Research Article

Positive Solutions for Higher Order Nonlocal Fractional Differential Equation with Integral Boundary Conditions

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Received 14 April 2018; Accepted 4 September 2018; Published 1 October 2018

Academic Editor: Xinguang Zhang

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In this paper, by using the spectral analysis of the relevant linear operator and Gelfand's formula, some properties of the first eigenvalue of a fractional differential equation were obtained; combining fixed point index theorem, sufficient conditions for the existence of positive solutions are established. An example is given to demonstrate the application of our main results.

1. Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, control theory, signal and image processing, and aerodynamics (see [1, 2]). In the last decade, boundary value problems of fractional calculus have received a great attention, based on various analytic techniques, and a variety of results concerning the existence of solutions can be found in the literature [3–19]. For example, Zhang [17] studied the following singular fractional boundary value problem (FBVP):

$$D_{0+}^{\alpha} y(t) + q(t) f(t, y(t), y'(t), \dots, y^{(n-2)}(t)) = 0, \quad 0 < t < 1, \quad (1)$$

$$y(0) = y'(0) = \dots = y^{(n-2)}(0) = y^{(n-2)}(1) = 0,$$

where $\alpha \in (n-1, n]$, $n \geq 2$, D_{0+}^{α} is the standard Riemann-Liouville derivative, the nonlinear term $f = g + h$, and g and h have different monotone properties. By using a mixed monotone method, a unique positive solution is obtained.

Integral boundary conditions arise in thermal conduction problems [20], semiconductor problems [21], and hydrodynamic problems [22]. Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations with integral boundary value problems have received a great deal of attention. To identify a few, see [23–27] and the references therein for integer order integral boundary

value problems and [28–33] for fractional order integral boundary value problems. In [28], authors using monotone iterative technique investigated the existence and uniqueness of the positive solutions of higher-order nonlocal fractional differential equations of the type

$$\begin{aligned} D_{0+}^{\alpha} y(t) + f(t, y(t)) &= 0, \quad 0 < t < 1, \quad n-1 < \alpha \leq n, \\ y(0) = y'(0) = \dots &= y^{(n-2)}(0) = 0, \quad (2) \\ y(1) &= \lambda [y], \end{aligned}$$

where $f \in C((0, 1) \times \mathbb{R}^+, \mathbb{R}^+)$. By means of Schauders fixed point theorem, FBVP (2) are also studied in [29]. In [30], the authors investigated problems (2) with $f(t, y(t))$ replaced by $q(t)f(t, y(t))$, and the existence and multiplicity of positive solutions are obtained by means of the fixed point index theory in cones.

Inspired by the work of the above papers, the aim of this paper is to establish the existence of positive solutions for the following nonlinear fractional differential equation involving Stieltjes integrals conditions:

$$\begin{aligned} -D_{0+}^{\alpha} y(t) &= f(t, y(t), y'(t), \dots, y^{(n-2)}(t)), \\ 0 < t < 1, \\ y(0) = y'(0) &= \dots = y^{(n-2)}(0) = 0, \\ y^{(n-2)}(1) &= \lambda [y^{(n-2)}], \end{aligned} \quad (3)$$

where D_{0+}^α is the Riemann-Liouville fractional derivative, $n-1 < \alpha \leq n$, $n \geq 2$, $\lambda[z] = \int_0^1 z(t) dA(t)$ is a linear functional on $C[0, 1]$ given by a Stieltjes integral with A representing a suitable function of bounded variation, and dA can be a signed measure. $f : (0, 1) \times (\mathbb{R}_0^+)^{n-1} \rightarrow \mathbb{R}^+$ is a continuous function, the nonlinearity $f(t, z_1, z_2, \dots, z_{n-1})$ may be singular at $t = 0, 1$, and $z_1 = z_2 = \dots = z_{n-1} = 0$. Here $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{R}_0^+ = (0, +\infty)$.

Our work presented in this paper has the following features. First of all, we discuss the boundary value problem with the Stieltjes integral boundary conditions $\lambda[z] = \int_0^1 z(t) dA(t)$ appearing in the boundary conditions of FBVP (2) as more general which was not considered in the literature [15], covering two-point, multipoint, and nonlocal problems as special cases. The second new feature is that the nonlinearity f is allowed to depend on higher order derivatives of unknown function $y(t)$ up to $n-2$ order which was not considered in the literature [28–30]. Thirdly, we allow that the nonlinearity $f(t, z_1, z_2, \dots, z_{n-1})$ may be singular at $t = 0, 1$ and $z_1 = z_2 = \dots = z_{n-1} = 0$. Therefore, our study improves and extends the previous results to some degree in the relevant literature [17, 28–30].

The rest of this paper is organized as follows. In Section 2, we present some lemmas that are used to prove our main results. In Section 3, the existence of positive solutions is established under some sufficient conditions. In Section 4, an example is given to demonstrate the application of our theoretical results.

2. Basic Definitions and Preliminaries

In this paper, our work is based on the theory of fractional calculus, and, for details on definitions of Riemann-Liouville fractional calculus, we refer the reader to [1, 2].

Let $E = C[0, 1]$, and then E is a Banach space with the norm $\|y\| = \max_{0 \leq t \leq 1} |y(t)|$, for any $y \in E$. Let $P = \{y \in E : y(t) \geq 0 \text{ for } t \in [0, 1]\}$ be a cone in E and construct a subcone of P as follows:

$$K = \{y \in P : y(t) \geq t^{\alpha-n+1} (1-t) \|y\| \text{ for } t \in [0, 1]\}. \quad (4)$$

For any $r > 0$, let $K_r = \{y \in K : \|y\| < r\}$, $\partial K_r = \{y \in K : \|y\| = r\}$, and $\bar{K}_r = \{y \in K : \|y\| \leq r\}$.

Set

$$G(t, s) = \frac{1}{\Gamma(\alpha - n + 2)} \begin{cases} [t(1-s)]^{\alpha-n+1} - (t-s)^{\alpha-n+1}, & 0 \leq s \leq t \leq 1, \\ [t(1-s)]^{\alpha-n+1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (5)$$

Lemma 1 (see [17]). *Let $q \in C_r[0, 1](C_r[0, 1] = \{q \in C[0, 1], t^r q \in C[0, 1], 0 \leq r < 1\})$. Then the boundary value problem*

$$\begin{aligned} -D_{0+}^{\alpha-n+2} u(t) &= q(t), \\ 0 < t < 1, \quad n-1 < \alpha \leq n, \quad n \geq 2, \end{aligned} \quad (6)$$

$$u(0) = 0, \quad u(1) = 0$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) q(s) ds. \quad (7)$$

According to the theory of fractional calculus, the unique solution of the problem

$$\begin{aligned} -D_{0+}^{\alpha-n+2} u(t) &= q(t), \\ 0 < t < 1, \quad n-1 < \alpha \leq n, \quad n \geq 2, \end{aligned} \quad (8)$$

$$u(0) = 0, \quad u(1) = \lambda[u]$$

is $\gamma(t) = t^{\alpha-n+1}$ with $\lambda[u]$ replaced by 1 and $q(t) \equiv 0$. As in [27], Green's function for boundary value problem (8) is given by

$$H(t, s) = \frac{\gamma(t)}{1 - \lambda[\gamma]} \mathcal{G}(s) + G(t, s), \quad (9)$$

where $\mathcal{G}(s) := \int_0^1 G(t, s) dA(t)$.

