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Research Article

Integral Inequalities Involving Strongly Convex Functions

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We study the notions of strongly convex function as well as F-strongly convex function. We present here some new integral inequalities of Jensen's type for these classes of functions. A refinement of companion inequality to Jensen's inequality established by Matić and Pečarić is shown to be recaptured as a particular instance. Counterpart of the integral Jensen inequality for strongly convex functions is also presented. Furthermore, we present integral Jensen-Steffensen and Slater's inequality for strongly convex functions.

1. Introduction and Preliminaries

The word "convexity" is the most important, natural, and fundamental notations in mathematics. Convex functions were presented by Johan Jensen over 100 years ago. Over the past few years, multiple generalizations and extensions have been made for convexity. These extensions and generalizations in the theory of inequalities have made valuable contributions in many areas of mathematics. Some new generalized concepts in this point of view are quasiconvex [1], strongly convex [2], approximately convex [3], logarithmically convex [4], midconvex functions [5], pseudoconvex [6], φ-convex [7], λ-convex [8], h-convex [9], delta-convex [10], Schur convex [11–15], and others [16–19].

The main ingredient of our investigation is the strongly convex function [2]. Let Ψ be the real function defined on interval I and c be positive number, then we say that the function Ψ is strongly convex with modulus c on I if

$$\Psi(\eta z+(1-\eta)y)\le\eta\Psi(z)+(1-\eta)\Psi(y)-c\eta(1-\eta)(z-y)^2 \tag{1}$$

for all $y,z\in I$ and $\eta\in[0,1]$.

Every strongly convex function is convex, but the converse is not true in general. Strongly convex functions have been utilized for proving the convergence of a gradient type algorithm for minimizing a function. They play a significant role in mathematical economics, approximation theory, and optimization theory. Many applications and properties of them can be found in [2, 9, 20]. In 2016, Adamek [21] further generalized the notion of strongly convex function. They replaced the nonnegative term $c(z-y)^2$ by a nonnegative real valued function F and defined it as follows: function Ψ is said to be F-strongly convex function if

$$\Psi(\eta z+(1-\eta)y)\le\eta\Psi(z)+(1-\eta)\Psi(y)-\eta(1-\eta)F(z-y) \tag{2}$$

for all $y,z\in I$ and $\eta\in[0,1]$. From [22], we also have

$$\Psi(y)+\Psi'_+(y)(z-y)+F(z-y)\le\Psi(z), \tag{3}$$

where Ψ is F-strongly convex function.

In literature the following inequality is well-known as Jensen inequality.

Theorem 1 (see [4]). *Let (Λ,X,μ) be a measure space with $0<\mu(\Lambda)<\infty$ and $\Psi:I\to\mathbb{R}$ be convex function. Suppose $\phi:\Lambda\to I$ is such that $\phi,\Psi(\phi)\in L^1(\mu)$, then one has*

$$\Psi\left(\frac{1}{\mu(\Lambda)}\int_\Lambda\phi\,d\mu\right)\le\frac{1}{\mu(\Lambda)}\int_\Lambda\Psi(\phi)\,d\mu. \tag{4}$$

In 1981, Slater proved a companion inequality to the Jensen inequality [23].

Theorem 2. *Let (Λ, X, μ) be a measure space with $0 < \mu(\Lambda) < \infty$ and $\Psi : I \to \mathbb{R}$ be increasing and convex function. Suppose $\phi : \Lambda \to I$ is such that $\Psi(\phi)$, $\Psi'_+(\phi)$, and $\phi\Psi'_+(\phi) \in L^1(\mu)$. If $\int_\Lambda \Psi'_+(\phi)d\mu \neq 0$, then one has*

$$\frac{1}{\mu(\Lambda)}\int_\Lambda \Psi(\phi)\,d\mu \le \Psi\left(\frac{\int_\Lambda \phi\Psi'_+(\phi)\,d\mu}{\int_\Lambda \Psi'_+(\phi)\,d\mu}\right). \tag{5}$$

In the case when Ψ is strictly convex, then equality holds in (5) if and only if ϕ is constant almost everywhere on Λ.

Remark 3. Some improvements and reversions of Slater's inequality are given in [24, 25].

The following inequality is the integral analogue of another companion inequality to the Jensen inequality.

Theorem 4 (see [26]). *Let (Λ, X, μ) be a measure space with $0 < \mu(\Lambda) < \infty$ and $\Psi : I \to \mathbb{R}$ be convex function. Suppose $\phi : \Lambda \to I$ is such that ϕ, $\Psi(\phi)$, $\Psi'_+(\phi)$, and $\phi\Psi'_+(\phi) \in L^1(\mu)$ and*

$$\begin{aligned}\overline{\phi} &= \frac{1}{\mu(\Lambda)}\int_\Lambda \phi\,d\mu\\ \overline{\Psi} &= \frac{1}{\mu(\Lambda)}\int_\Lambda \Psi(\phi)\,d\mu.\end{aligned} \tag{6}$$

Then the following inequalities hold:

$$0 \le \overline{\Psi} - \Psi\left(\overline{\phi}\right) \le \frac{1}{\mu(\Lambda)}\int_\Lambda \left(\phi - \overline{\phi}\right)\Psi'_+(\phi)\,d\mu. \tag{7}$$

If the function Ψ is strictly convex, then equality holds in (7) if and only if ϕ is constant almost everywhere on Λ.

