $d(x, A) = \inf_{y \in A} d(x, y)$ and $H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(a, b)\}$. Notice that $d(a, B) \leq d(a, b)$ and $d(a, B) \leq H(A, B)$, for each $a \in A$ and $b \in B$. Let $f : X \rightarrow X$, $F : X \rightarrow CL(X)$, and $C(f, F)$ be the set of the coincidence points of (f, F) .

Lemma 3 (see [14]). Let (X, d) be a b -metric space, $A, B \in CL(X)$; then $d(a, B) \leq H(A, B)$ for all $a \in A$.

Definition 4 (see [15]). A partially ordered set consists of a set X and a binary relation $\leq$ on X which satisfies the following conditions for all $x, y, z \in X$:

- (1) $x \leq x$ (reflexivity).
- (2) If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).
- (3) If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

A set with a partial order $\leq$ is called a partially ordered set. Let $(X, \leq)$ be a partially ordered set and $x, y \in X$. Elements x and y are said to be comparable elements of X if either $x \leq y$ or $y \leq x$.

Definition 5. Let A and B be two nonempty subsets of $(X, \leq)$; the relation $\leq_1$ between A and B is defined as $A \leq_1 B$: if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

Definition 6 (see [1]). A function $A : X \rightarrow L$ is said to be an $\mathcal{L}$ -fuzzy set if A is a mapping from a nonempty set X into a complete distributive lattice L .

Definition 7 (see [9, 10, 16]). Let A be an $\mathcal{L}$ -fuzzy set in X . The $\alpha_{\mathcal{L}}$ -level set of A (denoted by $A_{\alpha_{\mathcal{L}}}$) is defined as

$$\begin{aligned} A_{\alpha_{\mathcal{L}}} &= \{x \in X : \alpha_{\mathcal{L}} \leq_L A(x)\} \quad \text{if } \alpha_{\mathcal{L}} \in L \setminus \{0_{\mathcal{L}}\}, \\ A_{0_{\mathcal{L}}} &= \overline{\{x \in X : 0_{\mathcal{L}} \leq_L A(x)\}}, \end{aligned} \quad (1)$$

where $\overline{B}$ denotes the closure of the set B .

Definition 8 (see [9]). Let X and Y be two arbitrary nonempty sets and $\mathfrak{F}_{\mathcal{L}}(Y)$ denote the collection of all $\mathcal{L}$ -fuzzy sets in Y . A mapping F is called $\mathcal{L}$ -fuzzy mapping if F is a mapping from X into $\mathfrak{F}_{\mathcal{L}}(Y)$. An $\mathcal{L}$ -fuzzy mapping F is an $\mathcal{L}$ -fuzzy subset on $X \times Y$ with membership function $F(x)(y)$. The function $F(x)(y)$ is the grade of membership of y in $F(x)$.

Definition 9 (see [9]). Let F, G be $\mathcal{L}$ -fuzzy mappings from an arbitrary nonempty set X into $\mathfrak{F}_{\mathcal{L}}(X)$. A point $z \in X$ is called an $\mathcal{L}$ -fuzzy fixed point of F if $z \in \{Fz\}_{\alpha_{\mathcal{L}}}$, where $\alpha_{\mathcal{L}} \in L \setminus \{0_{\mathcal{L}}\}$. The point $z \in X$ is called a common $\mathcal{L}$ -fuzzy fixed point of F and G if $z \in \{Fz\}_{\alpha_{\mathcal{L}}} \cap \{Gz\}_{\alpha_{\mathcal{L}}}$.

Remark 10 (see [10, 16]). If $\alpha_{\mathcal{L}} = 1_{\mathcal{L}}$ then the $\mathcal{L}$ -fuzzy fixed point is just the fixed point for the L -fuzzy mapping.

Definition 11 (see [17]). Let (X, d) be a metric space, $f : X \rightarrow X$, and $F : X \rightarrow CL(X)$. Then the pair (f, F) is said to have (CLR) property if there exists a sequence $\{x_n\}$ in X and $A \in CL(X)$ such that $\lim_{n \rightarrow \infty} fx_n = u \in A = \lim_{n \rightarrow \infty} Fx_n$, with $u = fv$, for some $u, v \in X$.

Definition 12 (see [18]). Let (X, d) be a metric space, $Y \subseteq X$, $f : X \rightarrow X$, and $F : X \rightarrow CL(X)$. Two mappings (f, F) are said to be occasionally coincidentally idempotent if $f^2x = fx$ for some $x \in C(f, F)$.

Definition 13 (see [19]). Let (X, d) be a metric space, $f, g : X \rightarrow X$, and $F, G : X \rightarrow W(X)$. Let $x_0 \in X$; if there exists a sequence $\{y_n\}$ in X such that

$$\begin{aligned} \{y_{2n+1}\} &= \{gx_{2n+1}\} \subset Fx_{2n}, \\ \{y_{2n+2}\} &= \{fx_{2n+2}\} \subset Gx_{2n+1}, \end{aligned} \quad (2)$$

then $O(F, G, f, g, x_0)$ is called the orbit for the mappings (F, G, f, g) .

Definition 14 (see [19]). Metric space X is called x_0 joint orbitally complete, if every Cauchy sequence of each orbit at x_0 is convergent in X .

Definition 15 (see [20]). Let f be a mapping from a subset Y of a metric space (X, d) into X and $F : Y \rightarrow CL(X)$; then f is said to be F -weakly commuting at $x \in Y$ if $f^2x \in Ffx$.

3. Definitions in b -Metric Spaces

Let (X, d) be a b -metric space, $Y \subseteq X$, $f, g : Y \rightarrow X$ and F, G are two $\mathcal{L}$ -fuzzy mappings from Y into $\mathfrak{F}_{\mathcal{L}}(X)$ such that, for each $x \in Y$, $\alpha_{\mathcal{L}} \in L \setminus \{0_{\mathcal{L}}\}$, $\{Fx\}_{\alpha_{\mathcal{L}}}$ and $\{Gx\}_{\alpha_{\mathcal{L}}}$ are nonempty closed subsets of X . Similar to [11], first we rewrite some notions in b -metric spaces as follows.

Definition 16. The pairs (f, F) and (g, G) are said to have (JCLR) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A, B \in CL(X)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} fx_n &= \lim_{n \rightarrow \infty} gy_n = u, \\ \lim_{n \rightarrow \infty} \{Fx_n\}_{\alpha_{\mathcal{L}}} &= A, \\ \lim_{n \rightarrow \infty} \{Gy_n\}_{\alpha_{\mathcal{L}}} &= B, \end{aligned} \quad (3)$$

with $u = fv = gw$, $u \in A \cap B$, for some $u, v, w \in Y$.

Definition 17. Two mappings f and F are said to be occasionally coincidentally idempotent if $f^2x = fx$ for some $x \in C(f, F)$.

Definition 18. Two mappings f and F are said to be F -weakly commuting at $x \in Y$ if $ffx \in \{Ffx\}_{\alpha_{\mathcal{L}}}$.