Lemma 2. *Suppose that $\mathcal{G}(s) \geq 0$ for $s \in [0, 1]$ and $0 \leq \lambda[\gamma] < 1$, and then the functions $G(t, s)$ and $H(t, s)$ have the following properties:*

- (1) $G(t, s)$ and $H(t, s)$ are nonnegative and continuous for $(t, s) \in [0, 1] \times [0, 1]$.
- (2) $G(t, s)$ satisfies

$$\begin{aligned} \frac{t^{\alpha-n+1} (1-t) s (1-s)^{\alpha-n+1}}{\Gamma(\alpha - n + 1)} &\leq G(t, s) \\ &\leq \frac{s (1-s)^{\alpha-n+1}}{\Gamma(\alpha - n + 1)}, \quad \text{for } t, s \in [0, 1]. \end{aligned} \quad (10)$$

- (3) $H(t, s)$ satisfies

$$\begin{aligned} t^{\alpha-n+1} (1-t) \Phi(s) &\leq H(t, s) \leq \Phi(s), \\ &\text{for } t, s \in [0, 1], \end{aligned} \quad (11)$$

where

$$\Phi(s) = \frac{\mathcal{G}(s)}{1 - \lambda[\gamma]} + \frac{s (1-s)^{\alpha-n+1}}{\Gamma(\alpha - n + 1)}. \quad (12)$$

Throughout this paper, we adopt the following assumptions:

(H₀) A is a function of bounded variation, $\mathcal{G}(s) \geq 0$ for $s \in [0, 1]$ and $0 \leq \lambda[\gamma] < 1$.

(H₁) $f : (0, 1) \times (\mathbb{R}_0^+)^{n-1} \rightarrow \mathbb{R}^+$ is a continuous function and, for any $0 < r < R < +\infty$,

$$\begin{aligned} \lim_{m \rightarrow +\infty} \sup_{\substack{z_1 \in K_{R/(n-2)!} \\ \vdots \\ z_{n-2} \in K_R \\ z_{n-1} \in K_R \setminus K_r}} \int_{H(m)} \Phi(s) \\ \cdot f(s, z_1(s), z_2(s), \dots, z_{n-1}(s)) ds = 0, \end{aligned} \quad (13)$$

where $H(m) = [0, 1/m] \cup [(m-1)/m, 1]$.

In order to overcome the difficulty due to the dependance of f on derivatives, we consider the following modified problem:

$$\begin{aligned} & -D_{0+}^{\alpha-n+2}x(t) \\ & = f\left(t, I_{0+}^{n-2}x(t), I_{0+}^{n-3}x(t), \dots, I_{0+}^1x(t), x(t)\right), \end{aligned} \quad (14)$$

$$0 < t < 1,$$

$$x(0) = 0, \quad x(1) = \lambda[x]$$

where $n-1 < \alpha \leq n$, $n \geq 2$.

Lemma 3 (see [15]). *The nonlocal FBVP (3) has a positive solution $y(t) = I_{0+}^{n-2}x(t)$ if and only if the nonlinear fractional integrodifferential equation (14) has a positive solution $x(t)$.*

Define a nonlinear operator $T : K \setminus \{0\} \rightarrow P$ and a linear operator $L : E \rightarrow E$ as follows:

$$(Tx)(t) = \int_0^1 H(t, s) f\left(s, I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), \dots, I_{0+}^1x(s), x(s)\right) ds, \quad t \in [0, 1], \quad (15)$$

$$(Lx)(t) = \int_0^1 H(t, s) x(s) ds, \quad t \in [0, 1]. \quad (16)$$

Observe that problem (14) has solutions if the operator equation $x = Tx$ has fixed points.

Lemma 4 (Krein-Rutmann [34]). *Let $L : E \rightarrow E$ be a continuous linear operator, P be a total cone, and $L(P) \subset P$. If there exist $\psi \in E \setminus (-P)$ and a positive constant c such that $cL(\psi) \geq \psi$, then the spectral radius $r(L) \neq 0$ and L has a positive eigenfunction corresponding to its first eigenvalue $\lambda = (r(L))^{-1}$.*

Lemma 5 (Gelfand's formula [34]). *For a bounded linear operator L and the operator norm $\|\cdot\|$, the spectral radius of L satisfies*

$$r(L) = \lim_{m \rightarrow +\infty} \|L^m\|^{1/m}. \quad (17)$$

Lemma 6. *Assume that (H_0) holds. Then $L : K \rightarrow K$ defined by (16) is a completely continuous linear operator, and the spectral radius $r(L) \neq 0$; moreover L has a positive eigenfunction φ^* corresponding to its first eigenvalue $\lambda_1 = (r(L))^{-1}$.*

Proof. For any $x \in K$, by Lemma 2, we can obtain

$$\|Lx\| = \max_{t \in [0,1]} \int_0^1 H(t, s) x(s) ds \leq \int_0^1 \Phi(s) x(s) ds. \quad (18)$$

On the other hand, from Lemma 2, we also have

$$\begin{aligned} (Lx)(t) &= \int_0^1 H(t, s) x(s) ds \\ &\geq t^{\alpha-n+1} (1-t) \int_0^1 \Phi(s) x(s) ds. \end{aligned} \quad (19)$$

Then (18) and (19) yield that

$$(Lx)(t) \geq t^{\alpha-n+1} (1-t) \|Lx\|. \quad (20)$$

Consequently, $L : K \rightarrow K$. And, from the uniform continuity of $H(t, s)$, $(t, s) \in [0, 1] \times [0, 1]$, we have that $L : K \rightarrow K$ is a completely continuous linear operator.

Next we will show that L has the first eigenvalue $\lambda_1 > 0$ by using Krein-Rutmann's theorem. In fact, by Lemma 2, there exists $t_0 \in (0, 1)$ such that $H(t_0, t_0) > 0$. Thus there exists $[a, b] \subset (0, 1)$ such that $t_0 \in (a, b)$ and $H(t, s) > 0$ for all $t, s \in [a, b]$. Choose $x \in K$ such that $x(t_0) > 0$ and $x(t) = 0$ for all $t \notin [a, b]$. Then, for any $t \in [a, b]$, we have

$$\begin{aligned} (Lx)(t) &= \int_0^1 H(t, s) x(s) ds \geq \int_a^b H(t, s) x(s) ds \\ &> 0. \end{aligned} \quad (21)$$

So there exists $c > 0$ such that $c(Lx)(t) \geq x(t)$ for $t \in [0, 1]$. From Lemma 4, we know that the spectral radius $r(L) \neq 0$ and L has a positive eigenfunction φ^* corresponding to its first eigenvalue $\lambda_1 = (r(L))^{-1}$; i.e., $\lambda_1 L\varphi^* = \varphi^*$. The proof is completed. $\square$

Lemma 7. *Assume that $(H_0), (H_1)$ hold; then $T : \overline{K}_R \setminus K_r \rightarrow K$ is completely continuous.*

Proof. First, we prove $T(\overline{K}_R \setminus K_r) \subset K$. In fact, for any $x \in \overline{K}_R \setminus K_r$, Lemma 2 implies that

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t, s) f\left(s, I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), \dots, \right. \\ &\quad \left. I_{0+}^1x(s), x(s)\right) ds \leq \int_0^1 \Phi(s) f\left(s, I_{0+}^{n-2}x(s), \right. \\ &\quad \left. I_{0+}^{n-3}x(s), \dots, I_{0+}^1x(s), x(s)\right) ds, \quad t \in [0, 1]. \end{aligned} \quad (22)$$

Hence,

$$\begin{aligned} \|Tx\| &\leq \int_0^1 \Phi(s) f\left(s, I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), \dots, I_{0+}^1x(s), \right. \\ &\quad \left. x(s)\right) ds. \end{aligned} \quad (23)$$

On the other hand, from Lemma 2, we also have

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t, s) \\ &\quad \cdot f\left(s, I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), \dots, I_{0+}^1x(s), x(s)\right) ds \\ &\geq t^{\alpha-n+1} (1-t) \int_0^1 \Phi(s) \\ &\quad \cdot f\left(s, I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), \dots, I_{0+}^1x(s), x(s)\right) ds \\ &\geq t^{\alpha-n+1} (1-t) \|Tx\|. \end{aligned} \quad (24)$$

Therefore, $T(\overline{K}_R \setminus K_r) \subset K$.