Matić and Pečarić established a general inequality from which one can directly obtain inequalities (5) and (7).

Theorem 5 (see [27]). *Let all the assumptions of Theorem 4 be fulfilled. If $d_1, d_2 \in I$, then one has*

$$\begin{aligned}&\Psi(d_1) + \Psi'_+(d_1)\left(\overline{\phi} - d_1\right) \le \overline{\Psi}\\ &\quad\le \Psi(d_2) + \frac{1}{\mu(\Lambda)}\int_\Lambda (\phi - d_2)\Psi'_+(\phi)\,d\mu.\end{aligned} \tag{8}$$

Also, when Ψ is strictly convex, then equality in the left side in (8) holds if and only if $\phi = d_1$ almost everywhere on Λ, while equality in the right side in (8) holds if and only if $\phi = d_2$ almost everywhere on Λ.

Remark 6. Under the assumptions of Theorem 4, let $\int_\Lambda \Psi'_+(\phi)d\mu \neq 0$ and $\overline{\overline{\Psi}} = \int_\Lambda \phi\Psi'_+(\phi)d\mu / \int_\Lambda \Psi'_+(\phi)d\mu \in I$, then by setting $d_2 = \overline{\overline{\Psi}}$ in (8), we get Slater's inequality (5), and similarly by setting $d_2 = \overline{\phi}$ in (8), we get (7).

Merentes and Nikodem improved the Jensen inequality for strongly convex functions as follows.

Theorem 7 (see [28]). *Let (Λ, X, μ) be a probability measure space and $\Psi : I \to \mathbb{R}$ be strongly convex function with modulus c. Suppose $\phi : \Lambda \to I$ is a Lebesgue integrable function and $\overline{\theta} = \int_\Lambda \phi\,d\mu$. Then the following inequality holds:*

$$0 \le \int_\Lambda \Psi(\phi)\,d\mu - \Psi\left(\overline{\theta}\right) - c\int_\Lambda \left(\phi - \overline{\theta}\right)^2 d\mu. \tag{9}$$

For more recent results related to strongly convex function and Jensen type inequalities we recommend [22, 29–34].

This paper is organized as follows. In Section 2, we establish general inequalities for F-strongly convex function as well as strongly convex functions. As a consequence, we obtain integral Jensen inequality and Slater's inequality for strongly convex functions. Also by the virtue of these general inequalities we deduce converse of Jensen inequality. In Section 3, we give some properties of strongly convex functions. By using these properties of strongly convex functions we prove Jensen-Steffensen and Slater's type inequalities.

2. Jensen's Type Inequalities

We start this section to give the following general theorem.

Theorem 8. *Let (Λ, X, μ) be a measure space with $0 < \mu(\Lambda) < \infty$ and $\Psi : I \to \mathbb{R}$ be F-strongly convex function. Suppose $\phi : \Lambda \to I$ is such that , $\Psi(\phi)$, $\Psi'_+(\phi)$, and $\phi\Psi'_+(\phi) \in L^1(\mu)$ and also $\overline{\phi} = (1/\mu(\Lambda))\int_\Lambda \phi\,d\mu$, $\overline{\Psi} = (1/\mu(\Lambda))\int_\Lambda \Psi(\phi)d\mu$. If $d_1, d_2 \in I$, then one has*

$$\begin{aligned}&\Psi(d_1) + \Psi'_+(d_1)\left(\overline{\phi} - d_1\right) + \frac{1}{\mu(\Lambda)}\int_\Lambda F(\phi - d_1)\,d\mu\\ &\quad\le \overline{\Psi}\\ &\quad\le \Psi(d_2) + \frac{1}{\mu(\Lambda)}\int_\Lambda (\phi - d_2)\Psi'_+(\phi)\,d\mu\\ &\qquad - \frac{1}{\mu(\Lambda)}\int_\Lambda F(d_2 - \phi)\,d\mu.\end{aligned} \tag{10}$$

Proof. Since Ψ is strongly convex function, therefore

$$\Psi(y) - \Psi'_+(y)(y - z) + F(z - y) \le \Psi(z). \tag{11}$$

Letting $y \to \phi$ and $z \to d_2$, in (11), we get

$$\Psi(\phi) - \Psi'_+(\phi)(\phi - d_2) + F(d_2 - \phi) \le \Psi(d_2). \tag{12}$$

Taking integral of (12) and then dividing by $\mu(\Lambda)$, we obtain

$$\begin{aligned}\overline{\Psi} &\le \Psi(d_2) + \frac{1}{\mu(\Lambda)}\int_\Lambda (\phi - d_2)\Psi'_+(\phi)\,d\mu\\ &\quad - \frac{1}{\mu(\Lambda)}\int_\Lambda F(d_2 - \phi)\,d\mu.\end{aligned} \tag{13}$$

Similarly, rearranging (11), we get

$$\Psi(y) + \Psi'_+(y)(z - y) + F(z - y) \le \Psi(z). \tag{14}$$

Letting $y \to d_1$ and $z \to \phi$, in (14), we have

$$\Psi(d_1) + \Psi'_+(d_1)(\phi - d_1) + F(\phi - d_1) \le \Psi(\phi). \tag{15}$$

Taking integral of (15) and then dividing by $\mu(\Lambda)$, we get

$$\begin{aligned}\Psi(d_1) + \Psi'_+(d_1)\left(\overline{\phi} - d_1\right) + \frac{1}{\mu(\Lambda)}\int_\Lambda F(\phi - d_1)\,d\mu \\ \le \overline{\Psi}.\end{aligned} \tag{16}$$

Combining (13) and (16), we obtain (10). □

By virtue of Theorem 8, we can deduce some new and interesting consequences.