Let Φ be the family of all continuous mappings $\phi : [0, \infty)^6 \rightarrow [0, \infty)$ satisfying the following properties:

- (ϕ_1) F is nondecreasing in the 1st variable and nonincreasing in the 3rd, 4th, 5th, and 6th coordinate variables.
- (ϕ_2) There exists $h \in (0, 1)$ such that for every $u, v \geq 0$, $b \geq 1$ with $\phi(u, v, v, u, b(u+v), 0) \leq 0$ or $\phi(u, v, u, v, 0, b(u+v)) \leq 0$ implies $u \leq hv$.

Example 19. (i) $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h \max\{t_2, (t_3 + t_4)t_5t_6\}$.

(ii) $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - h \min\{t_2, (t_3 + t_4)/b, (t_5 + t_6)/b^2\}$, $b \geq 2$.

(iii) $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - ht_2$.

Let Ψ be the family of continuous mappings $\psi : [0, \infty)^3 \rightarrow \mathbb{R}$, which is nondecreasing in the first coordinate and satisfying the following condition for each $u, v \geq 0$: if $\psi(u, v, v) \leq 0$ or $\psi(u, v, u) \leq 0$, then $u \leq v$.

Example 20. (i) $\psi(t_1, t_2, t_3) = t_1 - \min\{t_2, (t_2 + t_3)/n\}$, $n \geq 2$.
(ii) $\psi(t_1, t_2, t_3) = t_1 - t_2$.
(iii) $\psi(t_1, t_2, t_3) = t_1 - (t_2 + t_3)/n$, $n \geq 2$.

4. Common Fixed Points in Ordered b -Metric Spaces

Theorem 21. Let (X, d) be a b -metric space, $Y \subseteq X$, and $\leq$ be a partial order defined on Y , $f, g : Y \rightarrow X$ such that $f(Y)$ and $g(Y)$ are closed. Suppose that F, G are two $\mathcal{L}$ -fuzzy mappings from Y into $\mathfrak{F}_{\mathcal{L}}(X)$ such that for each $x \in Y$ and $\alpha_{\mathcal{L}} \in L \setminus \{0_{\mathcal{L}}\}$, $\{Fx\}_{\alpha_{\mathcal{L}}}$ and $\{Gx\}_{\alpha_{\mathcal{L}}}$ are nonempty closed subsets of X which satisfy the following conditions:

- (1) $\{Fx\}_{\alpha_{\mathcal{L}}} \leq_1 g(Y)$ and $\{Gx\}_{\alpha_{\mathcal{L}}} \leq_1 f(Y)$.
- (2) $gy \in \{Fx\}_{\alpha_{\mathcal{L}}}$ or $fy \in \{Gx\}_{\alpha_{\mathcal{L}}}$ implies $x \leq y$.
- (3) If $y_n \rightarrow y$, then $y_n \leq y$ for all $n \in \mathbb{N}$.
- (4) (f, F) and (g, G) are weakly commuting and occasionally coincidentally idempotent.
- (5) For all comparable elements $x, y \in Y$ and $E > 0$, there exists $\phi \in \Phi$ such that

$$\phi(Q) + E \min(Q) \leq 0, \quad (4)$$

where

$$Q = (H(\{Fx\}_{\alpha_{\mathcal{L}}}, \{Gy\}_{\alpha_{\mathcal{L}}}), d(fx, gy), d(fx, \{Fx\}_{\alpha_{\mathcal{L}}}), d(gy, \{Gy\}_{\alpha_{\mathcal{L}}}), d(fx, \{Gy\}_{\alpha_{\mathcal{L}}}), d(gy, \{Fx\}_{\alpha_{\mathcal{L}}})) \quad (5)$$

- (6) One of $f(Y)$ or $g(Y)$ is x_0 joint orbitally complete for some $x_0 \in Y$;

then f and g have a common fixed point, while F and G have a common L -fuzzy fixed point

Proof. Let $x_0 \in X$ and $y_0 = fx_0$; then by (1), there exist $x_1, x_2 \in Y$ such that $y_1 = gx_1 \in \{Fx_0\}_{\alpha_{\mathcal{L}}}$ and $y_2 = fx_2 \in \{Gx_1\}_{\alpha_{\mathcal{L}}}$, from (2), $x_0 \leq x_1 \leq x_2$. Now, $d(y_1, y_2) = d(gx_1, fx_2) \leq H(\{Fx_0\}_{\alpha_{\mathcal{L}}}, \{Gx_1\}_{\alpha_{\mathcal{L}}})$; then

$$\begin{aligned} & \phi(H(\{Fx_0\}_{\alpha_{\mathcal{L}}}, \{Gx_1\}_{\alpha_{\mathcal{L}}}), d(fx_0, gx_1), \\ & d(fx_0, \{Fx_0\}_{\alpha_{\mathcal{L}}}), d(gx_1, \{Gx_1\}_{\alpha_{\mathcal{L}}}), \\ & d(fx_0, \{Gx_1\}_{\alpha_{\mathcal{L}}}), 0) \leq 0, \end{aligned} \quad (6)$$

but

$$\begin{aligned} & \phi(d(y_1, y_2), d(y_0, y_1), d(y_0, y_1), d(y_1, y_2), \\ & b(d(y_0, y_1) + d(y_1, y_2)), 0) \\ & \leq \phi(H(\{Fx_0\}_{\alpha_{\mathcal{L}}}, \{Gx_1\}_{\alpha_{\mathcal{L}}}), d(fx_0, gx_1), \\ & d(fx_0, \{Fx_0\}_{\alpha_{\mathcal{L}}}), d(gx_1, \{Gx_1\}_{\alpha_{\mathcal{L}}}), \\ & d(fx_0, \{Gx_1\}_{\alpha_{\mathcal{L}}}), 0) \leq 0; \end{aligned} \quad (7)$$

then from the property $\phi(u, v, v, u, b(u + v), 0) \leq 0$ which implies $u \leq hv$, there exists $h \in (0, 1)$ such that $d(y_1, y_2) \leq hd(y_0, y_1)$. Again, $y_2 = fx_2 \in \{Gx_1\}_{\alpha_{\mathcal{L}}}$, $y_3 = gx_3 \in \{Fx_2\}_{\alpha_{\mathcal{L}}}$, $x_2 \leq x_3$, and $d(y_3, y_2) = d(gx_3, fx_2) \leq H(\{Fx_2\}_{\alpha_{\mathcal{L}}}, \{Gx_1\}_{\alpha_{\mathcal{L}}})$; then

$$\begin{aligned} & \phi(H(\{Fx_2\}_{\alpha_{\mathcal{L}}}, \{Gx_1\}_{\alpha_{\mathcal{L}}}), d(fx_2, gx_1), \\ & d(fx_2, \{Fx_2\}_{\alpha_{\mathcal{L}}}), d(gx_1, \{Gx_1\}_{\alpha_{\mathcal{L}}}), 0, \\ & d(gx_1, \{Fx_2\}_{\alpha_{\mathcal{L}}})) \leq 0. \end{aligned} \quad (8)$$

