Research Article

Existence of Nontrivial Solutions for Some Second-Order Multipoint Boundary Value Problems

Hongyu Li and Junting Zhang

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong, 266590, China

Correspondence should be addressed to Hongyu Li; sdlhy1978@163.com

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By using fixed point theorems with lattice structure, the existence of negative solution and sign-changing solution for some second-order multipoint boundary value problems is obtained.

1. Introduction

In this paper, the following second-order ordinary differential equation will be considered:

$$-x''(t) = \varphi(t, x(t)), \quad 0 \leq t \leq 1, \quad (1)$$

subject to the multipoint boundary condition

$$\begin{aligned} x(0) &= 0, \\ x(1) &= \sum_{i=1}^{m-2} \beta_i x(\alpha_i), \end{aligned} \quad (2)$$

where $\beta_i > 0$, $i = 1, 2, \dots, m-2$; $\sum_{i=1}^{m-2} \beta_i < 1$; $0 < \alpha_1 < \alpha_2 < \dots < \alpha_{m-2} < 1$, and $\sum_{i=1}^{m-2} \beta_i \alpha_i < 1$.

The multipoint boundary value problems of ordinary differential equations arise in different areas of applied mathematics and physics. In 1992, Gupta studied nonlinear second-order three-point boundary value problems (see [1]). Since then, different types of nonlinear multipoint boundary value problems have been studied. Up to now, many great achievements about multipoint boundary value problems have been made. For example, many authors have investigated the existence of nontrivial solutions for nonlinear multipoint boundary value problems. Most of them have used upper and lower solution method, fixed point index theory, Guo-Krasnosel'skii fixed point theorem, bifurcation theory, fixed point theorems on cones, and so on (see [2–27] and references therein). For instance, in [2], the author

considered the second-order multipoint boundary value problem

$$\begin{aligned} y''(t) + f(y) &= 0, \quad 0 \leq t \leq 1, \\ y(0) &= 0, \end{aligned} \quad (3)$$

$$y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i).$$

By using fixed point index and Leray-Schauder degree methods, the author showed existence of multiple sign-changing solutions for the boundary value problem (3). In [14], the authors have considered the following multipoint boundary value problem:

$$\begin{aligned} -(L\varphi)(t) &= \lambda f(t, \varphi(t)), \quad 0 \leq t \leq 1, \\ \varphi'(0) &= 0, \end{aligned} \quad (4)$$

$$\varphi(1) = \sum_{i=1}^{m-2} \beta_i \varphi(\eta_i).$$

The authors have used global bifurcation method to obtain the existence of positive solution of the boundary value problem (4).

In recent years, some authors combine the theory of lattice and the theory of topological degree, so they have obtained some fixed point theorems with lattice structure

for nonlinear operators which are not assumed to be cone mappings (see [28–34]). At present, a few authors have used those fixed point theorems with lattice structure to study boundary value problems (see [6, 17, 28–37]). For example, in [35], by using fixed point theorems with lattice structure, the authors considered the existence of positive solution and sign-changing solution for integral boundary value problem under sublinear condition. In [37], the authors considered the existence of positive solution for fourth-order differential equation with fixed point theorems with lattice structure. In [6], the author considered the following second-order three-point boundary value problem:

$$\begin{aligned} -u''(t) &= g(t, u(t)), \quad 0 \leq t \leq 1, \\ u(0) &= 0, \\ u(1) &= \alpha u(\beta), \end{aligned} \quad (5)$$

where $g : [0, 1] \times (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ is continuous, $0 < \alpha < 1$, $0 < \beta < 1$. The author used fixed point theorems with lattice structure to study the existence of sign-changing solutions for the boundary value problem (5) under the unilaterally asymptotically linear condition.

