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Research Article

$A_p(\phi)$ Weights, $BMO(\phi)$, and Calderón-Zygmund Operators of ϕ-Type

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We introduce new classes of weights and *BMO* functions associated with a nondecreasing function ϕ of upper type β with $\beta > 0$ and obtain the weighted norm inequalities for Calderón-Zygmund operators of ϕ-type and their commutators.

1. Introduction

For a nonnegative and nondecreasing function ϕ mapping from $[0,\infty)$ to $[1,\infty)$, we shall mean that it is of upper type β with $\beta > 0$, if there exists a constant c_0 such that

$$\phi(\theta t) \le c_0 \theta^{\beta} \phi(t), \tag{1}$$

for all $\theta \ge 1$ and $t \ge 0$. In this paper, we always assume that $\phi(1) > 1$.

Let T be a continuous associated with the kernel function $K(x,y)$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ that satisfies

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy \tag{2}$$

for all $f \in C_0^{\infty}$ and x not in the support of f is said to be associated with K. Here, the kernel $K(x,y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,x) : x \in \mathbb{R}^n\}$ and satisfies that for any $N \ge 0$ and some $A > 0$ the size condition

$$|K(x,y)| \le \frac{A}{|x-y|^n} \frac{1}{\left[\phi\left(|x-y|^n\right)\right]^N}, \tag{3}$$

and for some $\delta > 0$ the regularity conditions

$$\begin{aligned} \left|K(x,y) - K(x',y)\right| &\le \frac{A\left|x-x'\right|^{\delta}}{\left(|x-y| + |x'-y|\right)^{n+\delta}} \\ &\cdot \frac{1}{\left[\phi\left(\left(|x-y| + |x'-y|\right)^n\right)\right]^N}, \end{aligned} \tag{4}$$

whenever $|x-x'| \le (1/2)\max(|x-y|, |x'-y|)$ and

$$\begin{aligned} \left|K(x,y) - K(x,y')\right| &\le \frac{A\left|x-x'\right|^{\delta}}{\left(|x-y| + |x-y'|\right)^{n+\delta}} \\ &\cdot \frac{1}{\left[\phi\left(\left(|x-y| + |x-y'|\right)^n\right)\right]^N}, \end{aligned} \tag{5}$$

whenever $|y-y'| \le (1/2)\max(|x-y|, |x-y'|)$. We also suppose that T is associated with K and it can be extended to be a bounded extension on $L^2(\mathbb{R}^n)$,

$$\|Tf\|_{L^2} \le B\|f\|_{L^2}. \tag{6}$$

In this paper, we will consider weighted estimates for these operators and their commutators.

Remark 1. It is clear that if T satisfies (3), (4), (5), and (6) then T falls within the scope of the Calderón-Zygmund theory. Then T has an extension that maps $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, and by interpolation, T also maps $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$.

Remark 2. When $\phi(t) = (1+t)$, from [1–3], we know that the pseudodifferential operators T_σ with the symbol $\sigma \in S^0_{1,0}$ defined by

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi) e^{2\pi i x\cdot\xi} \hat{f}(\xi)\, d\xi \tag{7}$$

satisfy (3), (4), (5), and (6), where $S^m_{1,\delta}$ denotes the set of all smooth functions $\sigma(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\left| D^\alpha_x D^\beta_\xi \sigma(x,\xi) \right| \le C_{\alpha,\beta} (1+|\xi|)^{m-\beta+\delta\alpha} \tag{8}$$

for all multi-indices α and β.

Definition 3. A weight will always mean a positive function which is locally integrable. We say that a weight ω belongs to the class $A^\theta_p(\phi)$ for $\theta \ge 0$ and $1 < p < \infty$, if there is a positive constant C such that, for all cubes Q.