Also

$$\begin{aligned} & \phi(d(y_2, y_3), d(y_1, y_2), d(y_2, y_3), d(y_1, y_2), 0, \\ & b(d(y_1, y_2) + d(y_2, y_3))) \\ & \leq \phi(H(\{Fx_2\}_{\alpha_{\mathcal{L}}}, \{Gx_1\}_{\alpha_{\mathcal{L}}}), d(fx_2, gx_1), \\ & d(fx_2, \{Fx_2\}_{\alpha_{\mathcal{L}}}), d(gx_1, \{Gx_1\}_{\alpha_{\mathcal{L}}}), 0, \\ & d(gx_1, \{Fx_2\}_{\alpha_{\mathcal{L}}})) \leq 0; \end{aligned} \quad (9)$$

then from the property $\phi(u, v, u, v, 0, b(u + v)) \leq 0$ which implies $u \leq hv$, there exists $h \in (0, 1)$ such that $d(y_2, y_3) \leq hd(y_1, y_2) \leq h^2 d(y_0, y_1)$. By induction we obtain $d(y_n, y_{n+1}) \leq h^n d(y_0, y_1)$. Continuing in this way, we obtain an orbit $O(F, G, f, g, x_0)$ such that $y_{2n+1} = gx_{2n+1} \in \{Fx_{2n}\}_{\alpha_{\mathcal{L}}}$ and $y_{2n+2} = fx_{2n+2} \in \{Gx_{2n+1}\}_{\alpha_{\mathcal{L}}}$. Further,

$$\begin{aligned} d(y_n, y_m) & \leq bd(y_n, y_{n+1}) + b^2 d(y_{n+1}, y_{n+2}) + \cdots \\ & \quad + b^{m-n-1} d(y_{m-1}, y_m) \\ & \leq bd(y_n, y_{n+1}) + b^2 d(y_{n+1}, y_{n+2}) + \cdots \\ & \quad + b^{m-n} d(y_{m-1}, y_m) \\ & \leq bh^n d(y_0, y_1) + b^2 h^{n+1} d(y_0, y_1) + \cdots \\ & \quad + b^{m-n} h^{m-1} d(y_0, y_1) \\ & = \frac{bh^n}{1 - bh} d(y_0, y_1). \end{aligned} \quad (10)$$

Therefore, $\lim_{n, m \rightarrow \infty} d(y_n, y_m) = 0$. Hence, $\{y_n\}$ is a Cauchy sequence. As $\{y_{2n+2}\}$ is a Cauchy sequence in $f(Y)$, and $f(Y)$ is joint orbitally complete, therefore, there exists $z \in X$ such

that $y_{2n+2} \rightarrow z = fu$, for $u \in Y$. Next, we show that $z \in \{Fu\}_{\alpha_{\mathcal{F}}}$. Since

$$\phi(Q) + E \min(Q) \leq 0, \quad (11)$$

where

$$\begin{aligned} Q = & \left(H\left(\{Fu\}_{\alpha_{\mathcal{F}}}, \{Gx_{2n+1}\}_{\alpha_{\mathcal{F}}}\right), d(fu, gx_{2n+1}), \right. \\ & d(fu, \{Fu\}_{\alpha_{\mathcal{F}}}), d(gx_{2n+1}, \{Gx_{2n+1}\}_{\alpha_{\mathcal{F}}}), \\ & \left. d(fu, \{Gx_{2n+1}\}_{\alpha_{\mathcal{F}}}), d(gx_{2n+1}, \{Fu\}_{\alpha_{\mathcal{F}}}) \right), \end{aligned} \quad (12)$$

then

$$\begin{aligned} & \phi\left(H\left(\{Fu\}_{\alpha_{\mathcal{F}}}, \{Gx_{2n+1}\}_{\alpha_{\mathcal{F}}}\right), d(fu, gx_{2n+1}), \right. \\ & d(fu, \{Fu\}_{\alpha_{\mathcal{F}}}), d(gx_{2n+1}, \{Gx_{2n+1}\}_{\alpha_{\mathcal{F}}}), \\ & \left. d(fu, \{Gx_{2n+1}\}_{\alpha_{\mathcal{F}}}), d(gx_{2n+1}, \{Fu\}_{\alpha_{\mathcal{F}}})\right) \leq 0, \\ & \phi\left(d(y_{2n+2}, \{Fu\}_{\alpha_{\mathcal{F}}}), d(z, y_{2n+1}), d(z, \{Fu\}_{\alpha_{\mathcal{F}}}), \right. \\ & d(y_{2n+1}, y_{2n+2}), d(z, y_{2n+2}), bd(y_{2n+1}, \{Fu\}_{\alpha_{\mathcal{F}}}) \\ & \left. \leq \phi\left(H\left(\{Fu\}_{\alpha_{\mathcal{F}}}, \{Gx_{2n+1}\}_{\alpha_{\mathcal{F}}}\right), d(fu, gx_{2n+1}), \right. \right. \\ & d(fu, \{Fu\}_{\alpha_{\mathcal{F}}}), d(gx_{2n+1}, \{Gx_{2n+1}\}_{\alpha_{\mathcal{F}}}), \\ & \left. \left. d(fu, \{Gx_{2n+1}\}_{\alpha_{\mathcal{F}}}), d(gx_{2n+1}, \{Fu\}_{\alpha_{\mathcal{F}}})\right) \leq 0. \right. \end{aligned} \quad (13)$$

As $n \rightarrow \infty$

$$\begin{aligned} & \phi\left(d(z, \{Fu\}_{\alpha_{\mathcal{F}}}), 0, d(z, \{Fu\}_{\alpha_{\mathcal{F}}}), 0, 0, bd(z, \{Fu\}_{\alpha_{\mathcal{F}}})\right) \\ & \leq 0. \end{aligned} \quad (14)$$

By $\phi(u, v, u, v, 0, b(u+v)) \leq 0$ which implies $u \leq hv$, we have $d(z, \{Fu\}_{\alpha_{\mathcal{F}}}) \leq h \cdot 0 = 0$. Thus, $z \in \{Fu\}_{\alpha_{\mathcal{F}}}$. As $z = fu \in \{Fu\}_{\alpha_{\mathcal{F}}} \in g(Y)$, therefore, there exists $v \in Y$ such that $z = gv$. Similarly, $z = gv \in \{Gv\}_{\alpha_{\mathcal{F}}}$. Since the pair (F, f) are weakly commuting and occasionally coincidentally idempotent and $z = fu \in \{Fu\}_{\alpha_{\mathcal{F}}}$, then $z = fu = ffu = fz$; therefore, $z = fz = ffu \in \{fFu\}_{\alpha_{\mathcal{F}}} \subseteq \{Ffu\}_{\alpha_{\mathcal{F}}} = \{Fz\}_{\alpha_{\mathcal{F}}}$. Also $z = gv \in \{Gv\}_{\alpha_{\mathcal{F}}}$; then $z = gv = ggv = gz$; therefore, $gz = ggv \in \{gGv\}_{\alpha_{\mathcal{F}}} \subseteq \{Ggv\}_{\alpha_{\mathcal{F}}} = \{Gz\}_{\alpha_{\mathcal{F}}}$. $\square$