Motivated by [6, 17, 28–37], we shall study the existence of nontrivial solutions for the boundary value problem (1), (2). In this paper, we assume that the nonlinear term satisfies superlinear conditions concerning the first eigenvalue corresponding to the relevant linear operator. The method we use is fixed point theorems with lattice structure. And we obtain the sufficient condition about the existence of negative solution and sign-changing solution for the boundary value problem (1), (2). The method is different from those of [2, 4]. And the main results are different from those of the work [2, 4]. This paper is arranged as follows. In Section 2, we give some definitions and fixed point theorems with lattice structure. In Section 3, we shall give some lemmas and the main results about the existence of nontrivial solutions (including negative solution and sign-changing solution) for the boundary value problem (1), (2). Finally, in Section 4, some examples are given to illustrate our main results.

2. Preliminaries

Let E be an ordered Banach space in which the partial ordering $\leq$ is induced by a cone $P \subset E$. P is called normal if there exists a positive constant $N > 0$ such that $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\|$. P is called solid if $\text{int } P \neq \emptyset$, i.e., P has nonempty interior. P is called total if $E = \overline{P} - \overline{P}$. If P is solid, then P is total. For the concepts and the properties about the cones, we refer to [31, 38, 39].

We call E a lattice under the partial ordering $\leq$, if $\sup\{u, v\}$ and $\inf\{u, v\}$ exist for arbitrary $u, v \in E$.

For $u \in E$, let

$$\begin{aligned} u^+ &= \sup\{u, \theta\}, \\ u^- &= \sup\{-u, \theta\}. \end{aligned} \quad (6)$$

u^+ and u^- are called positive part and negative part of u , respectively. Taking $|u| = u^+ + u^-$, then $|u| \in P$. For

the definition and the properties of the lattice, we refer to [40].

For convenience, we use the following notations:

$$\begin{aligned} u_+ &= u^+, \\ u_- &= -u^-, \end{aligned} \quad (7)$$

and clearly

$$\begin{aligned} u_+ &\in P, \\ u_- &\in (-P), \\ u &= u_+ + u_-. \end{aligned} \quad (8)$$

Definition 1 (see [28–31]). Let $D \subset E$ and $F : D \rightarrow E$ be a nonlinear operator. If there exists $u^* \in E$ such that

$$Fu = Fu_+ + Fu_- + u^*, \quad \forall u \in D, \quad (9)$$

then F is said to be quasi-additive on lattice.

Let $B : E \rightarrow E$ be a bounded linear operator. If $B(P) \subset P$, then the operator B is called to be positive.

In this section, we assume that E is a Banach space, P is a total cone, the partial ordering $\leq$ in E is induced by P , and E is a lattice in the partial ordering $\leq$.

Let $B : E \rightarrow E$ be a positive completely continuous linear operator; B^* the conjugated operator of B ; $r(B)$ a spectral radius of B ; and P^* the conjugated cone of P . Since $P \subset E$ is a total cone, by Krein-Rutman theorem, we can infer that if $r(B) \neq 0$, then there exist $\bar{u} \in P \setminus \{\theta\}$ and $f^* \in P^* \setminus \{\theta\}$, such that

$$\begin{aligned} B\bar{u} &= r(B)\bar{u}, \\ B^*f^* &= r(B)f^*. \end{aligned} \quad (10)$$

For $\delta > 0$. Let

$$P(f^*, \delta) = \{u \in P \mid f^*(u) \geq \delta \|u\|\}. \quad (11)$$

Then $P(f^*, \delta)$ is also a cone in E .

Definition 2 (see [30, 31, 41]). If there exist $\bar{u} \in P \setminus \{\theta\}$, $f^* \in P^* \setminus \{\theta\}$, and $\delta > 0$ such that (10) holds, and B maps P into $P(f^*, \delta)$, then the positive linear operator B is said to satisfy H condition.

Let P be a cone of a Banach space E . If $u \in (P \setminus \{\theta\})$ is a fixed point of A , then u is said to be a positive fixed point of A . If $u \in ((-P) \setminus \{\theta\})$ is a fixed point of operator A , then u is said to be a negative fixed point of operator A . If $u \in (P \setminus \{\theta\})$ is a fixed point of operator A , then u is said to be a positive fixed point of operator A . If $u \notin (P \cup (-P))$ is a fixed point of operator A , then u is said to be a sign-changing fixed point of operator A .