Proposition 9. *Suppose that all the assumptions of Theorem 8 are satisfied. Then*

$$\begin{aligned}&\frac{1}{\mu(\Lambda)}\int_\Lambda \Psi(\phi)\,d\mu \\ &\quad\le \frac{1}{\mu(\Lambda)}\int_\Lambda \phi\Psi'_+(\phi)\,d\mu \\ &\qquad + \inf_{z\in I}\left\{\Psi(z) - \frac{z}{\mu(\Lambda)}\int_\Lambda \Psi'_+(\phi)\,d\mu\right\} \\ &\qquad - \inf_{z\in I}\left\{\frac{1}{\mu(\Lambda)}\int_\Lambda F(z-\phi)\,d\mu\right\}.\end{aligned} \tag{17}$$

Proof. If we set $y = \phi$ in (11) and taking integral over Λ and then dividing by $\mu(\Lambda)$, we have

$$\begin{aligned}&\frac{1}{\mu(\Lambda)}\int_\Lambda \Psi(\phi)\,d\mu - \frac{1}{\mu(\Lambda)}\int_\Lambda \phi\Psi'_+(\phi)\,d\mu \\ &\quad + \frac{z}{\mu(\Lambda)}\int_\Lambda \Psi'_+(\phi)\,d\mu + \frac{1}{\mu(\Lambda)}\int_\Lambda F(z-\phi)\,d\mu \\ &\quad \le \Psi(z),\end{aligned} \tag{18}$$

or equivalently

$$\begin{aligned}\frac{1}{\mu(\Lambda)}\int_\Lambda \Psi(\phi)\,d\mu \le \frac{1}{\mu(\Lambda)}\int_\Lambda \phi\Psi'_+(\phi)\,d\mu + \Psi(z) \\ - \frac{z}{\mu(\Lambda)}\int_\Lambda \Psi'_+(\phi)\,d\mu \\ - \frac{1}{\mu(\Lambda)}\int_\Lambda F(z-\phi)\,d\mu.\end{aligned} \tag{19}$$

Taking the infimum over $z \in I$, we obtain (17). □

Proposition 10. *Suppose that all the assumptions of Theorem 8 are satisfied and $\overline{x} = (1/\mu(\Lambda))\int_\Lambda \Psi'_+(\phi)d\mu$, then*

$$\begin{aligned}0 &\le \overline{\Psi} - \Psi\left(\overline{\phi}\right) - \frac{1}{\mu(\Lambda)}\int_\Lambda F\left(\phi - \overline{\phi}\right)d\mu \\ &\le \inf_{z\in I}\{\Psi(z) - z\overline{x}\} + \frac{1}{\mu(\Lambda)}\int_\Lambda \phi\Psi'_+(\phi)\,d\mu - \Psi\left(\overline{\phi}\right) \\ &\quad - \inf_{z\in I}\left\{\frac{1}{\mu(\Lambda)}\int_\Lambda F(z-\phi)\,d\mu\right\} \\ &\quad - \frac{1}{\mu(\Lambda)}\int_\Lambda F\left(\phi - \overline{\phi}\right)d\mu \\ &\le \frac{1}{\mu(\Lambda)}\int_\Lambda \phi\Psi'_+(\phi)\,d\mu - \overline{\phi}\overline{x} \\ &\quad - \inf_{z\in I}\left\{\frac{1}{\mu(\Lambda)}\int_\Lambda F(z-\phi)\,d\mu\right\} \\ &\quad - \frac{1}{\mu(\Lambda)}\int_\Lambda F\left(\phi - \overline{\phi}\right)d\mu.\end{aligned} \tag{20}$$

Proof. By setting $d_1 = \overline{\phi}$ and $d_2 = z \in I$ in (10), we have

$$\begin{aligned}\Psi\left(\overline{\phi}\right) + \frac{1}{\mu(\Lambda)}\int_\Lambda F\left(\phi - \overline{\phi}\right)d\mu \le \overline{\Psi} \\ \le \Psi(z) + \frac{1}{\mu(\Lambda)}\int_\Lambda (\phi - z)\Psi'_+(\phi)\,d\mu \\ - \frac{1}{\mu(\Lambda)}\int_\Lambda F(z-\phi)\,d\mu.\end{aligned} \tag{21}$$

Indeed, the following equivalent form of (21) is

$$\begin{aligned}0 &\le \overline{\Psi} - \Psi\left(\overline{\phi}\right) - \frac{1}{\mu(\Lambda)}\int_\Lambda F\left(\phi - \overline{\phi}\right)d\mu \\ &\le \Psi(z) + \frac{1}{\mu(\Lambda)}\int_\Lambda (\phi - z)\Psi'_+(\phi)\,d\mu - \Psi\left(\overline{\phi}\right) \\ &\quad - \frac{1}{\mu(\Lambda)}\int_\Lambda F(z-\phi)\,d\mu \\ &\quad - \frac{1}{\mu(\Lambda)}\int_\Lambda F\left(\phi - \overline{\phi}\right)d\mu.\end{aligned} \tag{22}$$

