Positive Solutions for Boundary Value Problems of Fractional Differential Equation with Integral Boundary Conditions

Qiao Sun, 1 Hongwei Ji, 2 and Yujun Cui 3

1 Department of Applied Mathematics, Shandong University of Science and Technology, Qingdao 266590, China
2 Department of Mathematics and Physics, Nantong Normal College, Nantong 226010, China
3 State Key Laboratory of Mining Disaster Prevention and Control Co-Founded by Shandong Province and the Ministry of Science and Technology, Shandong University of Science and Technology, Qingdao 266590, China

Correspondence should be addressed to Yujun Cui; cyj720201@163.com

Received 29 November 2017; Accepted 22 February 2018; Published 22 April 2018

Abstract

By using two fixed-point theorems on cone, we discuss the existence results of positive solutions for the following boundary value problem of fractional differential equation with integral boundary conditions:

\[ \begin{align*}
&D_{0+}^{\alpha} x(t) + a(t) f(t, x(t)) = 0, \quad t \in (0, 1), \\
&x(0) = x'(0) = 0, \\
&x(1) = \int_0^1 x(t) dA(t).
\end{align*} \]

1. Introduction

Boundary value problem for fractional differential equation has aroused much attention in the past few years; many professors devoted themselves to the solvability of fractional differential equations, especially to the study of the existence of solutions for boundary value problems of fractional differential equation (see [1–28]). For example, Wang et al. [19] studied the existence of positive solutions for the following problem:

\[ \begin{align*}
&D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0, 1), \\
&u(0) = u'(0) = 0, \\
&u(1) = \sum_{i=1}^m \beta_i u(\xi_i),
\end{align*} \]

where \( m \geq 1 \) is integer and \( \xi_i, \beta_i > 0 \).

There have already been lots of books and papers involving the positive solutions for boundary value problems of fractional differential equation; however, only a few papers cover that for fractional differential equation boundary value problems with integral boundary conditions. Motivated by [14], we shall investigate the positive solutions of the following boundary value problem:

\[ \begin{align*}
&D_{0+}^{\alpha} x(t) + a(t) f(t, x(t)) = 0, \quad t \in (0, 1), \\
&x(0) = x'(0) = 0, \\
&x(1) = \int_0^1 x(t) dA(t),
\end{align*} \]

where \( D_{0+}^{\alpha} \) is the Riemann-Liouville differential operator of \( 2 < \alpha < 3 \), \( A(t) \) is right continuous on \([0, 1]\), left continuous at \( t = 1 \), and nondecreasing on \([0, 1]\) with \( A(0) = 0 \), and
\[ \int_0^1 u(t) dA(t) \] denotes the Riemann-Stieltjes integrals of \( u \) with respect to \( A \). And \( a(t), f(t, x(t)) \) satisfies the following conditions:

(H1) \( a \in L[0, 1] \) is nonnegative and not identically zero on any compact subset of \((0, 1)\), \( \sigma = \int_0^1 t^\alpha - 1 dA(t) < 1 \).

(H2) \( f : [0, 1] \times [0, +\infty) \to [0, +\infty) \) is continuous.

This paper consists of four sections. After the introduction, we recall some definitions, lemmas, and theorems in Section 2. And the main results of this paper are stated in Section 3. In the last section, we give two examples of the main results.

2. Preliminaries

Firstly, for convenience we recall some definitions, lemmas, and theorems.

**Definition 1** (see [29, 30]). Let \( f \in L^1([R^+]) \) define the Riemann-Liouville fractional integral of order \( \alpha > 0 \) for \( f \) as

\[
I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(s) (t-s)^{\alpha-1} ds,
\]

where \( \Gamma(\alpha) \) is Euler gamma function.

**Definition 2** (see [29, 30]). Define the Riemann-Liouville fractional derivative of order \( \alpha > 0 \) for \( f \) as

\[
D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \left( \frac{t^{\alpha+n-1}}{\Gamma(\alpha+n-1)} \int_0^t (t-s)^{\alpha+n-1} ds \right),
\]

where \( n = \lfloor \alpha \rfloor + 1 \),

**Lemma 3.** Let \( y \in C[0, 1], 2 < \alpha \leq 3; \) then the boundary value problem

\[
D_0^\alpha x(t) + y(t) = 0, \quad t \in (0, 1),
\]

\[
x(0) = x'(0) = 0,
\]

\[
x(1) = \int_0^1 x(t) dA(t)
\]

has the unique solution \( x(t) = \int_0^1 G_1(t, s)y(s)ds \), where

\[
G_1(t, s) = G(t, s) + \frac{t^{\alpha-1}}{1-\sigma} \int_0^1 G(\tau, s) dA(\tau),
\]

\[
G(t, s) = \begin{cases} t^{\alpha-1} (1-s)^{\alpha-1} -(t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1; \\ t^{\alpha-1} (1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases}
\]