Example 22. Let $Y \subseteq X = [0, 1]$ and $d(x, y) = |x - y|^2$; then (X, d) is a b -metric space with $b = 2$. Define the partial order $x \leq y$ as $x \leq y$ for each $x, y \in X$ and $X \leq_1 Y$ as follows: for

each $x \in X$, there exists $y \in Y$ such that $x \leq y$. Let $L = [0, 1]$ and define the maps f, g, F, G on X as $gx = x/2$, $fx = x/3$,

$$\begin{aligned} (Fx)(y) &= \begin{cases} 0, & \text{if } 0 \leq y \leq \frac{x}{5}, \\ \frac{1}{3}, & \text{if } \frac{x}{5} < y < \frac{x}{4}, \\ \frac{2}{3}, & \text{if } \frac{x}{4} \leq y \leq 1, \end{cases} \\ (Gx)(y) &= \begin{cases} 0, & \text{if } 0 \leq y \leq \frac{x}{7}, \\ \frac{1}{6}, & \text{if } \frac{x}{7} < y < \frac{x}{6}, \\ \frac{1}{4}, & \text{if } \frac{x}{6} \leq y \leq 1. \end{cases} \end{aligned} \quad (15)$$

Define the sequences x_{2n} , x_{2n+1} , and x_{2n+2} in Y as $x_{2n} = \{1/(2n+1)\}$, $x_{2n+1} = \{1/(2n+2)\}$, and $x_{2n+2} = \{1/(2n+3)\}$, $n \in \mathbb{N}$; then $y_{2n+1} = gx_{2n+1} = 1/2(2n+2)$ and $y_{2n+2} = fx_{2n+2} = 1/3(2n+3)$. Now, $\{Fx_{2n}\}_{2/3} = [1/4(2n+1), 1]$ and $\{Gx_{2n+1}\}_{1/4} = [1/6(2n+2), 1]$; then $gx_{2n+1} \in \{Fx_{2n}\}_{2/3}$ and $fx_{2n+2} \in \{Gx_{2n+1}\}_{1/4}$. Further, $\lim_{n \rightarrow \infty} fx_{2n+2} = \lim_{n \rightarrow \infty} gx_{2n+1} = 0 \in [0, 1] = \lim_{n \rightarrow \infty} \{Fx_{2n}\}_{2/3} = \lim_{n \rightarrow \infty} \{Gx_{2n+1}\}_{1/4}$. Let $\phi(t_1, \dots, t_6) = t_1 - ht_2$; then we have

$$\phi(Q) + E \min(Q) = 0, \quad (16)$$

where

$$\begin{aligned} Q = & \left(H\left(\{Fx_{2n}\}_{\alpha_L}, \{Gx_{2n+1}\}_{\alpha_L}\right), d(fx_{2n+2}, gx_{2n+1}), \right. \\ & d(fx_{2n+2}, \{Fx_{2n}\}_{\alpha_L}), d(gx_{2n+1}, \{Gx_{2n+1}\}_{\alpha_L}), \\ & \left. d(fx_{2n+2}, \{Gx_{2n+1}\}_{\alpha_L}), d(gx_{2n+1}, \{Fx_{2n}\}_{\alpha_L}) \right). \end{aligned} \quad (17)$$

Finally, $f0 = ff0 \in [0, 1] = Ff0$ and $g0 = g0 \in [0, 1] = Gg0$; that is, (f, F) and (g, G) are weakly commuting and occasionally coincidentally idempotent (f, F) and (g, G) are weakly commuting and occasionally coincidentally idempotent. Now, f, g, F, G satisfy all conditions of Theorem 21 and $f0 = g0 = 0 \in [0, 1] = \{F0\}_{2/3} = \{G0\}_{1/4}$ is a common fixed point.

Corollary 23. Let (X, d) be a complete b -metric space, $Y \subseteq X$, and $\leq$ be a partial order defined on Y , $f, g : Y \rightarrow X$ such that $f(Y)$ and $g(Y)$ are closed. Suppose that F, G are two $\mathcal{L}$ -fuzzy mappings from Y into $\mathfrak{F}_{\mathcal{F}}(X)$ such that, for each $x \in Y$ and $\alpha_{\mathcal{F}} \in L \setminus \{0_{\mathcal{F}}\}$, $\{Fx\}_{\alpha_{\mathcal{F}}}$ and $\{Gx\}_{\alpha_{\mathcal{F}}}$ are nonempty closed subsets of X which satisfy conditions ((1)–(5)); then f and g have a common fixed point, while F and G have a common L -fuzzy fixed point.

Remark 24. (1) Theorem 21 is a generalization of Theorem 2.2 [6], Theorem 14 [9], and Theorem 3.1 [8].

(2) In Theorem 21, one may put one of $f = g$ or $F = G$ or both; in this case, condition (4) has the following versions:

$$\begin{aligned}
& \phi \left(H \left(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Gy\}_{\alpha_{\mathcal{G}}} \right), d(fx, fy), d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}), d(fy, \{Gy\}_{\alpha_{\mathcal{G}}}), d(fx, \{Gy\}_{\alpha_{\mathcal{G}}}), d(fy, \{Fx\}_{\alpha_{\mathcal{F}}}) \right) \\
& + E \min \left(H \left(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Gy\}_{\alpha_{\mathcal{G}}} \right), d(fx, fy), d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}), d(fy, \{Gy\}_{\alpha_{\mathcal{G}}}), d(fx, \{Gy\}_{\alpha_{\mathcal{G}}}), d(fy, \{Fx\}_{\alpha_{\mathcal{F}}}) \right) \leq 0, \\
& \phi \left(H \left(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Fy\}_{\alpha_{\mathcal{F}}} \right), d(fx, gy), d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}), d(gy, \{Fy\}_{\alpha_{\mathcal{F}}}), d(fx, \{Fy\}_{\alpha_{\mathcal{F}}}), d(gy, \{Fx\}_{\alpha_{\mathcal{F}}}) \right) \\
& + E \min \left(H \left(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Fy\}_{\alpha_{\mathcal{F}}} \right), d(fx, gy), d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}), d(gy, \{Fy\}_{\alpha_{\mathcal{F}}}), d(fx, \{Fy\}_{\alpha_{\mathcal{F}}}), d(gy, \{Fx\}_{\alpha_{\mathcal{F}}}) \right) \leq 0, \\
& \phi \left(H \left(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Fy\}_{\alpha_{\mathcal{F}}} \right), d(fx, fy), d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}), d(fy, \{Fy\}_{\alpha_{\mathcal{F}}}), d(fx, \{Fy\}_{\alpha_{\mathcal{F}}}), d(gy, \{Fx\}_{\alpha_{\mathcal{F}}}) \right) \\
& + E \min \left(H \left(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Fy\}_{\alpha_{\mathcal{F}}} \right), d(fx, gy), d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}), d(fy, \{Fy\}_{\alpha_{\mathcal{F}}}), d(fx, \{Fy\}_{\alpha_{\mathcal{F}}}), d(fy, \{Fx\}_{\alpha_{\mathcal{F}}}) \right) \leq 0.
\end{aligned} \tag{18}$$