Taking the infimum over $z \in I$, we can easily derive the first and the second inequality in (20). The remaining third inequality in (20) follows because

$$\inf_{z\in I}\{\Psi(z) - z\overline{x}\} \le \Psi\left(\overline{\phi}\right) - \overline{\phi}\overline{x}. \tag{23}$$

□

Corollary 11. *Let (Λ, X, μ) be a measure space with $0 < \mu(\Lambda) < \infty$ and $\Psi : I \to \mathbb{R}$ be strongly convex function with modulus c. Suppose $\phi : \Lambda \to I$ is such that ϕ, $\Psi(\phi)$,*

$\Psi'_+(\phi)$, *and* $\phi\Psi'_+(\phi) \in L^1(\mu)$ *and also* $\overline{\phi} = (1/\mu(\Lambda)) \int_\Lambda \phi\, d\mu$, $\overline{\Psi} = (1/\mu(\Lambda)) \int_\Lambda \Psi(\phi) d\mu$. *If* $d_1, d_2 \in I$, *then one has*

$$\begin{aligned} &\Psi(d_1) + \Psi'_+(d_1)(\overline{\phi} - d_1) + \frac{c}{\mu(\Lambda)} \int_\Lambda (\phi - d_1)^2 d\mu \\ &\quad \le \overline{\Psi} \\ &\quad \le \Psi(d_2) + \frac{1}{\mu(\Lambda)} \int_\Lambda (\phi - d_2) \Psi'_+(\phi) d\mu \\ &\qquad - \frac{c}{\mu(\Lambda)} \int_\Lambda (\phi - d_2)^2 d\mu. \end{aligned} \tag{24}$$

Remark 12. If we put $d_1 = \overline{\phi}$ in (24) and take probability measure space, then we obtain integral Jensen inequality (9) for strongly convex function.

In the following corollary, we obtain integral Slater's inequality for strongly convex function.

Corollary 13. *Suppose* Ψ, ϕ, $\mu(\Lambda)$, $\overline{\phi}$, *and* $\overline{\Psi}$ *are stated as in Corollary 11 and assume that* $\int_\Lambda \Psi'_+(\phi) d\mu \neq 0$; *also if* $\overline{\overline{\phi}} = \int_\Lambda \phi\Psi'_+(\phi) d\mu / \int_\Lambda \Psi'_+(\phi) d\mu \in I$, *then*

$$\overline{\Psi} \le \Psi\left(\overline{\overline{\phi}}\right) - \frac{c}{\mu(\Lambda)} \int_\Lambda \left(\phi - \overline{\overline{\phi}}\right)^2 d\mu. \tag{25}$$

Proof. By setting $d_2 = \overline{\overline{\phi}}$ in (24), we deduced

$$\begin{aligned} \overline{\Psi} &\le \Psi\left(\frac{\int_\Lambda \phi\Psi'_+(\phi) d\mu}{\int_\Lambda \Psi'_+(\phi) d\mu}\right) \\ &\quad + \frac{1}{\mu(\Lambda)} \int_\Lambda \left(\phi - \frac{\int_\Lambda \phi\Psi'_+(\phi) d\mu}{\int_\Lambda \Psi'_+(\phi) d\mu}\right) \Psi'_+(\phi) d\mu \\ &\quad - \frac{c}{\mu(\Lambda)} \int_\Lambda \left(\phi - \frac{\int_\Lambda \phi\Psi'_+(\phi) d\mu}{\int_\Lambda \Psi'_+(\phi) d\mu}\right)^2 d\mu. \end{aligned} \tag{26}$$

Since $(1/\mu(\Lambda)) \int_\Lambda (\phi - \int_\Lambda \phi\Psi'_+(\phi) d\mu / \int_\Lambda \Psi'_+(\phi) d\mu) \Psi'_+(\phi) d\mu = 0$, therefore (26) is equivalent to (25). □

In the following corollary, we obtain a converse of the Jensen inequality for strongly convex function.

Corollary 14. *Suppose* Ψ, ϕ, $\mu(\Lambda)$, $\overline{\phi}$, *and* $\overline{\Psi}$ *are stated as in Corollary 11, then one has*

$$\begin{aligned} 0 &\le \overline{\Psi} - \Psi\left(\overline{\phi}\right) - \frac{c}{\mu(\Lambda)} \int_\Lambda \left(\phi - \overline{\phi}\right)^2 d\mu \\ &\le \frac{1}{\mu(\Lambda)} \int_\Lambda \left(\phi - \overline{\phi}\right) \Psi'_+(\phi) d\mu \\ &\quad - \frac{2c}{\mu(\Lambda)} \int_\Lambda \left(\phi - \overline{\phi}\right)^2 d\mu. \end{aligned} \tag{27}$$

Proof. By setting $d_1 = d_2 = \overline{\phi}$ in (24), we obtain (27). □

3. Jensen-Steffensen Inequality for Riemann-Stieltjes Integrals

To prove the main results of this section, first we prove the following lemma which will play a key role in the proof of main results.