In [30], Sun and Liu considered computation for the topological degree about superlinear operators which are not cone mappings and obtained the following results.

Lemma 3. Let the cone $P \subset E$ be solid, and $A : E \rightarrow E$ be a completely continuous operator, and $A = BF$, where B is

a positive completely continuous linear operator satisfying H condition and F is quasi-additive on lattice. Assume that

(i) there exist $c_1 > r^{-1}(B)$ and $u_1 \in P$ such that

$$Fu \geq c_1 u - u_1, \quad \forall u \in P; \quad (12)$$

(ii) there exist $0 < c_2 < r^{-1}(B)$ and $u_2 \in P$ such that

$$Fu \geq c_2 u - u_2, \quad \forall u \in (-P); \quad (13)$$

(iii) $A\theta = \theta$, the Fréchet derivative A'_θ of A at θ exists, and 1 is not an eigenvalue of A'_θ .

Then the operator A has at least one nonzero fixed point.

In [31], Sun further obtained the following result about the existence of sign-changing fixed points for superlinear operators.

Lemma 4. Let the conditions in Lemma 3 hold, and β denote the sum of the algebraic multiplicities for all eigenvalues of A'_θ lying in $(1, +\infty)$. In addition, assume that

(iv) $\beta \neq 0$, β is an even number;

(v) $A(P \setminus \{\theta\}) \subset \text{int}P$, $A((-P) \setminus \{\theta\}) \subset \text{int}(-P)$.

Then the operator A has at least one negative fixed point and one sign-changing fixed point.

3. Main Results

For convenience, we list the following conditions.

(C₁) $\varphi : [0, 1] \times R^1 \rightarrow R^1$ is continuous, $\varphi(t, 0) = 0$, $\forall t \in [0, 1]$.

(C₂) The sequence of positive solutions of the equation

$$\sin \sqrt{\gamma} = \sum_{i=1}^{m-2} \beta_i \sin(\alpha_i \sqrt{\gamma}) \quad (14)$$

is

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots. \quad (15)$$

(C₃) $\lim_{x \rightarrow 0} (\varphi(t, x)/x) = \eta$ uniformly on $t \in [0, 1]$.

Let $X = C[0, 1]$ with supremum norm $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$. Set $P = \{x \in X \mid x(t) \geq 0, t \in [0, 1]\}$, the P is a solid cone in X . And under the partial order $\leq$ which is induced by P , X is a lattice.

In the following, we define some operators A , B , and Φ :

$$(Ax)(t) = \int_0^1 K(t, s) \varphi(s, x(s)) ds, \quad t \in [0, 1], \quad (16)$$

$$(Bx)(t) = \int_0^1 K(t, s) x(s) ds, \quad t \in [0, 1], \quad (17)$$

$$(\Phi x)(t) = \varphi(t, x(t)), \quad t \in [0, 1], \quad (18)$$

where

$$K(t, s) = g(t, s) + \frac{t \sum_{i=1}^{m-2} \beta_i g(\alpha_i, s)}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i}, \quad (19)$$

$$g(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \quad (20)$$

Obviously, $A = B\Phi$, and the nontrivial fixed points of the operator A are nontrivial solutions of the boundary value problem (1), (2) (see [3]).

Lemma 5 (see [2]). Let μ be a positive number, and the linear operator B be defined by (17). Eigenvalues of the linear operator μB are

$$\frac{\mu}{\lambda_1}, \frac{\mu}{\lambda_2}, \dots, \frac{\mu}{\lambda_n}, \dots, \quad (21)$$

and algebraic multiplicity of μ/λ_n is equal to 1, where λ_n is defined by (C₂).

Lemma 6. The linear operator B satisfies H condition.