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)^{1/p} \left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{1/p'} \le C\left[\phi(|Q|)\right]^\theta. \tag{9}$$

We also say that a nonnegative function satisfies the $A^\theta_1(\phi)$ condition if there exists a constant $C > 0$ for all cubes Q,

$$\frac{1}{|Q|}\int_Q \omega(x)\,dx \le C\left[\phi(|Q|)\right]^\theta \inf_{x\in Q}\omega(x). \tag{10}$$

We also write $A^\theta_\infty(\phi) = \cup_{p\ge1} A^\theta_p(\phi)$, $A^\infty_p(\phi) = \cup_{\theta\ge0} A^\theta_p(\phi)$, and $A^\infty_\infty(\phi) = \cup_{\theta\ge0} A^\theta_\infty(\phi) = \cup_{p\ge1} A^\infty_p(\phi)$.

Remark 4. We remark that $A^0_p(\phi)$ coincides with Muckenhoupt's class of weights A_p for all $1 \le p < \infty$. However, in general, the class $A^\infty_p(\phi)$ is strictly larger than the class A_p for all $1 \le p < \infty$. On the other hand, when ϕ is a constant function, $A^\theta_p(\phi)$ also coincides with A_p for any $\theta \in (0,\infty)$. In particular, if $\phi \equiv 1$, $A^\infty_\infty(\phi) = A_\infty$.

Remark 5. It should be noted that $\omega(x)dx$ may not be a doubling measure; that is, for each positive constant C, there exists at least a cube Q such that

$$\omega(2Q) \ge C\omega(Q). \tag{11}$$

For example, let $\phi = 1 + |Q|$. If $0 \le \gamma \le n\theta$, then $\omega = (1+|x|)^{-(n+\gamma)} \notin A_\infty$ and $\omega(x)dx$ is a nondoubling measure, but $\omega = (1+|x|)^{-(n+\gamma)} \in A^\theta_1(\phi)$; see [1, 4]. We also need to point out that these weights have many differences from Orobitg-Pérez's undoubling weights [5] because they are against the reverse Hölder inequality by Proposition 15 (v).

Next we state our main theorem for these operators satisfying (3)–(6) as follows.

Theorem 6. *Assume that T satisfies (3), (4), (5), and (6). Let $\omega \in A^\infty_p(\phi)$ with $1 \le p < \infty$. Then T is bounded from $L^p(\omega)$ to $L^p(\omega)$ for $1 < p < \infty$ and $L^1(\omega)$ to $L^{1,\infty}(\omega)$.*

We also consider the commutator of Coifman-Rochberg-Weiss $[b,T]$ defined by

$$[b,T]f(x) = b(x)T(f)(x) - T(bf)(x). \tag{12}$$

Since our operator T is stronger than the standard Carlderón-Zygmund operator, we will generalize the symbol b from usual BMO to $BMO^\theta(\phi)$.

Definition 7. A locally integrable function f is said to be in $BMO^\theta_p(\phi)$ with $p \ge 1$ and $\theta \ge 0$, if there exists a positive constant C such that for any cube Q

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p\,dx\right)^{1/p} \le C\left[\phi(|Q|)\right]^\theta, \tag{13}$$

where f_Q is the average of f on Q. A norm for $b \in BMO^\theta(\phi)$, denoted by $\|f\|_{BMO^\theta(\phi)}$, is given by the infimum of the constants satisfying (13).

Remark 8. Clearly $BMO^{\theta_1}(\phi) \subset BMO^{\theta_2}(\phi)$ for $\theta_1 \le \theta_2$ and $BMO^\theta(\phi) = BMO$ if $\theta = 0$ or $\phi \equiv$ constant. We define $BMO^\infty(\phi) = \cup_{\theta\ge0} BMO^\theta(\phi)$. In [6], Morvidone proved that these spaces are independent of the scale p; that is, $BMO^\theta_1(\phi) = NMO^\theta_p(\phi)$ in the sense of norm. So we denote $BMO^\theta(\phi)$ simply.