Lemma 15. *Let* $\Psi : (\gamma, \beta) \to \mathbb{R}$ *be strongly convex function with modulus c. Suppose y is fixed element from interval* (γ, β), *then*

(a) *the function* $\Delta_y(\zeta) : (\gamma, \beta) \to \mathbb{R}$ *defined by*

$$\Delta_y(\zeta) = \Psi(\zeta) - \Psi(y) - \Psi'_+(y)(\zeta - y) - c(\zeta - y)^2 \tag{28}$$

(b) *and the function* $\overline{\Delta}_y(\zeta) : (\gamma, \beta) \to \mathbb{R}$ *defined by*

$$\overline{\Delta}_y(\zeta) = \Psi(y) - \Psi(\zeta) - \Psi'_+(\zeta)(y - \zeta) - c(y - \zeta)^2 \tag{29}$$

are nonnegative on (γ, β), *decreasing on* $(\gamma, y]$ *and increasing on* $[y, \beta)$.

Proof. By definition of strongly convexity we have

$$\begin{aligned} \Delta_y(\zeta) &= \Psi(\zeta) - \Psi(y) - \Psi'_+(y)(\zeta - y) - c(\zeta - y)^2 \\ &\ge 0, \quad \forall \zeta, y \in (\gamma, \beta) \end{aligned} \tag{30}$$

It means that $\Delta_y(\zeta)$ is nonnegative on (γ, β).

Let $\gamma < \zeta_1 < \zeta_2 \le y$. Since Ψ is strongly convex with modulus c, therefore $\Psi'_+(\zeta) - 2c\zeta$ is increasing and hence

$$\Psi'_+(\zeta_2) - 2c\zeta_2 \le \Psi'_+(y) - 2cy; \tag{31}$$

it follows

$$\begin{aligned} \Psi'_+(y)(\zeta_1 - \zeta_2) &\le \Psi'_+(\zeta_2)(\zeta_1 - \zeta_2) - 2c\zeta_2(\zeta_1 - \zeta_2) \\ &\quad + 2cy(\zeta_1 - \zeta_2). \end{aligned} \tag{32}$$

Setting $\zeta = \zeta_1$ and $\zeta = \zeta_2$ in (28) and then taking the difference, we get

$$\begin{aligned} \Delta_y(\zeta_1) - \Delta_y(\zeta_2) &= \Psi(\zeta_1) - \Psi(\zeta_2) \\ &\quad - \Psi'_+(y)(\zeta_1 - \zeta_2) - c(\zeta_1 - y)^2 \\ &\quad + c(\zeta_2 - y)^2 \\ &\ge \Psi(\zeta_1) - \Psi(\zeta_2) \\ &\quad - \Psi'_+(\zeta_2)(\zeta_1 - \zeta_2) \\ &\quad + 2c\zeta_2(\zeta_1 - \zeta_2) - 2cy(\zeta_1 - \zeta_2) \\ &\quad - c(\zeta_1 - y)^2 + c(\zeta_2 - y)^2 \\ &\qquad \text{(by using (32))} \\ &= \Psi(\zeta_1) - \Psi(\zeta_2) \\ &\quad - \Psi'_+(\zeta_2)(\zeta_1 - \zeta_2) - c(\zeta_1 - \zeta_2)^2 \\ &\ge 0. \end{aligned} \tag{33}$$

Hence, $\Delta_y(\zeta)$ is decreasing on $(\gamma, y]$.

Also, if $y \le \zeta_1 < \zeta_2 < \beta$, then similarly as above by strongly convexity we have

$$\Psi'_+(y) - 2cy \le \Psi'_+(\zeta_1) - 2c\zeta_1, \tag{34}$$

from which it follows that

$$\begin{aligned}\Psi'_+(y)(\zeta_1-\zeta_2) \ge{}& \Psi'_+(\zeta_1)(\zeta_1-\zeta_2) - 2c\zeta_1(\zeta_1-\zeta_2)\\ &+ 2cy(\zeta_1-\zeta_2).\end{aligned} \tag{35}$$

Setting $\zeta = \zeta_1$ and $\zeta = \zeta_2$ in (28), then taking the difference we get

$$\begin{aligned}\Delta_y(\zeta_1) - \Delta_y(\zeta_2) &= \Psi(\zeta_1) - \Psi(\zeta_2)\\ &\quad - \Psi'_+(y)(\zeta_1-\zeta_2) - c(\zeta_1-y)^2\\ &\quad + c(\zeta_2-y)^2\\ &\le \Psi(\zeta_1) - \Psi(\zeta_2)\\ &\quad - \Psi'_+(\zeta_1)(\zeta_1-\zeta_2)\\ &\quad + 2c\zeta_1(\zeta_1-\zeta_2) - 2cy(\zeta_1-\zeta_2)\\ &\quad - c(\zeta_1-y)^2 + c(\zeta_2-y)^2\\ &\qquad\qquad (\text{by using } (35))\\ &= \Psi(\zeta_1) - \Psi(\zeta_2)\\ &\quad - \Psi'_+(\zeta_1)(\zeta_1-\zeta_2) - c(\zeta_1-\zeta_2)^2\\ &\le 0.\end{aligned} \tag{36}$$

Hence, $\Delta_y(\zeta)$ is increasing on $[y, \beta)$.

In the same manner as above, we also obtain $\overline{\Delta}_y(\zeta)$ is nonnegative on (γ, β).