Next, for any $r > 0$, we assert that

$$\begin{aligned} & \sup_{x \in \partial K_r} \int_0^1 \Phi(s) \\ & \cdot f(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \dots, I_{0+}^1 x(s), x(s)) ds \\ & < +\infty, \end{aligned} \quad (25)$$

which implies that $T : K \setminus \{0\} \rightarrow P$ is well defined. In fact, it follows from (H_1) that there exists a natural number m such that

$$\begin{aligned} & \sup_{x \in \partial K_r} \int_{H(m)} \Phi(s) \\ & \cdot f(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \dots, I_{0+}^1 x(s), x(s)) ds \\ & < 1. \end{aligned} \quad (26)$$

For any $x \in \partial K_r$, as $x \in K$, we have

$$t^{\alpha-n+1} (1-t) \|x\| \leq x(t) \leq \|x\| = r, \quad t \in [0, 1], \quad (27)$$

and

$$\begin{aligned} & \frac{r t^{\alpha-n+i+1} (1-t) \Gamma(\alpha-n+2)}{\Gamma(\alpha-n+i+2)} \\ & = \frac{\|x\| t^{\alpha-n+i+1} (1-t) \mathbf{B}(i, \alpha-n+2)}{\Gamma(i)} \\ & \leq \frac{\|x\|}{\Gamma(i)} \int_0^t (t-s)^{i-1} s^{\alpha-n+1} (1-t) ds \leq I_{0+}^i x(t) \\ & = \frac{1}{\Gamma(i)} \int_0^t (t-s)^{i-1} x(s) ds \leq \frac{1}{i!} \|x\| = \frac{r}{i!}, \\ & \quad t \in [0, 1], \quad i = 1, 2, \dots, n-2, \end{aligned} \quad (28)$$

where $\mathbf{B}(\cdot, \cdot)$ is the beta function. So, for any $1/m \leq t \leq 1 - 1/m$, (27) and (28) yield that

$$\begin{aligned} & \frac{r}{m^{\alpha-n+2}} \leq x(t) \leq r, \\ & \frac{\Gamma(\alpha-n+2)r}{m^{\alpha-n+i+2}\Gamma(\alpha-n+i+2)} \leq I_{0+}^i x(t) \leq \frac{r}{i!}, \\ & \quad i = 1, 2, \dots, n-2. \end{aligned} \quad (29)$$

From (26) and (29), we have

$$\begin{aligned} & \sup_{x \in \partial K_r} \int_0^1 \Phi(s) \\ & \cdot f(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \dots, I_{0+}^1 x(s), x(s)) ds \\ & \leq \sup_{x \in \partial K_r} \int_{H(m)} \Phi(s) \\ & \cdot f(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \dots, I_{0+}^1 x(s), x(s)) ds \\ & + \sup_{x \in \partial K_r} \int_{1/m}^{1-1/m} \Phi(s) \\ & \cdot f(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \dots, I_{0+}^1 x(s), x(s)) ds \\ & < 1 + M_1 \int_0^1 \Phi(s) ds < +\infty, \end{aligned} \quad (30)$$

where

$$\begin{aligned} M_1 &= \max \left\{ f(t, z_1, z_2, \dots, z_{n-1}) : (t, z_1, z_2, \dots, z_{n-1}) \right. \\ & \in \left[\frac{1}{m}, 1 - \frac{1}{m} \right] \times \left[\frac{\Gamma(\alpha-n+2)r}{m^\alpha \Gamma(\alpha)}, \frac{r}{(n-2)!} \right] \\ & \times \left[\frac{\Gamma(\alpha-n+2)r}{m^{\alpha-1} \Gamma(\alpha-1)}, \frac{r}{(n-3)!} \right] \times \dots \\ & \left. \times \left[\frac{\Gamma(\alpha-n+2)r}{m^{\alpha-n+3} \Gamma(\alpha-n+3)}, r \right] \times \left[\frac{r}{m^{\alpha-n+2}}, r \right] \right\}. \end{aligned} \quad (31)$$

Thus (25) is true which implies that T is uniformly bounded on any bounded set.

Now we will show that $T : \overline{K_R} \setminus K_r \rightarrow K$ is continuous. Let $x_j, x_0 \in \overline{K_R} \setminus K_r$ and $\|x_j - x_0\| \rightarrow 0$ ($j \rightarrow \infty$). For any $\varepsilon > 0$, by (H_1) , there exists a natural number $k > 0$ such that

$$\sup_{x \in \overline{K_R} \setminus K_r} \int_{H(k)} \Phi(s) f(s, I_{0+}^{n-2} x(s), \dots, x(s)) ds < \frac{\varepsilon}{4}. \quad (32)$$

Since $f(t, z_1, z_2, \dots, z_{n-1})$ is uniformly continuous on $[1/k, 1 - 1/k] \times [\Gamma(\alpha-n+2)r/k^\alpha \Gamma(\alpha), r/(n-2)!] \times \dots \times [r/k^{\alpha-n+2}, r]$, we have that

$$\begin{aligned} & \lim_{j \rightarrow +\infty} \left| f(t, I_{0+}^{n-2} x_j(t), \dots, x_j(t)) \right. \\ & \left. - f(t, I_{0+}^{n-2} x_0(t), \dots, x_0(t)) \right| = 0 \end{aligned} \quad (33)$$

holds uniformly on $t \in [1/k, 1 - 1/k]$. It follows from the Lebesgue control convergence theorem that

$$\begin{aligned} & \int_{1/k}^{1-1/k} \Phi(s) \left| f(s, I_{0+}^{n-2} x_j(s), \dots, x_j(s)) \right. \\ & \left. - f(s, I_{0+}^{n-2} x_0(s), \dots, x_0(s)) \right| ds \rightarrow 0, \end{aligned} \quad (34)$$

as $j \rightarrow \infty$.

Thus, for the above $\varepsilon > 0$, there exists a natural number J such that, for $j > J$, we have

$$\int_{1/k}^{1-1/k} \Phi(s) \left| f(s, I_{0+}^{n-2} x_j(s), \dots, x_j(s)) - f(s, I_{0+}^{n-2} x_0(s), \dots, x_0(s)) \right| ds < \frac{\varepsilon}{2}. \quad (35)$$

It follows from (32) and (35) that when $j > J$,

$$\begin{aligned} \|Tx_j - Tx_0\| &\leq \int_{H(k)} \Phi(s) \left| f(s, I_{0+}^{n-2} x_j(s), \dots, x_j(s)) - f(s, I_{0+}^{n-2} x_0(s), \dots, x_0(s)) \right| ds \\ &\quad + \int_{1/k}^{1-1/k} \Phi(s) \left| f(s, I_{0+}^{n-2} x_j(s), \dots, x_j(s)) - f(s, I_{0+}^{n-2} x_0(s), \dots, x_0(s)) \right| ds \leq 2 \\ &\quad \cdot \sup_{x \in \overline{K_R} \setminus K_r} \int_{H(k)} \Phi(s) f(s, I_{0+}^{n-2} x(s), \dots, x(s)) ds \\ &\quad + \frac{\varepsilon}{2} < 2 \times \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \quad (36)$$

Hence, $T : \overline{K_R} \setminus K_r \rightarrow K$ is continuous.