**Proof.** The boundary value problem can be converted to an equivalent integral equation:

\[
x(t) = \int_0^1 y(s) (t-s)^{\alpha-1} ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},
\]

\[
c_1, c_2, c_3 \in R.
\]

Then the solution is

\[
x(t) = \int_0^1 y(s) (t-s)^{\alpha-1} ds + \int_0^1 x(s) dA(s).
\]

It follows from the boundary conditions \( x(0) = x'(0) = 0 \) that \( c_3 = c_2 = 0 \) and

\[
c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 y(s) (1-s)^{\alpha-1} ds + \int_0^1 x(s) dA(s).
\]

Thus we get

\[
x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^1 y(s) (t-s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int_0^1 y(s) (1-s)^{\alpha-1} ds \cdot t^{\alpha-1} + \int_0^1 x(s) dA(s) \cdot t^{\alpha-1}.
\]

Then we can obtain

\[
\int_0^1 x(s) dA(s) = \int_0^1 \int_0^1 G(s, \tau) y(\tau) d\tau dA(s)
\]

\[
+ \int_0^1 x(\tau) dA(\tau) \int_0^1 s^{\alpha-1} dA(s)
\]

\[
= \int_0^1 \int_0^1 G(s, \tau) y(\tau) d\tau dA(s)
\]

\[
+ \int_0^1 x(s) dA(s) \cdot \int_0^1 t^{\alpha-1} dA(t),
\]

which means

\[
\int_0^1 x(s) dA(s) = \frac{\int_0^1 \int_0^1 G(s, \tau) y(\tau) d\tau dA(s)}{1-\int_0^1 t^{\alpha-1} dA(t)}.
\]
\[ x(t) = \int_0^1 G(t, s) y(s) \, ds \\
+ \int_0^1 \int_0^1 G(t, r) y(r) \, d\tau \, dA(s) (1-\int_0^1 t^{\alpha-1} \, dA(s))^{-1/\alpha-1} \]
\[ = \int_0^1 \left[ G(t, s) + \frac{t^{\alpha-1}}{1-\alpha} \int_0^1 G(r, s) \, dA(r) \right] y(s) \, ds \]
\[ = \int_0^1 G_1(t, s) y(s) \, ds. \] 

Lemma 4 (see [20]). \( G(t, s) \) defined in (9) has the following properties:

(i) \( G(t, s) > 0 \), \( t, s \in (0, 1) \).
(ii) \( G(t, s) = G(1-s, 1-t) \), \( t, s \in [0, 1] \).
(iii) \( k(1-t)k(s) \leq \Gamma(\alpha)G(t, s) \leq (\alpha-1)k(s) \), \( t, s \in [0, 1] \) where \( k(t) = t(1-t)^{\alpha-1} \).

Lemma 5. If (H2) is satisfied, then \( G_1(t, s) \) defined in (8) has the following properties:

\[ k(1-t)k(s) \leq \Gamma(\alpha)G_1(t, s) \leq Lk(s), \quad t, s \in [0, 1], \] 
where \( L = (\alpha-1)(1+\int_0^1 dA(s)/(1-\alpha)). \)

Proof. The proof can be easily accomplished by Lemma 4, so we omitted it. \( \square \)

3. Main Results

We define the following notation: given \( \delta \in (0, 1/2) \), take
\[ h = \min_{\delta \leq t \leq 1-\delta} \Gamma(\alpha)G(t, s) \leq (\alpha-1)k(s), \quad t, s \in [0, 1]. \]
\[ M = \frac{\Gamma(\alpha)}{L \int_0^1 k(s) \, dA(s)} \]
\[ N = \frac{\Gamma(\alpha)}{h \int_{\delta}^{1-\delta} k(s) \, dA(s)} \]

Now we can obtain the following theorems.

Theorem 9. Suppose that (H1) and (H2) are satisfied; there exist two positive constants \( r_2 > r_1 > 0 \) such that

(H3) \( f(t, x) \leq Mr_2 \), \( (t, x) \in [0, 1] \times [0, r_2] \);

(H4) \( f(t, x) \geq Nr_1 \), \( (t, x) \in [0, 1] \times [0, r_1] \).

Then boundary value problem (4) has at least one positive solution \( x \in P \) such that \( r_1 \leq \|x\| \leq r_2 \).