Theorem 25. Let (X, d) be a b -metric space, $Y \subseteq X$, and $\leq$ be a partial order defined on Y , $f, g : Y \rightarrow X$ such that $f(Y)$ and $g(Y)$ are closed. Suppose that F_n is a sequence of $\mathcal{L}$ -fuzzy mappings from Y into $\mathfrak{F}_{\mathcal{L}}(X)$ such that, for each $x \in Y$ and $\alpha_{\mathcal{F}} \in L \setminus \{0_{\mathcal{F}}\}$, $\{F_n x\}_{\alpha_{\mathcal{F}}}$ are nonempty closed subsets of X which satisfy the following conditions:

- (1) $\{F_k x\}_{\alpha_{\mathcal{F}}} \preceq_1 g(Y)$ and $\{F_l x\}_{\alpha_{\mathcal{F}}} \preceq_1 f(Y)$.
- (2) $gy \in \{F_k x\}_{\alpha_{\mathcal{F}}}$ or $fy \in \{F_l x\}_{\alpha_{\mathcal{F}}}$ implies $x \leq y$.
- (3) If $y_n \rightarrow y$, then $y_n \leq y$ for all $n \in \mathbb{N}$.
- (4) (f, F_k) and (g, F_l) are weakly commuting and occasionally coincidentally idempotent.
- (5) For all comparable elements $x, y \in Y$, $k = 2n + 1$ and $l = 2n + 2$, $n \in \mathbb{N}$ and $E > 0$, there exists $\phi \in \Phi$ such that

$$\phi(Q) + E \min(Q) \leq 0, \tag{19}$$

where

$$\begin{aligned}
Q = & \left(H \left(\{F_k x\}_{\alpha_{\mathcal{F}}}, \{F_l y\}_{\alpha_{\mathcal{F}}} \right), d(fx, gy), \right. \\
& d(fx, \{F_k x\}_{\alpha_{\mathcal{F}}}), d(gy, \{F_l y\}_{\alpha_{\mathcal{F}}}), d(fx, \{F_l y\}_{\alpha_{\mathcal{F}}}), \\
& \left. d(gy, \{F_k x\}_{\alpha_{\mathcal{F}}}) \right).
\end{aligned} \tag{20}$$

- (6) One of $f(Y)$ or $g(Y)$ is x_0 joint orbitally complete for some $x_0 \in Y$,

then f and g have a common fixed point while F_k and F_l have a common L -fuzzy fixed point.

Proof. Put $F = F_k$ and $G = F_l$ in Theorem 21. This concludes the proof. $\square$

5. Common Fixed Points with (JCLR) Property

Theorem 26. Let (X, d) be a b -metric space, $Y \subseteq X$, and $\leq$ be a partial order defined on Y , $f, g : Y \rightarrow X$ such that

$f(Y)$ and $g(Y)$ are closed. Suppose that F, G are two $\mathcal{L}$ -fuzzy mappings from Y into $\mathfrak{F}_{\mathcal{L}}(X)$ such that, for each $x \in Y$ and $\alpha_{\mathcal{F}} \in L \setminus \{0_{\mathcal{F}}\}$, $\{Fx\}_{\alpha_{\mathcal{F}}}$ and $\{Gx\}_{\alpha_{\mathcal{F}}}$ are nonempty closed subsets of X which satisfy the following conditions:

- (1) (f, F) and (g, G) satisfy (JCLR) property.
- (2) (f, F) and (g, G) are weakly commuting and occasionally coincidentally idempotent.
- (3) For all comparable elements $x, y \in Y$ and $E > 0$ there exists $\phi \in \Phi$ such that

$$\phi(Q) + E \min(Q) \leq 0, \tag{21}$$

where

$$\begin{aligned}
Q = & \left(H \left(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Gy\}_{\alpha_{\mathcal{G}}} \right), d(fx, gy), \right. \\
& d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}), d(gy, \{Gy\}_{\alpha_{\mathcal{G}}}), d(fx, \{Gy\}_{\alpha_{\mathcal{G}}}), \\
& \left. d(gy, \{Fx\}_{\alpha_{\mathcal{F}}}) \right);
\end{aligned} \tag{22}$$

then f and g have a common fixed point, while F and G have a common L -fuzzy fixed point.

Proof. Since (f, F) and (g, G) satisfy (JCLR) property, there exist two sequences $\{x_n\}, \{y_n\}$ in Y and $A, B \in CL(X)$ such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = u,$$

$$\lim_{n \rightarrow \infty} \{Fx_n\}_{\alpha_{\mathcal{F}}} = A, \tag{23}$$

$$\lim_{n \rightarrow \infty} \{Gy_n\}_{\alpha_{\mathcal{G}}} = B.$$

With $u = fv = gw$ and $u \in A \cap B$, for some $u, v, w \in Y$, we show that $A = B$, since

$$\phi(Q) + E(Q) \leq 0, \tag{24}$$

where

$$\begin{aligned} Q = & \left(H \left(\{Fx_n\}_{\alpha_{\mathcal{F}}}, \{Gy_n\}_{\alpha_{\mathcal{F}}} \right), d \left(fx_n, gy_n \right), \right. \\ & d \left(fx_n, \{Fx_n\}_{\alpha_{\mathcal{F}}} \right), d \left(gy_n, \{Gy_n\}_{\alpha_{\mathcal{F}}} \right), \\ & \left. d \left(fx_n, \{Gy_n\}_{\alpha_{\mathcal{F}}} \right), d \left(gy_n, \{Fx_n\}_{\alpha_{\mathcal{F}}} \right) \right). \end{aligned} \quad (25)$$

As $n \rightarrow \infty$,

$$\begin{aligned} & \phi \left(H(A, B), 0, d(fv, A), d(fv, B), d(fv, B), \right. \\ & \left. d(fv, A) \right) \leq 0, \end{aligned} \quad (26)$$

which gives

$$\begin{aligned} & \phi \left(H(A, B), 0, 0, H(A, B), b(H(A, B) + 0), 0 \right) \\ & \leq \phi \left(H(A, B), 0, d(fv, A), d(fv, B), d(fv, B), \right. \\ & \left. d(fv, A) \right) \leq 0. \end{aligned} \quad (27)$$

From the property $\phi(u, v, u, v, b(u + v), 0) \leq 0$ which implies $u \leq hv$, there exists $h \in (0, 1)$ such that $H(A, B) \leq h \cdot 0 = 0$, so that $A = B$. Now, $fv \in \{Fv\}_{\alpha_{\mathcal{F}}}$; to prove this, since $fv \in A$, we show that $A = \{Fv\}_{\alpha_{\mathcal{F}}}$. As

$$\phi(Q) + E(Q) \leq 0, \quad (28)$$

where

$$\begin{aligned} Q = & \left(H \left(\{Fv\}_{\alpha_{\mathcal{F}}}, \{Gy_n\}_{\alpha_{\mathcal{F}}} \right), d(fv, gy_n), \right. \\ & d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}), d(gy_n, \{Gy_n\}_{\alpha_{\mathcal{F}}}), d(fv, \{Gy_n\}_{\alpha_{\mathcal{F}}}), \\ & \left. d(gy_n, \{Fv\}_{\alpha_{\mathcal{F}}}) \right), \end{aligned} \quad (29)$$

when $n \rightarrow \infty$, we have that

$$\begin{aligned} & \phi \left(H \left(\{Fv\}_{\alpha_{\mathcal{F}}}, A \right), 0, d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}), d(fv, A), \right. \\ & \left. d(fv, A), d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}) \right) \leq 0. \end{aligned} \quad (30)$$