For any bounded set $B \subset \overline{K_R} \setminus K_r$, we show that $T(B)$ is equicontinuous. In fact, by (H_1) , for any $\varepsilon > 0$, there exists a natural number l such that

$$\sup_{x \in \overline{K_R} \setminus K_r} \int_{H(l)} \Phi(s) f(s, I_{0+}^{n-2} x(s), \dots, x(s)) ds < \frac{\varepsilon}{4}. \quad (37)$$

Let

$$\begin{aligned} M_2 &= \max \left\{ f(t, z_1, z_2, \dots, z_{n-1}) : (t, z_1, z_2, \dots, z_{n-1}) \right. \\ &\quad \in \left[\frac{1}{l}, 1 - \frac{1}{l} \right] \times \left[\frac{\Gamma(\alpha - n + 2)r}{l^\alpha \Gamma(\alpha)}, \frac{r}{(n-2)!} \right] \\ &\quad \times \left[\frac{\Gamma(\alpha - n + 2)r}{l^{\alpha-1} \Gamma(\alpha-1)}, \frac{r}{(n-3)!} \right] \times \dots \\ &\quad \left. \times \left[\frac{\Gamma(\alpha - n + 2)r}{l^{\alpha-n+3} \Gamma(\alpha - n + 3)}, r \right] \times \left[\frac{r}{l^{\alpha-n+2}}, r \right] \right\}. \end{aligned} \quad (38)$$

Since $H(t, s)$ is uniformly continuous on $[0, 1] \times [0, 1]$, for the above $\varepsilon > 0$ and a fixed $s \in [1/l, 1 - 1/l]$, there exists $\delta > 0$ such that

$$\left| H(t, s) - H(t', s) \right| < \frac{\varepsilon}{2(M_2 + 1)}. \quad (39)$$

for $|t - t'| < \delta$, $t, t' \in [0, 1]$. Hence, for $|t - t'| < \delta$, $t, t' \in [0, 1]$, and $x \in B$, we have

$$\begin{aligned} |Tx(t) - Tx(t')| &\leq \int_{H(k)} |H(t, s) - H(t', s)| \\ &\quad \cdot f(s, I_{0+}^{n-2} x(s), \dots, I_{0+}^1 x(s), x(s)) ds \\ &\quad + \int_{1/k}^{1-1/k} |H(t, s) - H(t', s)| \\ &\quad \cdot f(s, I_{0+}^{n-2} x(s), \dots, I_{0+}^1 x(s), x(s)) ds \leq 2 \\ &\quad \cdot \sup_{x \in \overline{K_R} \setminus K_r} \int_{e(k)} \Phi(s) \\ &\quad \cdot f(s, I_{0+}^{n-2} x(s), \dots, I_{0+}^1 x(s), x(s)) ds \\ &\quad + \sup_{x \in \overline{K_R} \setminus K_r} \int_{1/k}^{1-1/k} |H(t, s) - H(t', s)| \\ &\quad \cdot f(s, I_{0+}^{n-2} x(s), \dots, I_{0+}^1 x(s), x(s)) ds < 2 \times \frac{\varepsilon}{4} \\ &\quad + \frac{\varepsilon}{2} = \varepsilon, \end{aligned} \quad (40)$$

which show that $T(B)$ is equicontinuous. According to the Ascoli-Arzelà theorem, $T : \overline{K_R} \setminus K_r \rightarrow K$ is completely continuous. The proof is completed. $\square$

Lemma 8 (see [35]). *Let K be a cone in Banach space E . Suppose that $T : \overline{K_r} \rightarrow K$ is a completely continuous operator.*

- (i) *If there exists $y_0 \in K \setminus \{\theta\}$ such that $y - Ty \neq \mu y_0$ for any $y \in \partial K_r$ and $\mu \geq 0$, then $i(T, K_r, K) = 0$.*
- (ii) *If $Ty \neq \mu y$ for any $y \in \partial K_r$ and $\mu \geq 1$, then $i(T, K_r, K) = 1$.*

3. Main Results

Theorem 9. *Suppose that the conditions (H_0) and (H_1) are satisfied, and*

$$\liminf_{\substack{z_1 \rightarrow 0 \\ \dots \\ z_{n-1} \rightarrow 0}} \frac{f(t, z_1, \dots, z_{n-1})}{z_1 + \dots + z_{n-1}} > \lambda_1, \quad (41)$$

uniformly on $t \in [0, 1]$,

$$\limsup_{\substack{z_1 + \dots + z_{n-1} \rightarrow +\infty \\ z_{n-1} \rightarrow +\infty}} \frac{f(t, z_1, \dots, z_{n-1})}{z_{n-1}} < \lambda_1, \quad (42)$$

uniformly on $t \in [0, 1]$,

where λ_1 is the first eigenvalue of L defined by (16). Then the FBVP (3) has at least one positive solution.

Proof. By (41), there exists $r > 0$ such that

$$f(t, z_1, z_2, \dots, z_{n-1}) \geq \lambda_1(z_1 + z_2 + \dots + z_{n-1}), \quad (43)$$

$$|z_1| \leq \frac{r}{(n-2)!}, \dots, |z_{n-1}| \leq r, \quad t \in [0, 1].$$

Let φ^* be the positive eigenfunction of L corresponding to λ_1 , and thus $\varphi^* = \lambda_1 L\varphi^*$. For any $x \in \partial K_r$, by virtue of (27) and (28), that is

$$|I_{0+}^{n-2} x(s)| \leq \frac{r}{(n-2)!}, \dots, |I_{0+}^1 x(s)| \leq r, \quad (44)$$

$$|x(s)| \leq r, \quad t \in [0, 1],$$

from (43) and (44), we have

$$(Tx)(t) = \int_0^1 H(t, s) \cdot f(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \dots, I_{0+}^1 x(s), x(s)) ds$$

$$\geq \lambda_1 \int_0^1 H(t, s) (I_{0+}^{n-2} x(s) + I_{0+}^{n-3} x(s) + \dots + I_{0+}^1 x(s) + x(s)) ds \geq \lambda_1 (Lx)(t), \quad (45)$$

$$t \in [0, 1].$$

We may suppose that T has no fixed point on ∂K_r (otherwise, the proof is finished). Now we show that

$$x - Tx \neq \mu \varphi^*, \quad x \in \partial K_r, \quad \mu \geq 0. \quad (46)$$

If, otherwise, there exist $x_0 \in \partial K_r$ and $\mu_0 \geq 0$ such that $x_0 - Tx_0 = \mu_0 \varphi^*$, obviously, $\mu_0 > 0$ and $x_0 = Tx_0 + \mu_0 \varphi^* \geq \mu_0 \varphi^*$. Let $\bar{\mu} = \sup\{\mu \mid x_0 \geq \mu \varphi^*\}$, and then $\bar{\mu} \geq \mu_0 > 0$ and $x_0 \geq \bar{\mu} \varphi^*$. Since $L(K) \subset K$, we have $\lambda_1 Lx_0 \geq \lambda_1 \bar{\mu} L\varphi^* = \bar{\mu} \varphi^*$. Therefore, by (45), we have

$$x_0 = Tx_0 + \mu_0 \varphi^* \geq \lambda_1 Lx_0 + \mu_0 \varphi^* \geq \bar{\mu} \varphi^* + \mu_0 \varphi^* = (\bar{\mu} + \mu_0) \varphi^*, \quad (47)$$

which contradicts the definition of $\bar{\mu}$. Hence (46) is true and it follows from Lemma 8 that

$$i(T, K_r, K) = 0. \quad (48)$$

Now we choose a constant $0 < \rho < 1$ and define a linear operator $\tilde{L}x = \rho \lambda_1 Lx$, and then $\tilde{L} : E \rightarrow E$ is a bounded linear operator and $\tilde{L}(K) \subset K$. Moreover $\tilde{L}\varphi^* = \rho \lambda_1 L\varphi^* = \rho \varphi^*$, so the spectral radius of $\tilde{L}$ is $r(\tilde{L}) = \rho$ and $\tilde{L}$ also has the first eigenvalue $(r(\tilde{L}))^{-1} = \rho^{-1} > 1$. By Gelfand's formula, we know

$$\rho = \lim_{m \rightarrow +\infty} \|\tilde{L}^m\|^{1/m}. \quad (49)$$

Let $\varepsilon_0 = (1/2)(1 - \rho)$; by (49), there exists a sufficiently large natural number M such that $m \geq M$ implies that $\|\tilde{L}^m\| \leq (\rho + \varepsilon_0)^m$. For any $x \in E$, define

$$\|x\|^* = \sum_{i=1}^M (\rho + \varepsilon_0)^{M-i} \|\tilde{L}^{i-1} x\|, \quad (50)$$

where $\tilde{L}^0 = I$ is the identity operator. Clearly, $\|\cdot\|^*$ is also the norm of E .