Proof. By (C₂), Lemma 5, and the definition of the spectral radius, we know that

$$r(B) = \sup_{\lambda \in \{1/\lambda_n, n=1,2,\dots\}} |\lambda| = \frac{1}{\lambda_1} > 0. \quad (22)$$

By (20), we have

$$g(\alpha_i, s) \geq \alpha_i(1-\alpha_i)s(1-s), \quad \forall s \in [0, 1]. \quad (23)$$

$$g(t, s) \leq s(1-s), \quad \forall t, s \in [0, 1]. \quad (24)$$

By (19) and (23), we have

$$\begin{aligned} K(t, s) &\geq \frac{t \sum_{i=1}^{m-2} \beta_i g(\alpha_i, s)}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i} \\ &\geq \frac{t \sum_{i=1}^{m-2} \beta_i \alpha_i (1-\alpha_i) s(1-s)}{1 + \sum_{i=1}^{m-2} \beta_i (1-\alpha_i)}, \end{aligned} \quad (25)$$

$\forall t, s \in [0, 1].$

From (24) and (25), we have

$$\begin{aligned} K(t, s) &\geq \frac{t \sum_{i=1}^{m-2} \beta_i \alpha_i (1-\alpha_i)}{1 + \sum_{i=1}^{m-2} \beta_i (1-\alpha_i)} g(\tau, s), \\ &\forall \tau, t, s \in [0, 1]. \end{aligned} \quad (26)$$

By (19), we have

$$\begin{aligned} K(t, s) &\geq \frac{t \sum_{i=1}^{m-2} \beta_i g(\alpha_i, s)}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i} \geq \frac{t \sum_{i=1}^{m-2} \beta_i \tau g(\alpha_i, s)}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i} \\ &\geq \frac{t \sum_{i=1}^{m-2} \beta_i \alpha_i (1-\alpha_i)}{1 + \sum_{i=1}^{m-2} \beta_i (1-\alpha_i)} \frac{\sum_{i=1}^{m-2} \beta_i \tau g(\alpha_i, s)}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i}, \end{aligned} \quad (27)$$

$\forall \tau, t, s \in [0, 1].$

Hence, by adding (26) to (27), we have

$$\begin{aligned} 2K(t, s) &\geq t \frac{\sum_{i=1}^{m-2} \beta_i \alpha_i (1-\alpha_i)}{1 + \sum_{i=1}^{m-2} \beta_i (1-\alpha_i)} \left(g(\tau, s) \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} \beta_i \tau g(\alpha_i, s)}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i} \right), \end{aligned} \quad (28)$$

i.e.,

$$K(t, s) \geq MtK(\tau, s), \quad \forall \tau, t, s \in [0, 1], \quad (29)$$

where $M = \sum_{i=1}^{m-2} \beta_i \alpha_i (1 - \alpha_i) / 2(1 + \sum_{i=1}^{m-2} \beta_i (1 - \alpha_i))$.
Let

$$(B^*x)(t) = \int_0^1 K^*(t, s)x(s)ds, \quad \forall t \in [0, 1], \quad (30)$$

where $K^*(t, s) = K(s, t)$. Obviously, $r(B^*) = r(B) = 1/\lambda_1 > 0$. By Krein-Rutman theorem, there exist $x(t) \in P \setminus \{\theta\}$ and $x^*(t) \in P \setminus \{\theta\}$ such that

$$(Bx)(t) = r(B)x(t), \quad (31)$$

$$(B^*x^*)(t) = r(B)x^*(t). \quad (32)$$

By (29) and (32), we obtain

$$\begin{aligned} x^*(s) &= r^{-1}(B)(B^*x^*)(s) \\ &= r^{-1}(B) \int_0^1 K^*(s, t)x^*(t)dt \\ &= r^{-1}(B) \int_0^1 K(t, s)x^*(t)dt \\ &\geq Mr^{-1}(B) \int_0^1 tK(\tau, s)x^*(t)dt \\ &= \left[Mr^{-1}(B) \int_0^1 tx^*(t)dt \right] K(\tau, s), \end{aligned} \quad (33)$$