Proof. The solution of boundary value problem (4) is equivalent to the fixed point of operator \( T \). Let \( \Omega_2 = \{ x \in P : \|x\| < r_2 \} \), when \( x \in \partial \Omega_2 \), for any \( t \in [0, 1] \) we have \( 0 \leq x(t) \leq r_2 \). By Lemma 5 and (H3) we get
\[ (Tx)(t) = \int_0^1 G_1(t, s) a(s) f(s, x(s)) \, ds \]
\[ \leq \frac{LMr_2}{\Gamma(\alpha)} \int_{\delta}^{1-\delta} k(s) \, dA(s) r_2 = \|x\|, \] 
which means when \( x \in \partial \Omega_2 \), \( \|Tx\| \leq \|x\| \).
Let \( \Omega = \{ x \in P : \| x \| < r_1 \} \); when \( x \in \partial \Omega \), for any \( t \in [0, 1] \) we have \( 0 \leq x(t) \leq r_1 \). By Lemma 5 and (H4) we get
\[
(Tx)(t) = \int_0^t G_1(t, s)a(s)f(s, x(s))\,ds
\]
\[
\geq k(1 - t) \frac{N_{r_1}}{\Gamma(\alpha)} \int_0^1 k(s)a(s)\,ds
\]
\[
\geq \min_{\delta \leq s \leq 1 - \delta} k(1 - t) \frac{N_{r_1}}{\Gamma(\alpha)} \int_0^1 k(s)a(s)\,ds
\]
\[
= r_1 = \| x \|,
\]
which means when \( x \in \partial \Omega \), \( \| Tx \| \geq \| x \| \).

It follows from Theorem 6 that we know that \( T \) has at least one fixed point in \( (\Omega_1 \setminus \Omega) \), which means that the boundary value problem (4) has at least one solution.

**Theorem 10.** Suppose that (H1) and (H2) are satisfied; there exist four positive constants \( a, b, c, d \) with \( 0 < a < b < (h/L)d < c, \) such that
\[
\begin{align*}
&\text{(H5)} \ f(t, x) < Ma, \ (t, x) \in [0, 1] \times [0, a]; \\
&\text{(H6)} \ f(t, x) > Nb, \ (t, x) \in [\delta, 1 - \delta] \times [b, d]; \\
&\text{(H7)} \ f(t, x) \leq Mc, \ (t, x) \in [0, 1] \times [0, c].
\end{align*}
\]

Then boundary value problem (4) has at least three positive solutions \( x_1, x_2, x_3, \) such that
\[
\begin{align*}
&\max_{0 \leq t \leq 1} |x_1(t)| < a, \\
&b < \min_{\delta \leq t \leq 1 - \delta} |x_2(t)| < \max_{0 \leq t \leq 1} |x_2(t)| \leq c, \\
&a < \max_{0 \leq t \leq 1} |x_3(t)| \leq c, \\
&\min_{\delta \leq t \leq 1 - \delta} |x_3(t)| < b.
\end{align*}
\]

**Proof.** Define a nonnegative continuous concave function \( \theta \) on \( P \) as
\[
\theta(x) = \min_{\delta \leq t \leq 1 - \delta} x(t).
\]
If \( x \in \overline{P} = \{ x \in P : \| x \| \leq c \} \), then \( \| x \| \leq c \); it follows from (H7) that \( f(t, x) \leq Mc \); hence
\[
\| Tx \| \leq L \frac{c}{\Gamma(\alpha)} \int_0^1 k(s)a(s)f(s, x(s))\,ds
\]
\[
\leq LMc \frac{c}{\Gamma(\alpha)} \int_0^1 k(s)a(s)\,ds = c.
\]
Thus, \( T(\overline{P}) \subset \overline{P} \). It follows from Lemma 8 that \( T \) is completely continuous. In the same way, let \( x \in \partial \overline{P} \); it follows from (H5) that \( f(t, x) < Ma \) for any \( t \in [0, 1] \), which shows that condition (c2) of Theorem 7 is fulfilled.

Let \( x(t) = (b + d)/2 \); it is easy to know that \( x \in P(\theta, b, d) \) and \( \{ x \in P(\theta, b, d) : \theta(x) > b \} \neq \emptyset \). If \( x \in P(\theta, b, d) \), we have \( b \leq x(t) \leq d \) for any \( t \in [\delta, 1 - \delta] \). We know \( f(t, x(t)) > Nb \) for \( \delta \leq t \leq 1 - \delta \) by (H6). So we get
\[
\theta(Tx) = \min_{\delta \leq t \leq 1 - \delta} x(t)
\]
\[
= \min_{\delta \leq t \leq 1 - \delta} \int_0^1 G_1(t, s)a(s)f(s, x(s))\,ds
\]
\[
\geq \min_{\delta \leq t \leq 1 - \delta} \frac{k(1 - t)}{\Gamma(\alpha)} \int_0^1 k(s)a(s)f(s, x(s))\,ds
\]
\[
> \frac{Nh}{\Gamma(\alpha)} \int_0^1 k(s)a(s)\,ds = b.
\]
So condition (c1) of Theorem 7 holds.