On the other hand, it follows from (42) that there exists $R_0 > r$ such that

$$f(t, z_1, z_2, \dots, z_{n-1}) \leq \rho \lambda_1 z_{n-1}, \quad (51)$$

$$|z_1 + \dots + z_{n-1}| \geq R_0, \quad |z_{n-1}| \geq R_0.$$

Let

$$M_0 = \sup_{x \in \partial K_{R_0}} \int_0^1 \Phi(s) \cdot f(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \dots, I_{0+}^1 x(s), x(s)) ds, \quad (52)$$

and then, by (25), we know that $M_0 < +\infty$. For the sake of convenience, let $M_0^* = \|M_0\|^*$ and choose

$$R > \max \left\{ R_0, \frac{2M_0^*}{\varepsilon_0(\rho + \varepsilon_0)^{M-1}} \right\}. \quad (53)$$

In the following we will prove that

$$Tx \neq \mu x, \quad x \in \partial K_R, \quad \mu \geq 1. \quad (54)$$

If, otherwise, there exist $x_1 \in \partial K_R$ and $\mu_1 \geq 1$ such that $Tx_1 = \mu_1 x_1$. It follows from $R > R_0$ and $x_1 \in C[0, 1]$ that there exists $0 < t_1 \leq 1$ such that $x_1(t_1) = R_0$. Thus let $\tilde{x}(t) = \min\{x_1(t), R_0\}$, and then we have $\|\tilde{x}\| = R_0$ and $\tilde{x} \in \partial K_{R_0}$. Now set

$$D(x_1) = \{t \in [0, 1] : x_1(t) > R_0\}, \quad (55)$$

and then, for any $t \in D(x_1)$, we have $|x_1(t)| > R_0$ and $|I_{0+}^{n-2} x_1(t) + I_{0+}^{n-3} x_1(t) + \dots + I_{0+}^1 x_1(t) + x_1(t)| > R_0$. It follows from (51) and Lemma 2 that

$$(Tx_1)(t) = \int_0^1 H(t, s) f(s, I_{0+}^{n-2} x_1(s), I_{0+}^{n-3} x_1(s), \dots, I_{0+}^1 x_1(s), x_1(s)) ds \leq \int_{D(x_1)} H(t, s) f(s, I_{0+}^{n-2} x_1(s), I_{0+}^{n-3} x_1(s), \dots, I_{0+}^1 x_1(s), x_1(s)) ds$$

$$+ \int_{[0,1] \setminus D(x_1)} H(t, s) f(s, I_{0+}^{n-2} x_1(s), I_{0+}^{n-3} x_1(s), \dots, I_{0+}^1 x_1(s), x_1(s)) ds \leq \rho \lambda_1 \int_0^1 H(t, s) x_1(s) ds$$

$$+ \int_0^1 \Phi(s) f(s, I_{0+}^{n-2} x_1(s), I_{0+}^{n-3} x_1(s), \dots, I_{0+}^1 x_1(s), x_1(s)) ds \leq (\tilde{L}x_1)(t) + M_0, \quad t \in [0, 1]. \quad (56)$$

Noticing that $\tilde{L}(K) \subset K$ is a bounded linear operator, and from (56), we have

$$0 \leq (\tilde{L}^j(Tx_1))(t) \leq (\tilde{L}^j(\tilde{L}x_1 + M_0))(t), \quad (57)$$

$$j = 0, 1, 2, \dots, M-1.$$

Then (57) yields that

$$\|\tilde{L}^j(Tx_1)\| \leq \|\tilde{L}^j(\tilde{L}x_1 + M_0)\|, \quad (58)$$

$$j = 0, 1, 2, \dots, M-1,$$

which leads to

$$\begin{aligned} \|Tx_1\|^* &= \sum_{i=1}^M (\rho + \varepsilon_0)^{M-i} \|\tilde{L}^{i-1}(Tx_1)\| \\ &\leq \sum_{i=1}^M (\rho + \varepsilon_0)^{M-i} \|\tilde{L}^{i-1}(\tilde{L}x_1 + M_0)\| \\ &= \|\tilde{L}x_1 + M_0\|^*. \end{aligned} \quad (59)$$

Since $x_1 \in \partial K_R$ and $\|x_1\| = R$, from (50), we have

$$\|x_1\|^* > (\rho + \varepsilon_0)^{M-1} \|x_1\| = (\rho + \varepsilon_0)^{M-1} R > \frac{2}{\varepsilon_0} M_0^*, \quad (60)$$

which implies that

$$M_0^* < \frac{\varepsilon_0}{2} \|x_1\|^*. \quad (61)$$

By (50), (59), and (61), we have

$$\begin{aligned} \mu_1 \|x_1\|^* &= \|Tx_1\|^* \leq \|\tilde{L}x_1\| + M_0^* \\ &= \sum_{i=1}^M (\rho + \varepsilon_0)^{M-i} \|\tilde{L}^{i-1}(Lx_1)\| + M_0^* \\ &= (\rho + \varepsilon_0) \sum_{i=1}^{M-1} (\rho + \varepsilon_0)^{M-i-1} \|\tilde{L}^i x_1\| \\ &\quad + \|\tilde{L}^M x_1\| + M_0^* \\ &\leq (\rho + \varepsilon_0) \sum_{i=1}^{M-1} (\rho + \varepsilon_0)^{M-i-1} \|\tilde{L}^i x_1\| \\ &\quad + (\rho + \varepsilon_0)^M \|x_1\| + M_0^* \\ &= (\rho + \varepsilon_0) \sum_{i=1}^M (\rho + \varepsilon_0)^{M-i} \|\tilde{L}^{i-1} x_1\| + M_0^* \\ &= (\rho + \varepsilon_0) \|x_1\|^* + M_0^* \\ &\leq (\rho + \varepsilon_0) \|x_1\|^* + \frac{\varepsilon_0}{2} \|x_1\|^* = \frac{\rho + 3}{4} \|x_1\|^*. \end{aligned} \quad (62)$$

Notice that $\mu_1 \geq 1$, we have $(\rho + 3)/4 \geq 1$, and then $\rho \geq 1$, which is a contradiction with $0 < \rho < 1$. Thus (54) is indeed true, and, by Lemma 8, we have

$$i(T, K_R, K) = 1. \quad (63)$$

It follows from (48) and (63) that

$$i(T, K_R \setminus \bar{K}_r, K) = i(T, K_R, K) - i(T, K_r, K) = 1. \quad (64)$$

Then T has at least one fixed point on $K_R \setminus \bar{K}_r$. Consequently, problem (14) has at least one positive solution, which implies that the FBVP (3) has at least one positive solution. $\square$

Now we consider another case of problem (14). For this, we define a linear operator L_τ for any sufficiently small $0 < \tau < 1$ as follows:

$$(L_\tau x)(t) = \int_\tau^{1-\tau} H(t, s) x(s) ds, \quad t \in [0, 1]. \quad (65)$$

From Lemma 6, we know L_τ is also a completely continuous linear operator, and the spectral radius $r(L_\tau) \neq 0$, and moreover L_τ has a positive eigenfunction φ_τ corresponding to its first eigenvalue $\lambda_\tau = (r(L_\tau))^{-1}$.