$\forall \tau, s \in [0, 1].$

Set

$$f^*(u) = \int_0^1 x^*(t)u(t)dt, \quad \forall u \in X, t \in [0, 1]. \quad (34)$$

Obviously, $f^* \in P^* \setminus \{\theta\}$, and by (34), for $u \in X$, we have

$$\begin{aligned} f^*(Bu) &= \int_0^1 x^*(t)(Bu)(t)dt \\ &= \int_0^1 x^*(t)dt \int_0^1 K(t, s)u(s)ds \\ &= \int_0^1 \int_0^1 K(t, s)x^*(t)u(s)dt ds \\ &= \int_0^1 \left(\int_0^1 K^*(s, t)x^*(t)dt \right) u(s)ds \\ &= \int_0^1 r(B)x^*(s)u(s)ds = r(B)f^*(u). \end{aligned} \quad (35)$$

That is,

$$B^*f^* = r(B)f^*. \quad (36)$$

From (33) and (35), we have

$$\begin{aligned} f^*(Bu) &= r(B) \int_0^1 x^*(s)u(s)ds \\ &\geq M \int_0^1 x^*(t)tdt \int_0^1 K(\tau, s)u(s)ds \\ &= \left(M \int_0^1 tx^*(t)dt \right) (Bu)(\tau), \end{aligned} \quad (37)$$

$\forall \tau \in [0, 1].$

By (37), we have

$$f^*(Bu) \geq \delta \|Bu\|, \quad (38)$$

where $\delta = M \int_0^1 tx^*(t)dt > 0$.

Therefore, from (31), (36), and (38), it is easy to know that the linear operator B satisfies H condition. $\square$

Theorem 7. Suppose that (C_1) , (C_2) , and (C_3) hold. In addition, assume that there exists $\gamma > 0$ such that

$$\liminf_{x \rightarrow +\infty} \frac{\varphi(t, x)}{x} \geq \lambda_1 + \gamma, \quad \text{uniformly on } t \in [0, 1]; \quad (39)$$

$$\limsup_{x \rightarrow -\infty} \frac{\varphi(t, x)}{x} \leq \lambda_1 - \gamma, \quad \text{uniformly on } t \in [0, 1]. \quad (40)$$

If $\eta \neq \lambda_1, \lambda_2, \dots, \lambda_n, \dots$, where λ_i is defined by (C_2) , then the boundary value problem (1), (2) has at least one nontrivial solution.

Proof. By (C_1) , we easily know that $A : X \rightarrow X$ is a completely continuous operator, and $B : X \rightarrow X$ is a bounded positive linear completely continuous operator (see [3]). By Lemma 6, we know that the linear operator B satisfies H condition.

For $x \in X$, let

$$\begin{aligned} x_+(t) &= \begin{cases} x(t), & x(t) \geq 0, \\ 0, & x(t) < 0, \end{cases} \\ x_-(t) &= \begin{cases} x(t), & x(t) \leq 0, \\ 0, & x(t) > 0, \end{cases} \end{aligned} \quad (41)$$

and then $x(t) = x_+(t) + x_-(t)$.

By $\varphi(t, 0) = 0$, we have

$$\begin{aligned} \Phi(x) &= \varphi(t, x(t)) = \varphi(t, x_+(t) + x_-(t)) \\ &= \varphi(t, x_+(t)) + \varphi(t, x_-(t)) \\ &= \Phi(x_+) + \Phi(x_-). \end{aligned} \quad (42)$$

From (42), we know that Φ is quasi-additive on lattice.