Let $\gamma < \zeta_1 < \zeta_2 \le y$, then by strongly convexity we have

$$\Psi'_+(\zeta_1) - 2c\zeta_1 \le \Psi'_+(\zeta_2) - 2c\zeta_2, \tag{37}$$

$$\begin{aligned}&\text{that is, } \Psi'_+(\zeta_2)(y-\zeta_2)\\ &\quad \ge \Psi'_+(\zeta_1)(y-\zeta_2) - 2c\zeta_1(y-\zeta_2) + 2c\zeta_2(y-\zeta_2).\end{aligned} \tag{38}$$

Setting $\zeta = \zeta_1$ and $\zeta = \zeta_2$ in (29) and then taking the difference we get

$$\begin{aligned}\overline{\Delta}_y(\zeta_1) - \overline{\Delta}_y(\zeta_2) &= \Psi(\zeta_2) - \Psi(\zeta_1)\\ &\quad - \Psi'_+(\zeta_1)(y-\zeta_1)\\ &\quad + \Psi'_+(\zeta_2)(y-\zeta_2) - c(y-\zeta_1)^2\\ &\quad + c(y-\zeta_2)^2\\ &\ge \Psi(\zeta_2) - \Psi(\zeta_1)\\ &\quad - \Psi'_+(\zeta_1)(y-\zeta_1)\\ &\quad + \Psi'_+(\zeta_1)(y-\zeta_2)\\ &\quad - 2c\zeta_1(y-\zeta_2) + 2c\zeta_2(y-\zeta_2)\\ &\quad - c(y-\zeta_1)^2 + c(y-\zeta_2)^2\\ &\qquad\qquad (\text{by } (38))\\ &= \Psi(\zeta_2) - \Psi(\zeta_1)\\ &\quad - \Psi'_+(\zeta_1)(\zeta_2-\zeta_1) - c(\zeta_2-\zeta_1)^2\\ &\ge 0.\end{aligned} \tag{39}$$

Hence, $\overline{\Delta}_y(\zeta)$ is decreasing on $(\gamma, y]$.

Also, if $y \le \zeta_1 < \zeta_2 < \beta$, then by strongly convexity we have

$$\Psi'_+(\zeta_1) - 2c\zeta_1 \le \Psi'_+(\zeta_2) - 2c\zeta_2, \tag{40}$$

$$\begin{aligned}\text{i.e., } \Psi'_+(\zeta_1)(y-\zeta_1) \ge{}& \Psi'_+(\zeta_2)(y-\zeta_1)\\ &- 2c\zeta_2(y-\zeta_1)\\ &+ 2c\zeta_1(y-\zeta_1).\end{aligned} \tag{41}$$

Setting $\zeta = \zeta_1$ and $\zeta = \zeta_2$ in (29) and then taking the difference we get

$$\begin{aligned}\overline{\Delta}_y(\zeta_1) - \overline{\Delta}_y(\zeta_2) &= \Psi(\zeta_2) - \Psi(\zeta_1)\\ &\quad - \Psi'_+(\zeta_1)(y-\zeta_1)\\ &\quad + \Psi'_+(\zeta_2)(y-\zeta_2) - c(y-\zeta_1)^2\\ &\quad + c(y-\zeta_2)^2\\ &\le \Psi(\zeta_2) - \Psi(\zeta_1)\\ &\quad - \Psi'_+(\zeta_2)(y-\zeta_1)\\ &\quad + \Psi'_+(\zeta_2)(y-\zeta_2)\\ &\quad - 2c\zeta_1(y-\zeta_1) + 2c\zeta_2(y-\zeta_1)\\ &\quad - c(y-\zeta_1)^2 + c(y-\zeta_2)^2\\ &\qquad\qquad (\text{by } (41))\\ &= \Psi(\zeta_2) - \Psi(\zeta_1)\\ &\quad - \Psi'_+(\zeta_2)(\zeta_1-\zeta_2) + c(\zeta_1-\zeta_2)^2\\ &\le 0.\end{aligned} \tag{42}$$

Hence, $\overline{\Delta}_y(\zeta)$ is increasing on $[y, \beta)$. This completes the proof. □

The following lemma is given in [35].

Lemma 16. *Let $\phi : [\gamma, \beta] \to \mathbb{R}$ be a nonnegative function and suppose $\xi : [\gamma, \beta] \to \mathbb{R}$ is either a bounded variation or*

continuous. Also assume that the functions ϕ and ξ have no common discontinuity points.

(a) *If ϕ is increasing on $[\gamma,\beta]$, then*

$$\phi(\beta)\inf_{\gamma\le\tau\le\beta}\int_\tau^\beta d\xi(x)\le\int_\gamma^\beta\phi(x)\,d\xi(x)\le\phi(\beta)\sup_{\gamma\le\tau\le\beta}\int_\tau^\beta d\xi(x). \tag{43}$$

(b) *If ϕ is decreasing on $[\gamma,\beta]$, then*

$$\phi(\gamma)\inf_{\gamma\le\tau\le\beta}\int_\gamma^\tau d\xi(x)\le\int_\gamma^\beta\phi(x)\,d\xi(x)\le\phi(\gamma)\sup_{\gamma\le\tau\le\beta}\int_\gamma^\tau d\xi(x). \tag{44}$$

In the next result, we prove some general integral inequalities for strongly convex functions.