Lemma 10. Suppose that (H_0) holds, and there exists an eigenvalue $\tilde{\lambda}_1$ of L such that

$$\lim_{\tau \rightarrow 0^+} \lambda_\tau = \tilde{\lambda}_1. \quad (66)$$

Proof. Take $\tau_1 \geq \tau_2 \geq \dots \geq \tau_k \geq \dots$ and $\tau_k \rightarrow 0$ ($k \rightarrow +\infty$). So, for any $j > k$ and $\varphi \in E$, we have

$$(L_{\tau_k} \varphi)(t) \leq (L_{\tau_j} \varphi)(t) \leq (L\varphi)(t), \quad t \in [0, 1], \quad (67)$$

and

$$(L_{\tau_k}^i \varphi)(t) \leq (L_{\tau_j}^i \varphi)(t) \leq (L^i \varphi)(t), \quad (68)$$

$$t \in [0, 1], \quad i = 2, 3, \dots,$$

where $L_{\tau_k}^i = L(L_{\tau_k}^{i-1})$, $i = 2, 3, \dots$. Consequently, $\|L_{\tau_k}^i\| \leq \|L_{\tau_j}^i\| \leq \|L^i\|$, $i = 1, 2, \dots$. From Gelfand's formula, we get $\lambda_1 \leq \lambda_{\tau_j} \leq \lambda_{\tau_k}$, where λ_1 is the first eigenvalue of L . Since $\{\lambda_{\tau_k}\}$ is monotonous with lower boundedness λ_1 , let

$$\lim_{k \rightarrow +\infty} \lambda_{\tau_k} = \tilde{\lambda}_1. \quad (69)$$

Now we shall show that $\tilde{\lambda}_1$ is an eigenvalue of L . Suppose φ_{τ_k} is a positive eigenfunction of L_{τ_k} corresponding to λ_{τ_k} with $\|\varphi_{\tau_k}\| = 1$, $k = 1, 2, \dots$; i.e.,

$$\varphi_{\tau_k}(t) = \lambda_{\tau_k} \int_{\tau_k}^{1-\tau_k} H(t, s) \varphi_{\tau_k}(s) ds = \lambda_{\tau_k} L_{\tau_k} \varphi_{\tau_k}(t), \quad (70)$$

$$t \in [0, 1].$$

Notice that

$$\|\lambda_{\tau_k} \varphi_{\tau_k}\| = \max_{0 \leq t \leq 1} \int_{\tau_k}^{1-\tau_k} H(t, s) \varphi_{\tau_k}(s) ds \leq \int_0^1 \Phi(s) ds, \quad (71)$$

$$k = 1, 2, \dots,$$

and thus $\{\lambda_{\tau_k} \varphi_{\tau_k}\} \subset E$ is uniformly bounded.

On the other hand, for any k and $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned} &|L_{\tau_k} \varphi_{\tau_k}(t_1) - L_{\tau_k} \varphi_{\tau_k}(t_2)| \\ &\leq \int_{\tau_k}^{1-\tau_k} |H(t_1, s) - H(t_2, s)| \varphi_{\tau_k}(s) ds. \end{aligned} \quad (72)$$

It follows from (72) that $\{L_{\tau_k} \varphi_{\tau_k}\} \subset E$ is equicontinuous since $H(t, s)$ is uniformly continuous on $[0, 1] \times [0, 1]$. Without loss of generality, we may suppose that, by Arzela-Ascoli theorem and $\lim_{k \rightarrow +\infty} \lambda_{\tau_k} = \tilde{\lambda}_1$, we get that $\varphi_{\tau_k} \rightarrow \varphi_0$ as $k \rightarrow +\infty$. This leads to $\|\varphi_0\| = 1$ and then, by (70), we have

$$\varphi_0(t) = \tilde{\lambda}_1 \int_0^1 H(t, s) \varphi_0(s) ds, \quad t \in [0, 1]; \quad (73)$$

that is, $\varphi_0 = \tilde{\lambda}_1 L \varphi_0$. The proof is completed. $\square$

Theorem 11. Suppose that conditions (H_0) and (H_1) are satisfied, and

$$\liminf_{\substack{z_1 \rightarrow 0 \\ \dots \\ z_{n-1} \rightarrow 0}} \frac{f(t, z_1, \dots, z_{n-1})}{z_{n-1}} < \lambda_1, \quad (74)$$

uniformly on $t \in [0, 1]$,

$$\limsup_{z_1 + \dots + z_{n-1} \rightarrow +\infty} \frac{f(t, z_1, \dots, z_{n-1})}{z_1 + \dots + z_{n-1}} > \tilde{\lambda}_1, \quad (75)$$

uniformly on $t \in [0, 1]$,

where $\lambda_1, \tilde{\lambda}_1$ are the eigenvalues of L and λ_1 is the first eigenvalue of L . Then the FBVP (3) has at least one positive solution.

Proof. It follows from (74) that there exists $r > 0$ such that

$$\begin{aligned} f(t, z_1, z_2, \dots, z_{n-1}) &\leq \lambda_1 z_{n-1}, \\ |z_1| &\leq \frac{r}{(n-2)!}, \dots, |z_{n-1}| \leq r, \quad t \in [0, 1]. \end{aligned} \quad (76)$$

Thus for any $x \in \partial K_r$, noticing

$$\begin{aligned} |I_{0+}^{n-2} x(s)| &\leq \frac{r}{(n-2)!}, \dots, |I_{0+}^1 x(s)| \leq r, \\ |x(s)| &\leq r, \quad t \in [0, 1], \end{aligned} \quad (77)$$

by (76), we have

$$\begin{aligned} (Tx)(t) &= \int_0^1 H(t, s) \\ &\cdot f(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \dots, I_{0+}^1 x(s), x(s)) ds \\ &\leq \lambda_1 \int_0^1 H(t, s) x(s) ds = \lambda_1 (Lx)(t), \\ &\quad t \in [0, 1]. \end{aligned} \quad (78)$$

In fact, we may suppose that T has no fixed point on ∂K_r (otherwise, the proof is finished). Now we show that

$$Tx \neq \mu x, \quad x \in \partial K_r, \quad \mu \geq 1. \quad (79)$$

Otherwise, there exist $x_0 \in \partial K_r$ and $\mu_0 \geq 1$ such that $Tx_0 = \mu_0 x_0$. We know $\mu_0 > 1$ and from (78) we have

$$\mu_0 x_0 = Tx_0 \leq \lambda_1 Lx_0. \quad (80)$$

By using induction for (80), we have

$$\mu_0^m x_0 \leq \lambda_1^m L^m x_0, \quad m = 1, 2, \dots \quad (81)$$

which implies that

$$\|L^m\| \geq \frac{\|L^m x_0\|}{\|x_0\|} \geq \frac{\mu_0^m \|x_0\|}{\lambda_1^m \|x_0\|} = \frac{\mu_0^m}{\lambda_1^m}. \quad (82)$$

By Gelfand's formula, we know

$$r(L) = \lim_{m \rightarrow +\infty} \|L^m\|^{1/m} \geq \frac{\mu_0}{\lambda_1} > \frac{1}{\lambda_1}, \quad (83)$$

which is a contradiction with $r(L) = \lambda_1^{-1}$. Hence (79) is true and, by Lemma 8, we have

$$i(T, K_r, K) = 1. \quad (84)$$

For a fixed sufficiently small $0 < \tau < 1$, let

$$\begin{aligned} \chi_\tau &= \frac{\tau^\alpha \Gamma(\alpha - n + 2)}{\Gamma(\alpha)} + \frac{\tau^{\alpha-1} \Gamma(\alpha - n + 2)}{\Gamma(\alpha - 1)} + \dots \\ &+ \frac{\tau^{\alpha-n+3} \Gamma(\alpha - n + 2)}{\Gamma(\alpha - n + 3)} + \tau^{\alpha-n+2}. \end{aligned} \quad (85)$$

So it follows from (74) and $\lim_{\tau \rightarrow 0^+} \lambda_\tau = \tilde{\lambda}_1$ that there exist a sufficiently small $\tau > 0$ and $R > r$ such that

$$\begin{aligned} f(t, z_1, z_2, \dots, z_{n-1}) &\geq \lambda_\tau (z_1 + z_2 + \dots + z_{n-1}), \\ z_1 + z_2 + \dots + z_{n-1} &\geq \chi_\tau R, \quad t \in [0, 1], \end{aligned} \quad (86)$$

where λ_τ is the first eigenvalue of L_τ .