So that

$$\begin{aligned} & \phi \left(H \left(\{Fv\}_{\alpha_{\mathcal{F}}}, A \right), 0, H(A, \{Fv\}_{\alpha_{\mathcal{F}}}), 0, 0, \right. \\ & \left. b(0 + H(A, \{Fv\}_{\alpha_{\mathcal{F}}})) \right) \leq \phi \left(H \left(\{Fv\}_{\alpha_{\mathcal{F}}}, A \right), 0, \right. \\ & \left. d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}), d(fv, A), d(fv, A), d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}) \right) \\ & \leq 0. \end{aligned} \quad (31)$$

From the property $\phi(u, v, v, u, 0, b(u + v)) \leq 0$ which implies $u \leq hv$, there exists $h \in (0, 1)$ such that $H(\{Fv\}_{\alpha_{\mathcal{F}}}, A) \leq h \cdot 0 = 0$; this gives $H(A, \{Fv\}_{\alpha_{\mathcal{F}}}) = 0$; then $A = \{Fv\}_{\alpha_{\mathcal{F}}}$, and $fv \in \{Fv\}_{\alpha_{\mathcal{F}}}$. By a similar way, one can find $gw \in \{Gw\}_{\alpha_{\mathcal{F}}}$ for $w \in Y$. Further, $ffv = fv$ and $ffv \in \{Ffv\}_{\alpha_{\mathcal{F}}}$ so that $u = fu \in \{Fu\}_{\alpha_{\mathcal{F}}}$. Also, $ggw = gw$ and $ggw \in \{Ggw\}_{\alpha_{\mathcal{F}}}$ imply $u = gu \in \{Gu\}_{\alpha_{\mathcal{F}}}$. Then f, g, F, G have a common fixed point. $\square$

Theorem 27. In Theorem 26, we may replace condition (3) by another: for all $x, y \in Y$ and $E > 0$ there exist $\psi \in \Psi$ such that

$$\begin{aligned} & \psi \left(H \left(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Gy\}_{\alpha_{\mathcal{F}}} \right), d(fx, gy), d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}) \right. \\ & \left. + E \min \left(H \left(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Gy\}_{\alpha_{\mathcal{F}}} \right), d(fx, gy), \right. \right. \\ & \left. \left. d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}) \right) \right) \leq 0; \end{aligned} \quad (32)$$

then f and g have a common fixed point, while F and G have a common L -fuzzy fixed point.

Proof. Since (f, F) and (g, G) satisfy (JCLR) property, there exist two sequences $\{x_n\}, \{y_n\}$ in Y and $A, B \in CL(X)$ such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = u, \lim_{n \rightarrow \infty} \{Fx_n\}_{\alpha_{\mathcal{F}}} = A, \lim_{n \rightarrow \infty} \{Gy_n\}_{\alpha_{\mathcal{F}}} = B$, with $u = fv = gw$ and $u \in A \cap B$, for some $u, v, w \in Y$. Now, we show that $fv \in \{Fv\}_{\alpha_{\mathcal{F}}}$; otherwise, since

$$\begin{aligned} & \psi \left(H \left(\{Fv\}_{\alpha_{\mathcal{F}}}, \{Gy_n\}_{\alpha_{\mathcal{F}}} \right), d(fv, gy_n), d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}) \right. \\ & \left. + E \min \left(H \left(\{Fv\}_{\alpha_{\mathcal{F}}}, \{Gy_n\}_{\alpha_{\mathcal{F}}} \right), d(fv, gy_n), \right. \right. \\ & \left. \left. d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}) \right) \right) \leq 0. \end{aligned} \quad (33)$$

When $n \rightarrow \infty$, we have $\psi(H(\{Fv\}_{\alpha_{\mathcal{F}}}, B), 0, d(fv, \{Fv\}_{\alpha_{\mathcal{F}}})) \leq 0$. But $fv \in B$ implies $d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}) = d(\{Fv\}_{\alpha_{\mathcal{F}}}, fv) \leq H(\{Fv\}_{\alpha_{\mathcal{F}}}, B)$; then

$$\begin{aligned} & \psi \left(d(\{Fv\}_{\alpha_{\mathcal{F}}}, fv), 0, d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}) \right) \\ & \leq \psi \left(H \left(\{Fv\}_{\alpha_{\mathcal{F}}}, B \right), 0, d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}) \right) \leq 0. \end{aligned} \quad (34)$$

By $\psi(u, v, u) \leq 0 \Rightarrow u \leq v$, this gives $d(\{Fv\}_{\alpha_{\mathcal{F}}}, fv) \leq 0$ which is a contradiction of $d(x, y) \geq 0, \forall x, y \in Y$; then $d(fv, \{Fv\}_{\alpha_{\mathcal{F}}}) = 0$ and $fv \in \{Fv\}_{\alpha_{\mathcal{F}}}$. Also, $gw \in \{Gw\}_{\alpha_{\mathcal{F}}}$ for $w \in Y$. We prove this by contradiction; otherwise, since

$$\begin{aligned} & \psi \left(H \left(\{Fx_n\}_{\alpha_{\mathcal{F}}}, \{Gw\}_{\alpha_{\mathcal{F}}} \right), d(fx_n, gw), d(fx_n, \right. \\ & \left. \{Fx_n\}_{\alpha_{\mathcal{F}}}) \right) + E \min \left(H \left(\{Fx_n\}_{\alpha_{\mathcal{F}}}, \{Gw\}_{\alpha_{\mathcal{F}}} \right), \right. \\ & \left. d(fx_n, gw), d(fx_n, \{Fx_n\}_{\alpha_{\mathcal{F}}}) \right) \leq 0. \end{aligned} \quad (35)$$

Letting $n \rightarrow \infty$, we have $\psi(H(A, \{Gw\}_{\alpha_{\mathcal{F}}}), 0, d(fv, A)) \leq 0$, but $gw \in A$ implies $d(gw, \{Gw\}_{\alpha_{\mathcal{F}}}) \leq H(A, \{Gw\}_{\alpha_{\mathcal{F}}})$; then

$$\begin{aligned} & \psi \left(d(gw, \{Gw\}_{\alpha_{\mathcal{F}}}), 0, 0 \right) \leq \psi \left(H \left(A, \{Gw\}_{\alpha_{\mathcal{F}}} \right), 0, 0 \right) \\ & \leq 0. \end{aligned} \quad (36)$$

From $\psi(u, v, v) \leq 0 \Rightarrow u \leq v$, this gives $d(gw, \{Gw\}_{\alpha_{\mathcal{F}}}) \leq 0$ which is a contradiction; then $gw \in \{Gw\}_{\alpha_{\mathcal{F}}}$. Further, being weakly commuting and occasionally coincidentally idempotent imply that, respectively, $u = fv = ffv = fu$ and $ffv \in \{Ffv\}_{\alpha_{\mathcal{F}}}$, so that $u = fu \in \{Fu\}_{\alpha_{\mathcal{F}}}$. Also, $ggw = gw$ and $ggw \in \{Ggw\}_{\alpha_{\mathcal{F}}}$ imply $u = gu \in \{Gu\}_{\alpha_{\mathcal{F}}}$. Then f, g, F, G have a common fixed point. $\square$

6. An Application to Integral Contractive Condition

Suppose that $\alpha, \beta : [0, \infty) \rightarrow [0, \infty)$ are summable nonnegative Lebesgue integrable functions such that for each $\epsilon > 0$, $\int_0^\epsilon \alpha(s) ds \geq 0$ and $\int_0^\epsilon \beta(e) de \geq 0$. As a simple application of Theorems 21, 25, and 26, we give the following result.