Theorem 17. *Suppose $\phi:[\gamma,\beta]\to(\kappa,\nu)$ is monotonic and continuous function and let $\Psi:(\kappa,\nu)\to\mathbb{R}$ be a strongly convex function with modulus c. If $\xi:[\gamma,\beta]\to\mathbb{R}$ is either a bounded variation or continuous and satisfying $\xi(\gamma)\le\xi(x)\le\xi(\beta)$ for all $x\in[\gamma,\beta]$, $\xi(\beta)-\xi(\gamma)>0$, then $\overline{\phi}$ and $\overline{\Psi}$ given by*

$$\overline{\phi}=\frac{1}{\xi(\beta)-\xi(\gamma)}\int_\gamma^\beta\phi(x)\,d\xi(x),\qquad \overline{\Psi}=\frac{1}{\xi(\beta)-\xi(\gamma)}\int_\gamma^\beta\Psi(\phi(x))\,d\xi(x) \tag{45}$$

are well defined and $\overline{\phi}\in(\kappa,\nu)$. Also, if $\Psi'_+(\phi(x))$ and ξ have no common discontinuity points, then for $d_1,d_2\in(\kappa,\nu)$, one has

$$\begin{aligned}&\Psi(d_1)+\Psi'_+(d_1)\left(\overline{\phi}-d_1\right)+\frac{c}{\xi(\beta)-\xi(\gamma)}\int_\gamma^\beta(\phi(x)-d_1)^2\,d\xi(x)\le\overline{\Psi}\\&\le\Psi(d_2)+\frac{1}{\xi(\beta)-\xi(\gamma)}\int_\gamma^\beta(\phi(x)-d_2)\Psi'_+(\phi(x))\,d\xi(x)\\&-\frac{c}{\xi(\beta)-\xi(\gamma)}\int_\gamma^\beta(\phi(x)-d_2)^2\,d\xi(x).\end{aligned} \tag{46}$$

Proof. Under the given conditions in [35], it has been shown that

$$\phi([\gamma,\beta])=[\phi(\gamma),\phi(\beta)]\subseteq(\kappa,\nu). \tag{47}$$

We define the function $\lambda:[\gamma,\beta]\to\mathbb{R}$ by $\lambda(x)=\Delta_{d_1}(\phi(x))$, where Δ_{d_1} is defined as in Lemma 15(a), i.e.,

$$\begin{aligned}\lambda(x)&=\Delta_{d_1}(\phi(x))\\&=\Psi(\phi(x))-\Psi(d_1)-\Psi'_+(d_1)(\phi(x)-d_1)\\&\quad-c(\phi(x)-d_1)^2.\end{aligned} \tag{48}$$

From Lemma 15(a) it follows that λ is nonnegative and since ϕ and $\Psi(\phi)$ are continuous therefore the integral $\int_\gamma^\beta\lambda(x)d\xi(x)$ exists. Thus we discuss the following three cases:

(i) If $\phi(\beta)\le d_1$, since ϕ is increasing on $[\gamma,\beta]$ and by Lemma 15(a) Δ_{d_1} is decreasing on $(\kappa,d_1]$, therefore $\lambda=\Delta_{d_1}(\phi(x))$ is decreasing on $[\gamma,\beta]$. So using Lemma 16 (b), we obtain

$$\int_\gamma^\beta\lambda(x)\,d\xi(x)\ge\lambda(\gamma)\inf_{\gamma\le\tau\le\beta}\{\xi(\tau)-\xi(\gamma)\}=0. \tag{49}$$

(ii) If $d_1\le\phi(\gamma)$, since Δ_{d_1} is increasing on $[d_1,\nu)$ by Lemma 15(a), therefore $\lambda=\Delta_{d_1}(\phi(x))$ is increasing on $[\gamma,\beta]$. So using Lemma 16 (a), we have

$$\int_\gamma^\beta\lambda(x)\,d\xi(x)\ge\lambda(\beta)\inf_{\gamma\le\tau\le\beta}\{\xi(\beta)-\xi(\tau)\}=0. \tag{50}$$

(iii) If $\phi(\gamma)<d_1<\phi(\beta)$, since ϕ is continuous on $[\gamma,\beta]$, there exists at least one point $\overline{x}\in(\gamma,\beta)$ such that $\phi(\overline{x})=d_1$. Also by Lemma 15(a), λ is decreasing on $[\gamma,\overline{x}]$ and λ is increasing on $[\overline{x},\beta]$. Using Lemma 16, we have

$$\begin{aligned}\int_\gamma^\beta\lambda(x)\,d\xi(x)&=\int_\gamma^{\overline{x}}\lambda(x)\,d\xi(x)+\int_{\overline{x}}^\beta\lambda(x)\,d\xi(x)\\&\ge\lambda(\gamma)\inf_{\gamma\le\tau\le\overline{x}}\{\xi(\overline{x})-\xi(\gamma)\}\\&\quad+\lambda(\beta)\inf_{\overline{x}\le\tau\le\beta}\{\xi(\beta)-\xi(\overline{x})\}=0.\end{aligned} \tag{51}$$