Let φ_τ be the positive eigenfunction of L_τ corresponding to λ_τ , and then $\varphi_\tau = \lambda_\tau L_\tau \varphi_\tau$. For any $x \in \partial K_R$, $s \in [\tau, 1 - \tau]$, we have

$$\begin{aligned} I_{0+}^{n-2} x(s) + I_{0+}^{n-3} x(s) + \dots, I_{0+}^1 x(s) + x(s) \\ = \frac{1}{\Gamma(n-2)} \int_0^s (s-\tau)^{n-3} x(\tau) d\tau + \frac{1}{\Gamma(n-3)} \\ \cdot \int_0^s (s-\tau)^{n-4} x(\tau) d\tau + \dots + \frac{1}{\Gamma(1)} \\ \cdot \int_0^s (s-\tau)^{1-1} x(\tau) d\tau + x(s) \geq \frac{1}{\Gamma(n-2)} \\ \cdot \int_0^s (s-\tau)^{n-3} \tau^{\alpha-n+1} (1-\tau) d\tau \|x\| \\ + \frac{1}{\Gamma(n-3)} \int_0^s (s-\tau)^{n-4} \tau^{\alpha-n+1} (1-\tau) d\tau \|x\| + \dots \\ + \frac{1}{\Gamma(1)} \int_0^s (s-\tau)^{1-1} \tau^{\alpha-n+1} (1-\tau) d\tau \|x\| \\ + s^{\alpha-n+1} (1-s) \|x\| \geq \left(\frac{\tau^\alpha \Gamma(\alpha - n + 2)}{\Gamma(\alpha)} \right. \\ + \frac{\tau^{\alpha-1} \Gamma(\alpha - n + 2)}{\Gamma(\alpha - 1)} + \dots + \frac{\tau^{\alpha-n+3} \Gamma(\alpha - n + 2)}{\Gamma(\alpha - n + 3)} \\ \left. + \tau^{\alpha-n+2} \right) R = \chi_\tau R. \end{aligned} \quad (87)$$

By (86) and (87), we have

$$\begin{aligned}
 (Tx)(t) &= \int_0^1 H(t, s) \\
 &\cdot f\left(s, I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), \dots, I_{0+}^1x(s), x(s)\right) ds \\
 &\geq \int_\tau^{1-\tau} H(t, s) \\
 &\cdot f\left(s, I_{0+}^{n-2}x(s), I_{0+}^{n-3}x(s), \dots, I_{0+}^1x(s), x(s)\right) ds \quad (88) \\
 &\geq \lambda_\tau \int_\tau^{1-\tau} H(t, s) \left(I_{0+}^{n-2}x(s) + I_{0+}^{n-3}x(s)\right. \\
 &\quad \left.+ \dots, I_{0+}^1x(s) + x(s)\right) ds \geq \lambda_\tau \int_\tau^{1-\tau} H(t, s) \\
 &\cdot x(s) ds = \lambda_\tau (L_\tau x)(t), \quad t \in [0, 1].
 \end{aligned}$$

Proceeding as for the proof of Theorem 9, we get

$$x - Tx \neq \mu \varphi_\tau, \quad x \in \partial K_R, \quad \mu \geq 0. \quad (89)$$

By Lemma 8, we have

$$i(T, K_R, K) = 0. \quad (90)$$

It follows from (84) and (90) that

$$i(T, K_R \setminus \overline{K}_r, K) = i(T, K_R, K) - i(T, K_r, K) = -1. \quad (91)$$

Then T has at least one fixed point on $K_R \setminus \overline{K}_r$. Consequently, problem (14) has at least one positive solution, which implies that the FBVP (3) has at least one positive solution. $\square$

Remark 12. In this work, the nonlinearity $f(t, z_1, z_2, \dots, z_{n-1})$ may be singular at $t = 0, 1$ and $z_1 = z_2 = \dots = z_{n-1} = 0$. To the best of our knowledge, very little work has been done for the case where f can be singular at $z_1 = z_2 = \dots = z_{n-1} = 0$.

4. An Example

Example 1. Consider the following problem:

$$D_{0+}^{7/2}y(t) + \frac{(y + y' + y'')^{-1/4} + \ln y''}{\sqrt{1-t}} = 0, \quad 0 < t < 1, \quad (92)$$

$$y(0) = y'(0) = y''(0) = 0, \quad y''(1) = \lambda[y''],$$

where $\alpha = 7/2$, $f(t, z_1, z_2, z_3) = ((z_1 + z_2 + z_3)^{-1/4} + \ln z_3)/\sqrt{1-t}$, and $(t, z_1, z_2, z_3) \in (0, 1) \times (\mathbb{R}_0^+)^3$. Obviously, f is singular at $t = 1$ and $z_1 = z_2 = z_3 = 0$. Clearly,

$$\begin{aligned}
 G(t, s) &= \begin{cases} G_1(t, s) = \frac{[t(1-s)]^{1/2}}{\Gamma(3/2)}, & 0 \leq t \leq s \leq 1, \\ G_2(t, s) = \frac{[t(1-s)]^{1/2} - (t-s)^{1/2}}{\Gamma(3/2)}, & 0 \leq s \leq t \leq 1. \end{cases} \quad (93)
 \end{aligned}$$

In the following we discuss that condition (H_0) holds when $\lambda[\cdot]$ take different cases.

(1) Let $\lambda[y''] = 0$. In this case, we have

$$\begin{aligned}
 \lambda[y] &= 0, \\
 \mathcal{G}(s) &= 0, \\
 \Phi(s) &= \frac{s(1-s)^{1/2}}{\Gamma(1/2)} \leq \frac{13(1-s)^{1/2}}{\Gamma(1/2)}. \quad (94)
 \end{aligned}$$

(2) Now, let $\lambda[y''] = \int_0^1 y''(t)(4-5t)dt$. Note that the function $h(t) = 4-5t$ changes the sign on the interval $[0, 1]$. In this case, we have

$$\begin{aligned}
 \lambda[y] &= \int_0^1 t^{1/2} dA(t) = \int_0^1 t^{1/2} (4-5t) dt = \frac{2}{3} < 1, \\
 \mathcal{G}(s) &= \int_0^1 G(t, s) (4-5t) dt \\
 &= \frac{1}{\Gamma(5/2)} (1-s)^{1/2} s (3-2s) \geq 0, \quad s \in [0, 1], \\
 \Phi(s) &= \frac{\mathcal{G}(s)}{1-\lambda[y]} + \frac{s(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+1)} \quad (95) \\
 &= \frac{3s(1-s)^{1/2} (3-2s)}{\Gamma(5/2)} + \frac{s(1-s)^{1/2}}{\Gamma(1/2)} \\
 &\leq \frac{13(1-s)^{1/2}}{\Gamma(1/2)}.
 \end{aligned}$$

(3) Let $\lambda[y''] = (1/2)y''(1/2)$. In this case, we have

$$\begin{aligned}
 \lambda[y] &= \int_0^1 t^{1/2} dA(t) = \frac{\sqrt{2}}{4} \approx 0.353553 < 1, \\
 0 \leq \mathcal{G}(s) &= \frac{1}{2} G\left(\frac{1}{2}, s\right) \\
 &= \begin{cases} \frac{[(1/2)(1-s)]^{1/2}}{\Gamma(3/2)}, & \frac{1}{2} \leq s \leq 1, \\ \frac{[(1/2)(1-s)]^{1/2} - (1/2-s)^{1/2}}{\Gamma(3/2)}, & 0 \leq s \leq \frac{1}{2}. \end{cases} \quad (96) \\
 &\leq \frac{(\sqrt{2}/4)(1-s)^{1/2}}{\Gamma(3/2)}, \quad s \in [0, 1], \\
 \Phi(s) &= \frac{\mathcal{G}(s)}{1-\lambda[y]} + \frac{s(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+1)} \\
 &\leq \frac{\sqrt{2}(1-s)^{1/2}}{(4-\sqrt{2})\Gamma(3/2)} + \frac{s(1-s)^{1/2}}{\Gamma(1/2)} \leq \frac{13(1-s)^{1/2}}{\Gamma(1/2)}.
 \end{aligned}$$

(4) Let $\lambda[y''] = 2y''(1/2) - y''(3/4)$. Note that the coefficients $b_1 = 2$, $b_2 = -1$; i.e., not all of the

coefficients must be positive, some coefficients of b_i can be negative. In this case, we have

$$\lambda[\gamma] = \int_0^1 t^{1/2} dA(t) = \sqrt{2} - \frac{\sqrt{3}}{2} \approx 0.548188 < 1,$$

$$\mathcal{G}(s) = \begin{cases} 2G_2\left(\frac{1}{2}, s\right) - G_2\left(\frac{3}{4}, s\right), & 0 \leq s < \frac{1}{2}, \\ 2G_1\left(\frac{1}{2}, s\right) - G_2\left(\frac{3}{4}, s\right), & \frac{1}{2} \leq s \leq \frac{3}{4}, \\ 2G_1\left(\frac{1}{2}, s\right) - G_1\left(\frac{3}{4}, s\right), & \frac{3}{4} < s \leq 1. \end{cases} \quad (97)$$