From (39) and (40), there exists $C > 0$ such that

$$\frac{\varphi(t, x)}{x} \geq \lambda_1 + \frac{\gamma}{4}, \quad \forall x \geq C, t \in [0, 1], \quad (43)$$

$$\frac{\varphi(t, x)}{x} \leq \lambda_1 - \frac{\gamma}{4}, \quad \forall x \leq -C, t \in [0, 1]. \quad (44)$$

By (43) and (44), we have

$$\varphi(t, x) \geq \left(\lambda_1 + \frac{\gamma}{4}\right)x, \quad \forall x \geq C, \quad t \in [0, 1], \quad (45)$$

$$\varphi(t, x) \geq \left(\lambda_1 - \frac{\gamma}{4}\right)x, \quad \forall x \leq -C, \quad t \in [0, 1]. \quad (46)$$

Let $\widetilde{C} = \max_{0 \leq t \leq 1, |x| \leq C} |\varphi(t, x)|$. Then by (45) and (46), we have

$$\varphi(t, x) \geq \left(\lambda_1 + \frac{\gamma}{4}\right)x - \widetilde{C}, \quad \forall x \geq 0, \quad t \in [0, 1], \quad (47)$$

$$\varphi(t, x) \geq \left(\lambda_1 - \frac{\gamma}{4}\right)x - \widetilde{C}, \quad \forall x \leq 0, \quad t \in [0, 1],$$

i.e.,

$$\Phi x \geq h_1 x - \widetilde{C}, \quad \forall x \in P, \quad (48)$$

$$\Phi x \geq h_2 x - \widetilde{C}, \quad \forall x \in (-P),$$

where $h_1 = \lambda_1 + \gamma/4$, $h_2 = \lambda_1 - \gamma/4$. Obviously, we have

$$h_1 > r^{-1}(B), \quad (49)$$

$$h_2 < r^{-1}(B).$$

In the following, we prove that $A'_\theta = \eta B$.

In fact, by $\varphi(t, 0) = 0$, $\forall t \in [0, 1]$, we have $A\theta = \theta$. From (C_3) , $\forall \epsilon > 0$, $\exists \delta > 0$, when $0 < |x| < \delta$, we have

$$\left| \frac{\varphi(t, x)}{x} - \eta \right| < \epsilon, \quad (50)$$

i.e.,

$$|\varphi(t, x) - \eta x| < \epsilon |x|, \quad \forall t \in [0, 1], \quad 0 < |x| < \delta. \quad (51)$$

So

$$\|\Phi x - \eta x\| \leq \epsilon \|x\|, \quad \forall \|x\| < \delta. \quad (52)$$

Therefore, by (52), we have

$$\begin{aligned} \|Ax - A\theta - \eta Bx\| &= \|B(\Phi x - \eta x)\| \\ &\leq \|B\| \cdot \|(\Phi x - \eta x)\| \leq \epsilon \|B\| \cdot \|x\|, \quad (53) \\ &\quad \forall \|x\| < \delta. \end{aligned}$$

So

$$\lim_{\|x\| \rightarrow 0} \frac{\|Ax - A\theta - \eta Bx\|}{\|x\|} = 0, \quad (54)$$

i.e.,

$$A'_\theta = \eta B. \quad (55)$$

Since $r(A'_\theta) = \eta r(B)$, we know that 1 is not the eigenvalue of A'_θ by Lemma 6 and (C_2) .

By the above proof, we know that the conditions of Lemma 3 hold. So by Lemma 3, the boundary value problem (1), (2) has at least one nontrivial solution. $\square$

Theorem 8. Assume that $(C_1) - (C_3)$, (39), and (40) are satisfied. In addition, suppose that $\varphi(t, x)x > 0$, $\forall t \in [0, 1], x \neq 0$, and $\lambda_{2n_0} < \eta < \lambda_{2n_0+1}$, where n_0 is a natural number. Then the boundary value problem (1), (2) has at least one negative solution and one sign-changing solution.