From the above three subcases we conclude that

$$\begin{aligned}\int_\gamma^\beta\lambda(x)\,d\xi(x)&=\int_\gamma^\beta\{\Psi(\phi(x))-\Psi(d_1)\\&\quad-\Psi'_+(d_1)(\phi(x)-d_1)\\&\quad-c(\phi(x)-d_1)^2\}\,d\xi(x)\ge0.\end{aligned} \tag{52}$$

$$\begin{aligned}&\text{i.e., }\Psi(d_1)\{\xi(\beta)-\xi(\gamma)\}+\Psi'_+(d_1)\\&\cdot\left\{\int_\gamma^\beta(\phi(x))\,d\xi(x)-d_1\{\xi(\beta)-\xi(\gamma)\}\right\}\\&+c\int_\gamma^\beta(\phi(x)-d_1)^2\,d\xi(x)\\&\le\int_\gamma^\beta\Psi(\phi(x))\,d\xi(x).\end{aligned} \tag{53}$$

Dividing (53) by $\xi(\beta)-\xi(\gamma)>0$, we obtain the left side of the inequality (46). Similarly, if ϕ is decreasing we consider the cases $\phi(\gamma)\le d_1$ (λ is increasing on $[\gamma,\beta]$), $d_1\le\phi(\beta)$ (λ is decreasing on $[\gamma,\beta]$), and $\phi(\gamma)<d_1<\phi(\beta)$ (λ is decreasing on $[\gamma,\overline{x}]$ and increasing on $[\overline{x},\beta]$). In all three cases we obtain $\int_\gamma^\beta\lambda(x)d\xi(x)\ge0$ from the first inequality in (46) which directly follows.

Similarly, we can prove the right side of the inequality (46). We define the function $\overline{\lambda} : [\gamma, \beta] \to \mathbb{R}$ by $\overline{\lambda}(x) = \overline{\Delta}_{d_2}(\phi(x))$, where $\overline{\Delta}_{d_2}$ is defined as in Lemma 15(b), i.e.,

$$\begin{aligned}\overline{\lambda}(x) &= \overline{\Delta}_{d_2}(\phi(x)) \\ &= \Psi(d_2) - \Psi(\phi(x)) - \Psi'_+(\phi(x))(d_2 - \phi(x)) \\ &\quad - c(d_2 - \phi(x))^2.\end{aligned} \tag{54}$$

Since ϕ is monotonic and continuous, $\Psi(\phi)$ is continuous and $\Psi'_+(\phi)$ is monotonic which have no common discontinuity point with ξ. Therefore the integral $\int_\gamma^\beta \lambda(x)d\xi(x)$ exists. Using the same process as above we have $\int_\gamma^\beta \overline{\lambda}(x)d\xi(x) \geq 0$, which means that

$$\begin{aligned}&\int_\gamma^\beta \{\Psi(d_2) - \Psi(\phi(x)) - \Psi'_+(\phi(x))(d_2 - \phi(x)) \\ &\quad - c(d_2 - \phi(x))^2\} d\xi(x) \geq 0. \\ &\text{i.e., } \{\xi(\beta) - \xi(\gamma)\}\Psi(d_2) - \int_\gamma^\beta \Psi(\phi(x))d\xi(x) \\ &\quad + \int_\gamma^\beta (\phi(x) - d_2)\Psi'_+(\phi(x))d\xi(x) \\ &\quad - c\int_\gamma^\beta (\phi(x) - d_2)^2 d\xi(x) \geq 0.\end{aligned} \tag{55}$$

If we divide by $\xi(\beta)-\xi(\gamma) > 0$, we obtain the second inequality of (46). So, both inequalities of (46) are proved. □

Now, we are in a situation to obtain the following result.

Corollary 18. *Suppose all the assumptions of Theorem 17 are satisfied, then one has*

$$\begin{aligned}0 &\leq \overline{\Psi} - \Psi(\overline{\phi}) - \frac{c}{\xi(\beta) - \xi(\gamma)}\int_\gamma^\beta (\phi(x) - \overline{\phi})^2 d\xi(x) \\ &\leq \frac{1}{\xi(\beta) - \xi(\gamma)}\int_\gamma^\beta (\phi(x) - \overline{\phi})\Psi'_+(\phi(x))d\xi(x) \\ &\quad - \frac{2c}{\xi(\beta) - \xi(\gamma)}\int_\gamma^\beta (\phi(x) - \overline{\phi})^2 d\xi(x).\end{aligned} \tag{56}$$

Proof. By setting $d_1 = d_2 = \overline{\phi}$ in (46), we obtain (56). □

Remark 19. If we set $c = 0$ in (56), we obtain Theorem 3.2 in [35].

In the following corollary, we obtain integral Slater's inequality for strongly convex functions.

Corollary 20. *Suppose all the assumptions of Theorem 17 are satisfied and assume* $\int_\gamma^\beta \Psi'_+(\phi(x))d\xi(x) \neq 0$; *also if* $\overline{\overline{\phi}} = \int_\gamma^\beta \phi(x)\Psi'_+(\phi(x))d\xi(x) / \int_\gamma^\beta \Psi'_+(\phi(x))d\xi(x) \in I$, *then*

$$\overline{\Psi} \leq \Psi(\overline{\overline{\phi}}) - \frac{c}{\xi(\beta) - \xi(\gamma)}\int_\gamma^\beta (\phi(x) - \overline{\overline{\phi}})^2 d\xi(x). \tag{57}$$

Proof. Similar to the proof of Corollary 13, setting $d_2 = \overline{\overline{\phi}}$ in the right hand side of (46), we get (57). □

Remark 21. If we set $c = 0$ in (57), we obtain Slater's inequality for convex functions given in [35].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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