Then $0 \leq \mathcal{G}(s) < 1$, $s \in [0, 1]$, and

$$\begin{aligned} \Phi(s) &= \frac{\mathcal{G}(s)}{1 - \lambda[\gamma]} + \frac{s(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+1)} \\ &\leq \frac{2\sqrt{2}(1-s)^{1/2}}{(2-2\sqrt{2}+\sqrt{3})\Gamma(3/2)} + \frac{s(1-s)^{1/2}}{\Gamma(1/2)} \\ &\leq \frac{13(1-s)^{1/2}}{\Gamma(1/2)}. \end{aligned} \quad (98)$$

Seen from above, condition (H_0) holds in different cases 1-4. Now we define a cone

$$K = \{y \in C[0, 1] : y(t) \geq t^{1/2}(1-t)\|y\| \text{ for } t \in [0, 1]\}. \quad (99)$$

For any $0 < r < R < +\infty$ and $z_3 \in \bar{K}_R \setminus K_r$, we have

$$\begin{aligned} 0 &\leq rt^{5/2}(1-t) \leq t^{1/2}(1-t)r = t^{1/2}(1-t)\|z_3\| \\ &\leq z_3(t) \leq R, \quad t \in [0, 1], \\ 0 &\leq \frac{2}{3}rt^{5/2}(1-t) \leq \frac{2}{3}rt^{3/2}(1-t) \end{aligned}$$

$$= \frac{rt^{3/2}(1-t)\Gamma(3/2)}{\Gamma(5/2)} \leq z_2(t) = I_{0+}^1 z_3(t) \leq R,$$

$$t \in [0, 1],$$

$$\begin{aligned} \lim_{m \rightarrow +\infty} \sup_{\substack{z_1 \in K_{R/2} \\ z_2 \in K_R \\ z_3 \in \bar{K}_R \setminus K_r}} \int_{H(m)} \Phi(s) f(s, z_1(s), z_2(s), z_3(s)) ds &\leq \lim_{m \rightarrow +\infty} \sup_{\substack{z_1 \in K_{R/2} \\ z_2 \in K_R \\ z_3 \in \bar{K}_R \setminus K_r}} \int_{H(m)} \frac{13(1-s)^{1/2}}{\Gamma(1/2)} \\ &\times \frac{(z_1(s) + z_2(s) + z_3(s))^{-1/4} + |\ln z_3(s)|}{\sqrt{1-s}} ds \\ &\leq \lim_{m \rightarrow +\infty} \sup_{\substack{z_1 \in K_{R/2} \\ z_2 \in K_R \\ z_3 \in \bar{K}_R \setminus K_r}} \int_{H(m)} \frac{13}{\Gamma(1/2)} \left[|\ln R| + |\ln r| + |\ln s^{1/2}(1-s)| + \left(\frac{29}{15}r\right)^{-1/4} s^{-5/8}(1-s)^{-1/4} \right] ds = \frac{26}{\Gamma(1/2)} [|\ln R| \\ &+ |\ln r|] \lim_{m \rightarrow +\infty} \frac{1}{m} + \frac{13}{\Gamma(1/2)} \lim_{m \rightarrow +\infty} \int_{H(m)} \left[|\ln s^{1/2}(1-s)| + \left(\frac{29}{15}r\right)^{-1/4} s^{-5/8}(1-s)^{-1/4} \right] ds = 0, \end{aligned} \quad (104)$$

$$0 \leq \frac{4}{15}rt^{5/2}(1-t) = \frac{rt^{5/2}(1-t)\Gamma(3/2)}{\Gamma(7/2)} \leq z_1(t)$$

$$= I_{0+}^2 z_3(t) \leq \frac{R}{2}, \quad t \in [0, 1].$$

(100)

Noticing that $\ln z_3$ is decreasing on $(0, 1)$ and is increasing on $[1, +\infty)$, so we have

$$\begin{aligned} |\ln z_3(t)| &\leq |\ln rt^{5/2}(1-t)| + |\ln R| \\ &\leq |\ln r| + |\ln R| + |\ln t^{5/2}(1-t)|, \\ (z_1(t) + z_2(t) + z_3(t))^{-1/4} &\end{aligned} \quad (101)$$

$$\leq \left(\frac{29}{15}r\right)^{-1/4} t^{-5/8}(1-t)^{-1/4}.$$

Consequently

$$\begin{aligned} &\int_0^1 \left[|\ln t^{1/2}(1-t)| + \left(\frac{29}{15}r\right)^{-1/4} t^{-5/8}(1-t)^{-1/4} \right] dt \\ &\leq \frac{1}{2} \int_0^1 |\ln t| dt + \int_0^1 |\ln(1-t)| dt \\ &\quad + \left(\frac{29}{15}r\right)^{-1/4} \int_0^1 t^{-5/8}(1-t)^{-1/4} dt \\ &\leq \frac{3}{2} + \left(\frac{29}{15}r\right)^{-1/4} \mathbf{B}\left(\frac{3}{8}, \frac{3}{4}\right) < +\infty. \end{aligned} \quad (102)$$

The absolute continuity of integral leads to

$$\begin{aligned} \lim_{m \rightarrow +\infty} \int_{H(m)} &\left[|\ln t^{1/2}(1-t)| \right. \\ &\left. + \left(\frac{29}{15}r\right)^{-1/4} t^{-5/8}(1-t)^{-1/4} \right] dt = 0. \end{aligned} \quad (103)$$

Hence

On the other hand, it is obvious that

$$\begin{aligned}
 & \liminf_{\substack{z_1 \rightarrow 0 \\ z_2 \rightarrow 0 \\ z_3 \rightarrow 0}} \frac{f(t, z_1, z_2, z_3)}{z_1 + z_2 + z_3} \\
 &= \liminf_{\substack{z_1 \rightarrow 0 \\ z_2 \rightarrow 0 \\ z_3 \rightarrow 0}} \frac{(z_1 + z_2 + z_3)^{-1/2} + \ln z_3}{\sqrt{t(1-t)}(z_1 + z_2 + z_3)} = +\infty, \\
 & \limsup_{\substack{z_1 + z_2 + z_3 \rightarrow +\infty \\ z_3 \rightarrow +\infty}} \frac{f(t, z_1, z_2, z_3)}{z_3} \\
 &= \limsup_{\substack{z_1 + z_2 + z_3 \rightarrow +\infty \\ z_3 \rightarrow +\infty}} \frac{(z_1 + z_2 + z_3)^{-1/2} + \ln z_3}{z_3 \sqrt{t(1-t)}} = 0,
 \end{aligned} \tag{105}$$

uniformly on $t \in [0, 1]$, which implies that

$$\begin{aligned}
 & \limsup_{\substack{z_1 + z_2 + z_3 \rightarrow +\infty \\ z_3 \rightarrow +\infty}} \frac{f(t, z_1, z_2, z_3)}{z_3} < \lambda_1 \\
 & < \liminf_{\substack{z_1 \rightarrow 0 \\ z_2 \rightarrow 0 \\ z_3 \rightarrow 0}} \frac{f(t, z_1, z_2, z_3)}{z_1 + z_2 + z_3}.
 \end{aligned} \tag{106}$$

Therefore, all conditions of Theorem 9 are satisfied. Thus Theorem 9 ensures that FBVP (92) has at least one positive solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The authors were supported financially by China Postdoctoral Science Foundation funded project (2017M612231), the Natural Science Foundation of Shandong Province of China (ZR2017MA036), Doctoral Scientific Research Foundation of Qufu Normal University and Youth Foundation of Qufu Normal University (BSQD20130140), and Research Fund of Qufu Normal University (XKJ201303).

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