Proof. By (17), for $\forall x \in P \setminus \{\theta\}$, we have

$$\begin{aligned} (Bx)(t) &= \int_0^1 K(t, s)x(s)ds \\ &= \int_0^1 g(t, s)x(s)ds \\ &\quad + \frac{t \sum_{i=1}^{m-2} \beta_i \int_0^1 g(\alpha_i, s)x(s)ds}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i} \\ &\leq t(1-t) \int_0^1 x(s)ds \end{aligned} \quad (56)$$

$$\begin{aligned} &\quad + \frac{t \sum_{i=1}^{m-2} \beta_i}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i} \int_0^1 x(s)ds \\ &= \left[t(1-t) + \frac{t \sum_{i=1}^{m-2} \beta_i}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i} \right] \int_0^1 x(s)ds. \end{aligned}$$

$$\begin{aligned} (Bx)(t) &= \int_0^1 g(t, s)x(s)ds \\ &\quad + \frac{t \sum_{i=1}^{m-2} \beta_i \int_0^1 g(\alpha_i, s)x(s)ds}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i} \\ &\geq \frac{t \sum_{i=1}^{m-2} \beta_i \int_0^1 g(\alpha_i, s)x(s)ds}{1 - \sum_{i=1}^{m-2} \beta_i \alpha_i}. \end{aligned} \quad (57)$$

From (56) and (57), we obtain that

$$B(P \setminus \{\theta\}) \subset \text{int } P. \quad (58)$$

Similarly, we know that

$$B(-P \setminus \{\theta\}) \subset \text{int } (-P). \quad (59)$$

Since $\varphi(t, x)x > 0$, $\forall t \in [0, 1], x \neq 0$, we have $\varphi(t, x) > 0$, $\forall x > 0, t \in [0, 1]$, and $\varphi(t, x) < 0$, $\forall x < 0, t \in [0, 1]$. So we have

$$(\Phi x) \in P \setminus \{\theta\}, \quad \forall x \in P \setminus \{\theta\}. \quad (60)$$

$$(\Phi x) \in (-P) \setminus \{\theta\}, \quad \forall x \in (-P) \setminus \{\theta\}. \quad (61)$$

By (58)-(61), we have

$$\begin{aligned} A(P \setminus \{\theta\}) &\subset \text{int } P, \\ A((-P) \setminus \{\theta\}) &\subset \text{int } (-P). \end{aligned} \quad (62)$$

Let β be the sum of algebraic multiplicities for all the eigenvalues of A'_θ , lying in the interval $(1, \infty)$. By (55), Lemma 5, and $\lambda_{2n_0} < \eta < \lambda_{2n_0+1}$, we know that

$$\beta = 2n_0. \quad (63)$$

By (62) and (63), we know that the conditions (iv) and (v) in Lemma 4 hold. By the proof of Theorem 7, the conditions (i), (ii), and (iii) in Lemma 4 are satisfied. Therefore, by Lemma 4, the boundary value problem (1), (2) has at least one negative solution and one sign-changing solution. $\square$

4. Examples

We consider second-order four-point boundary value problem

$$\begin{aligned} -x''(t) &= \varphi(t, x(t)), \quad 0 \leq t \leq 1, \\ x(0) &= 0, \\ x(1) &= \frac{1}{3}x\left(\frac{1}{3}\right) + \frac{1}{2}x\left(\frac{1}{2}\right). \end{aligned} \quad (64)$$

By simple calculations, $\lambda_1 \approx 5.602$, $\lambda_2 \approx 42.32$, $\lambda_3 \approx 99.97$, and $\lambda_4 \approx 148.87$ are solutions of the equation

$$\sin \sqrt{x} = \frac{1}{3} \sin \frac{\sqrt{x}}{3} + \frac{1}{2} \sin \frac{\sqrt{x}}{2}. \quad (65)$$

Example 1. Choose

$$\begin{aligned} \varphi(t, x) &= \begin{cases} 8x + (t-1)\sqrt{x}, & t \in [0, 1], x \in [4, +\infty), \\ \frac{30+5t}{3}(x-1) - 3t, & t \in [0, 1], x \in (1, 4), \\ 3x - 3(1+t)x^2, & t \in [0, 1], x \in [-1, 1], \\ \frac{8-t}{7}(x+1) - (6+3t), & t \in [0, 1], x \in (-8, -1), \\ 2x + (t-1)\sqrt[3]{x}, & t \in [0, 1], x \in (-\infty, -8]. \end{cases} \end{aligned} \quad (66)$$