Theorem 28. *Let (X, d) be a b-metric space, $Y \subseteq X$, and let $\leq$ be a partial order defined on Y , $f, g : Y \rightarrow X$ such that $f(Y)$ and $g(Y)$ are closed. Suppose that F, G are two $\mathcal{L}$ -fuzzy mappings from Y into $\mathfrak{F}_{\mathcal{L}}(X)$ such that for each $x \in Y$ and $\alpha_{\mathcal{L}} \in L \setminus \{0_{\mathcal{L}}\}$, $\{Fx\}_{\alpha_{\mathcal{L}}}$ and $\{Gx\}_{\alpha_{\mathcal{L}}}$ are nonempty closed subsets of X and satisfying the following conditions*

- (1) (f, F_k) and (g, F_l) satisfy (JCLR) property.
- (2) (f, F_k) and (g, F_l) are weakly commuting and occasionally coincidentally idempotent.
- (3) For all comparable elements $x, y \in Y$ and $E > 0$, there exist $\phi \in \Phi$ such that

$$\int_0^{\phi(Q)} \alpha(s) ds + E \int_0^{\min(Q)} \beta(e) de \leq 0, \quad (37)$$

where

$$\begin{aligned} Q = & \left(H \left(\{F_k x\}_{\alpha_{\mathcal{L}}}, \{F_l y\}_{\alpha_{\mathcal{L}}} \right), d(fx, gy), \right. \\ & d(fx, \{F_k x\}_{\alpha_{\mathcal{L}}}), d(gy, \{F_l y\}_{\alpha_{\mathcal{L}}}), d(fx, \{F_l y\}_{\alpha_{\mathcal{L}}}), \\ & \left. d(gy, \{F_k x\}_{\alpha_{\mathcal{L}}}) \right), \end{aligned} \quad (38)$$

where $k = 2n + 1$, $l = 2n + 2$, $n \in \mathbb{N}$; then f and g have a common fixed point, while F_k and F_l have a common L -fuzzy fixed point.

Proof. Since (f, F_k) and (g, F_l) satisfy the property (JCLR), there exist two sequences $\{x_{kn}\}, \{y_{kn}\}$ in Y and $A_k, B_k \in CL(X)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} f x_{kn} &= \lim_{n \rightarrow \infty} g y_{kn} = u_k, \\ \lim_{n \rightarrow \infty} \{F_k x_{kn}\}_{\alpha_{\mathcal{L}}} &= A_k, \\ \lim_{n \rightarrow \infty} \{F_l y_{kn}\}_{\alpha_{\mathcal{L}}} &= B_k, \end{aligned} \quad (39)$$

with $u_k = f v_k = g w_k$, $u_k \in A_k \cap B_k$, for some $u_k, v_k, w_k \in Y$. Now, we show that $A_k = B_k$. As

$$\int_0^{\phi(Q)} \alpha(s) ds + E \int_0^{\min(Q)} \beta(e) de \leq 0, \quad (40)$$

where

$$\begin{aligned} Q = & \left(H \left(\{F_k x_{kn}\}_{\alpha_{\mathcal{L}}}, \{F_l y_{kn}\}_{\alpha_{\mathcal{L}}} \right), d(fx_{kn}, g y_{kn}), \right. \\ & d(fx_{kn}, \{F_k x_{kn}\}_{\alpha_{\mathcal{L}}}), d(g y_{kn}, \{F_l y_{kn}\}_{\alpha_{\mathcal{L}}}), \\ & \left. d(fx_{kn}, \{F_l y_{kn}\}_{\alpha_{\mathcal{L}}}), d(g y_{kn}, \{F_k x_{kn}\}_{\alpha_{\mathcal{L}}}) \right). \end{aligned} \quad (41)$$

As $n \rightarrow \infty$, we find that

$$\int_0^{\phi(H(A_k, B_k), 0, 0, d(f v_k, B_k), d(f v_k, B_k), 0))} \alpha(s) ds \leq 0. \quad (42)$$

So that

$$\phi(H(A_k, B_k), 0, 0, d(f v_k, B_k), d(f v_k, B_k), 0) \leq 0, \quad (43)$$

which give

$$\begin{aligned} & \int_0^{\phi(H(A_k, B_k), 0, 0, H(A_k, B_k), b(H(A_k, B_k) + 0), 0))} \alpha(s) ds \\ & \leq \int_0^{\phi(H(A_k, B_k), 0, d(f v_k, A_k), d(f v_k, B_k), d(f v_k, B_k), d(f v_k, A_k))} \alpha(s) ds \\ & \leq 0. \end{aligned} \quad (44)$$

By (ϕ) , we have $H(A_k, B_k) = 0$; that is, $A_k = B_k$. As $u_k \in A_k$, we show that $\{F_k v_k\}_{\alpha_{\mathcal{L}}} = A_k$. As

$$\int_0^{\phi(Q)} \alpha(s) ds + E \int_0^{\min(Q)} \beta(e) de \leq 0, \quad (45)$$

where

$$\begin{aligned} Q = & \left(H \left(\{F_k v_k\}_{\alpha_{\mathcal{L}}}, \{F_l y_{kn}\}_{\alpha_{\mathcal{L}}} \right), d(f v_k, g y_{kn}), \right. \\ & d(f v_k, \{F_k v_k\}_{\alpha_{\mathcal{L}}}), d(g y_{kn}, \{F_l y_{kn}\}_{\alpha_{\mathcal{L}}}), \\ & \left. d(f v_k, \{F_l y_{kn}\}_{\alpha_{\mathcal{L}}}), d(g y_{kn}, \{F_k v_k\}_{\alpha_{\mathcal{L}}}) \right), \end{aligned} \quad (46)$$

which on making $n \rightarrow \infty$ reduces to

$$\begin{aligned} & \int_0^{\phi(H(\{F_k v_k\}_{\alpha_{\mathcal{L}}}, A_k), 0, 0, d(\{F_k v_k\}_{\alpha_{\mathcal{L}}}, A_k), d(\{F_k v_k\}_{\alpha_{\mathcal{L}}}, A_k), 0))} \alpha(s) ds \\ & \leq 0. \end{aligned} \quad (47)$$

By a similar way, we have $\{F_k v_k\}_{\alpha_{\mathcal{L}}} = A_k$. The remaining parts are easy to prove. This concludes the proof. $\square$

Now, we may apply Theorem 28 with the following example.