When \( x \in P(\theta, b, c) \) with \( \| Tx \| > d \), noting that
\[
\| Tx \| \leq L \frac{c}{\Gamma(\alpha)} \int_0^1 k(s)a(s)f(s, x(s))\,ds,
\]
thus
\[
(Tx)(t) \geq k(1 - t) \frac{\min_{\delta \leq t \leq 1 - \delta} k(s)a(s)f(s, x(s))\,ds}{L}
\]
\[
\geq k(1 - t) \frac{\min_{\delta \leq t \leq 1 - \delta} k(s)a(s)f(s, x(s))\,ds}{L} \| Tx \|,
\]
so we obtain
\[
\theta(Tx) = \min_{\delta \leq t \leq 1 - \delta} x(t) \geq \frac{\min_{\delta \leq t \leq 1 - \delta} k(1 - t) \| Tx \|}{L}
\]
\[
= \frac{h}{L}d > b.
\]
That is to say, (c3) is satisfied.

All conditions of Theorem 7 are satisfied, so \( T \) has at least three fixed points \( x_1, x_2, x_3, \) which means that the boundary value problem (4) has at least three positive solutions \( x_1, x_2, x_3, \) such that
\[
\begin{align*}
&\max_{0 \leq t \leq 1} |x_1(t)| < a, \\
&b < \min_{\delta \leq t \leq 1 - \delta} |x_2(t)| < \max_{0 \leq t \leq 1} |x_2(t)| \leq c, \\
&a < \max_{0 \leq t \leq 1} |x_3(t)| \leq c, \\
&\min_{\delta \leq t \leq 1 - \delta} |x_3(t)| < b.
\end{align*}
\]

The proof of this theorem is finished.

**4. Some Examples**

Now we present two examples to illustrate our main results.

**Example 1.** Let us see the following problem:
\[
D^{5/2}_{0^+} x(t) + \frac{1}{4} \sin t + x + 3 = 0, \quad t \in (0, 1),
\]
\[
x(0) = x'(0) = 0, \quad x(1) = \int_0^1 x(t)\,dt.
\]
Choose $\delta = 1/3$; we obtain that $L = 4$, $h = 2/\sqrt{243}$, $M = 105 \sqrt{7}/64 \approx 2.91$, and $N = 76545 \sqrt{7}/(704 \sqrt{2} - 256) \approx 183.48$. Then for any $(t, x) \in [0, 1] \times [0, 2]$, we have $f(t, x) = (1/4) \sin t + x + 3 \leq 5.25 \leq Mr_\\delta \approx 5.82$, and for any $(t, x) \in [0, 1] \times [0, 0.01]$, we get $f(t, x) = (1/4) \sin t + x + 3 \geq 3 \geq Nr_\\delta \approx 1.83$.

Then the boundary value problem has at least one positive solution $x \in P$ such that $0.01 \leq |x| \leq 2$.

Example 2. We now study the following problem:

$$D_0^\frac{\delta}{2} x(t) + f(t, x) = 0, \quad t \in (0, 1),$$

$$x(0) = x'(0) = 0,$$

$$x(1) = \int_0^1 x(t) \, dt,$$

where

$$f(t, x) = \begin{cases} 
\frac{t}{3} + x^2, & x \leq 1; \\
183 + \frac{t}{3} + x, & x > 1.
\end{cases}$$

Choose $\delta = 1/3$; we have $M \approx 2.91$, $N \approx 183.48$, $L = 4$, and $h = 0.1283$. Let $a = 1/2$, $b = 1$, $c = 100$, and $d = 35$; then for any $(t, x) \in [0, 1] \times [0, 1/2]$, we have $f(t, x) = t/3 + x^2 \leq 0.59 < Ma \approx 1.46$, for any $(t, x) \in [1/3, 2/3] \times [1, 35]$, we have $f(t, x) = 183 + t/3 + x \geq 18.411 > Nb \approx 183.48$, and for any $(t, x) \in [0, 1] \times [0, 100]$, we have $f(t, x) = 183 + t/3 + x \leq 283.34 < Mc \approx 291$. Then by Theorem 10, we conclude that this boundary value problem has at least three positive solutions $x_1, x_2, x_3$, such that

$$\max_{0 \leq t \leq 1} |x_1(t)| < \frac{1}{2},$$

$$1 < \min_{0 \leq t \leq 1-\delta} |x_2(t)| < \max_{0 \leq t \leq 1} |x_2(t)| \leq 100,$$

$$\frac{1}{2} < \max_{0 \leq t \leq 1-\delta} |x_3(t)| \leq 100,$$

$$\min_{0 \leq t \leq 1-\delta} |x_3(t)| < 1.$$

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This project is supported by NNSF of China (11371221, 11571207), SDNSF (ZR2018MA011), and the Tai’shan Scholar Engineering Construction Fund of Shandong Province of China.

**References**