By (66), it is easy to know that $\varphi : [0, 1] \times (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ is continuous, and $\varphi(t, 0) = 0, \forall t \in [0, 1]$. By calculation, $\eta = 3 < \lambda_1$. We can choose $\gamma = 2$. Then we have

$$\begin{aligned} \liminf_{x \rightarrow +\infty} \frac{\varphi(t, x)}{x} &= 8 \geq \lambda_1 + \gamma, \\ \limsup_{x \rightarrow -\infty} \frac{\varphi(t, x)}{x} &= 2 \leq \lambda_1 - \gamma. \end{aligned} \quad (67)$$

So by Theorem 7, the boundary value problem (64) has at least one nontrivial solution.

Example 2. Choose

$$\begin{aligned} \varphi(t, x) &= \begin{cases} 10x + (1-t)\sqrt[3]{x}, & t \in [0, 1], x \in [8, +\infty), \\ \frac{31-3t}{7}(x-1) + (51+t), & t \in [0, 1], x \in (1, 8), \\ 50x + (1+t)x^{5/3}, & t \in [0, 1], x \in [-1, 1], \\ \frac{-21-4t}{26}(x+1) - (51+t), & t \in [0, 1], x \in (-27, -1), \\ x + (1-t)\sqrt[3]{x}, & t \in [0, 1], x \in (-\infty, -27]. \end{cases} \end{aligned} \quad (68)$$

By (68), we know that $\varphi : [0, 1] \times (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ is continuous, $\varphi(t, 0) = 0, \forall t \in [0, 1]$, and

$\varphi(t, x)x > 0, \forall t \in [0, 1], x \neq 0$. By calculation, $\lambda_2 \leq \eta = 50 < \lambda_3$. We can choose $\gamma = 4$. Then we have

$$\begin{aligned} \liminf_{x \rightarrow +\infty} \frac{\varphi(t, x)}{x} &= 10 \geq \lambda_1 + \gamma, \\ \limsup_{x \rightarrow -\infty} \frac{\varphi(t, x)}{x} &= 1 \leq \lambda_1 - \gamma. \end{aligned} \quad (69)$$

So by Theorem 8, the boundary value problem (64) has at least one negative solution and one sign-changing solution.

Example 3. Choose

$$\begin{aligned} \varphi(t, x) &= \begin{cases} \frac{45}{32}x^2 + (1-t)\sqrt[3]{x}, & t \in [0, 1], x \in [8, +\infty), \\ \frac{31-3t}{7}(x-1) + (61+t), & t \in [0, 1], x \in (1, 8), \\ 60x + (1+t)x^{5/3}, & t \in [0, 1], x \in [-1, 1], \\ \frac{-31-4t}{26}(x+1) - (61+t), & t \in [0, 1], x \in (-27, -1), \\ x + (1-t)\sqrt[3]{x}, & t \in [0, 1], x \in (-\infty, -27]. \end{cases} \end{aligned} \quad (70)$$

By (70), we know that $\varphi : [0, 1] \times (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ is continuous, $\varphi(t, 0) = 0, \forall t \in [0, 1]$, and $\varphi(t, x)x > 0, \forall t \in [0, 1], x \neq 0$. By calculation, $\lambda_2 \leq \eta = 60 < \lambda_3$. We can choose $\gamma = 3$. Then we have

$$\begin{aligned} \liminf_{x \rightarrow +\infty} \frac{\varphi(t, x)}{x} &= +\infty \geq \lambda_1 + \gamma, \\ \limsup_{x \rightarrow -\infty} \frac{\varphi(t, x)}{x} &= 1 \leq \lambda_1 - \gamma. \end{aligned} \quad (71)$$

So by Theorem 8, the boundary value problem (64) has at least one negative solution and one sign-changing solution.

Data Availability

The data underlying the findings of our manuscript can be obtained by simple calculations. We did not quote other data.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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