Example 29. Let $X = [0, 1]$, $Y \subseteq X$, and $d(x, y) = |x - y|^2$; then (X, d) is a b -metric space with $b = 2$. Define the partial order $x \leq y$ as $x \leq y$ for each $x, y \in X$ and $X \leq_1 Y$ as follows: for each $x \in X$, there exist $y \in Y$ such that $x \leq y$. Let $L = [0, 1]$ and define the maps f, g, F, G on X as

$$\begin{aligned} fx &= \begin{cases} \frac{1}{6}, & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{x}{3}, & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \\ gx &= \begin{cases} \frac{1}{6}, & \text{if } 0 \leq x < \frac{1}{3}, \\ \frac{x}{2}, & \text{if } \frac{1}{3} \leq x \leq 1, \end{cases} \\ (Fx)(y) &= \begin{cases} \frac{5}{6}, & \text{if } 0 \leq y \leq \frac{1}{10}, \\ \frac{4}{5}, & \text{if } \frac{1}{10} < y < \frac{x+1}{9}, \\ \frac{2}{3}, & \text{if } \frac{x+1}{9} \leq y \leq 1, \end{cases} \\ (Gx)(y) &= \begin{cases} \frac{3}{4}, & \text{if } 0 \leq y < \frac{1}{9}, \\ \frac{1}{2}, & \text{if } \frac{1}{9} \leq y < \frac{x+1}{8}, \\ \frac{1}{4}, & \text{if } \frac{x+1}{8} \leq y \leq 1, \end{cases} \end{aligned} \quad (48)$$

for all $x, y \in X$. Define two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that $\{x_n\} = \{1/2 + 1/4n\}$, $\{y_n\} = \{1/3 + 1/6n\}$, $n \in \mathbb{N}$. Now, we show that for $x, y \in Y$ there exist $\phi \in \Phi$ such that

$$\int_0^{\phi(Q)} \alpha(s) ds + E \int_0^{\min(Q)} \beta(e) de \leq 0, \quad (49)$$

where

$$\begin{aligned} Q &= \left(H(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Gy\}_{\alpha_{\mathcal{G}}}), d(fx, gy), \right. \\ &\quad d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}), d(gy, \{Gy\}_{\alpha_{\mathcal{G}}}), d(fx, \{Gy\}_{\alpha_{\mathcal{G}}}), \\ &\quad \left. d(gy, \{Fx\}_{\alpha_{\mathcal{F}}}) \right). \end{aligned} \quad (50)$$

Since $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gy_n = 1/6 \in [1/6, 1] = \lim_{n \rightarrow \infty} Fx_n = \lim_{n \rightarrow \infty} Gy_n$, the hybrid pairs (f, F) and (g, G) satisfy the property (JCLR). Also $f(1/6) = ff(1/6) = 1/6$ and $g(1/6) = gg(1/6) = 1/6$ (occasionally coincidentally idempotent). Finally, $ff(1/6) = 1/6 \in [7/54, 1] = \{Ff(1/6)\}_{2/3}$ and $gg(1/6) = 1/6 \in [7/48, 1] = \{Gg(1/6)\}_{2/3}$ (weakly commuting). Let $\phi(t_1, \dots, t_6) = t_1 - h \max\{t_2, (t_3 + t_4)t_5 t_6\}$. Further,

$$\begin{aligned} &\phi\left(H(\{Fx_n\}_{\alpha_{\mathcal{F}}}, \{Gy_n\}_{\alpha_{\mathcal{G}}}), d(fx_n, gy_n), \right. \\ &\quad d(fx_n, \{Fx_n\}_{\alpha_{\mathcal{F}}}), d(gy_n, \{Gy_n\}_{\alpha_{\mathcal{G}}}), \\ &\quad \left. d(fx_n, \{Gy_n\}_{\alpha_{\mathcal{G}}}), d(gy_n, \{Fx_n\}_{\alpha_{\mathcal{F}}})\right) = 0, \end{aligned} \quad (51)$$

which gives

$$\int_0^{\phi(H(\{Fx_n\}_{\alpha_{\mathcal{F}}}, \{Gy_n\}_{\alpha_{\mathcal{G}}}), d(fx_n, gy_n), d(fx_n, \{Fx_n\}_{\alpha_{\mathcal{F}}}), d(gy_n, \{Gy_n\}_{\alpha_{\mathcal{G}}}), d(fx_n, \{Gy_n\}_{\alpha_{\mathcal{G}}}), d(gy_n, \{Fx_n\}_{\alpha_{\mathcal{F}}}))} \alpha(s) ds = 0. \quad (52)$$

Then

$$\int_0^{\phi(Q)} \alpha(s) ds + E \int_0^{\min(Q)} \beta(e) de \leq 0, \quad (53)$$

where

$$\begin{aligned} Q &= \left(H(\{Fx_n\}_{\alpha_{\mathcal{F}}}, \{Gy_n\}_{\alpha_{\mathcal{G}}}), d(fx_n, gy_n), \right. \\ &\quad d(fx_n, \{Fx_n\}_{\alpha_{\mathcal{F}}}), d(gy_n, \{Gy_n\}_{\alpha_{\mathcal{G}}}), \\ &\quad \left. d(fx_n, \{Gy_n\}_{\alpha_{\mathcal{G}}}), d(gy_n, \{Fx_n\}_{\alpha_{\mathcal{F}}}) \right). \end{aligned} \quad (54)$$

Thus, the maps f, g, F, G satisfy the conditions in Theorem 28. Now, the maps f, g, F, G have a common fixed point.

Remark 30. In Theorem 28, condition (37), one may put $\alpha(s) = 1$ or $\beta(e) = 1$ or both, which give the following:

$$\begin{aligned} \phi(Q) + E \int_0^{\min(Q)} \beta(e) de &\leq 0, \\ \int_0^{\phi(Q)} \alpha(s) ds + E \min(Q) &\leq 0, \\ \phi(Q) + E \min(Q) &\leq 0. \end{aligned} \quad (55)$$

Theorem 31. Let (X, d) be a b -metric space, $Y \subseteq X$, $f, g : Y \rightarrow X$ and F, G are two $\mathcal{L}$ -fuzzy mappings from Y into $\mathfrak{F}_{\mathcal{L}}(X)$ such that for each $x \in Y$, $\alpha_{\mathcal{F}} \in L \setminus \{0_{\mathcal{F}}\}$, $\{Fx\}_{\alpha_{\mathcal{F}}}$ and $\{Gx\}_{\alpha_{\mathcal{G}}}$ are nonempty closed subsets of X which satisfy the following condition: for all $x, y \in Y$ and $E > 0$, there exists $\phi \in \Phi$ such that

$$\int_0^{\psi(Q)} \alpha(s) ds + E \int_0^{\min(Q)} \beta(e) de \leq 0, \quad (56)$$

where

$$\begin{aligned} Q &= \left(H(\{Fx\}_{\alpha_{\mathcal{F}}}, \{Gy\}_{\alpha_{\mathcal{G}}}), d(fx, gy), d(fx, \{Fx\}_{\alpha_{\mathcal{F}}}) \right). \end{aligned} \quad (57)$$

If (f, F) and (g, G) satisfy (JCLR) property, weakly commuting and occasionally coincidentally idempotent, then f and g have a common fixed point, while F and G have a common L -fuzzy fixed point.

Proof. It is similar to Theorem 27. $\square$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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