Remark 9. Recently Harboure et al. [7] established an intimate relationship between these spaces and ϕ-Carleson measures similar to well-known Fefferman-Stein's theorem. Precisely, let Ψ be a function in $\mathcal{S}$ with a null integral. $f \in BMO^1(\phi)$ if and only if $d\mu = |\Psi_t * f|^2(x)dx(dt/t)$ is a ϕ-Carleson measure; that is, $d\mu(\widehat{Q}) \le |Q|\phi^2(|Q|)$, where $\widehat{Q}$ is a Carleson box.

Theorem 10. *Assume that T satisfies (3), (4), (5), and (6). Let $b \in BMO^\theta(\phi)$ for $\theta > 0$ and $\omega \in A^\infty_p(\phi)$ with $1 \le p < \infty$. Then there exists a positive constant C such that for any $1 < p < \infty$*

$$\|[b,T](f)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}. \tag{14}$$

To consider the endpoint case, we introduce the Orlicz norm. For $\Phi(t) = t(1+\log^+ t)$ and a cube Q on $\mathbb{R}^n$, by the Luxemburg norm of f it means that

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}. \tag{15}$$

We recall that $\|f\|_{\Phi,Q} > 1$ if and only if $(1/|Q|)\int_Q \Phi(|f(x)|)dx > 1$.

Theorem 11. *Assume that T satisfies (3), (4), (5), and (6). Let $b \in BMO^\theta(\phi)$ for $\theta \ge 0$ and $\omega \in A^\infty_1(\phi)$ with $1 \le p < \infty$.*

Then there exists a positive constant C such that for any $1 < p < \infty$

$$\omega\left(\left\{x \in \mathbb{R}^n : \left|[b,T](f)\right| > \lambda\right\}\right) \le C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \omega(x)\, dx. \tag{16}$$

Remark 12. One of our original motivations is to procure some detailed operators that satisfy our conditions like pseudodifferential operators. Restricted with our knowledge, we have found nothing to date.

2. Some Preliminaries and Notations

In this section, we begin with defining the weighted maximal function by

$$M_\omega f(x) = \sup_{Q \ni x} \frac{1}{\omega(5Q)} \int_Q |f(y)|\, \omega(y)\, dy. \tag{17}$$

Proposition 13. *Let* $\omega \in A_\infty^\infty(\phi)$. *Then*

$$\omega\left(\left\{x \in \mathbb{R}^n : M_\omega f(x) > \lambda\right\}\right) \le \frac{C}{\lambda} \|f\|_{L^1(\omega)}, \quad \forall \lambda > 0,\ f \in L^1(\omega), \tag{18}$$

and for $1 < p < \infty$,

$$\|M_\omega f\|_{L^p(\omega)} \le C_p \|f\|_{L^p(\omega)}. \tag{19}$$

Proof. By the Marcinkiewicz interpolation theorem and the fact that $M_{\phi,\omega}$ is bounded on $L^\infty(\omega)$, we only need to prove (18). For any $\lambda > 0$, let $E_\lambda = \{x \in \mathbb{R}^n : M_\omega f(x) > \lambda\}$. Fixing $x \in E_\lambda$, then there exists a cube $Q_x \ni x$ such that

$$\frac{1}{\omega(5Q_x)} \int_{Q_x} |f(y)|\, \omega(y)\, dy > \lambda. \tag{20}$$

Thus, $\{Q_x\}_{x \in E_\lambda}$ covers E_λ. By Vitali's lemma, there exists a class of disjoint cubes $\{Q_{x_j}\}$ such that $Q_{x_j} \subset E_\lambda \subset 5Q_{x_j}$ and

$$\begin{aligned} \omega(E_\lambda) &\le \sum_j \omega\left(5Q_{x_j}\right) \le \left(\frac{C}{\lambda}\right) \sum_j \int_{Q_{x_j}} |f(y)|\, \omega(y)\, dy \\ &\le \left(\frac{C}{\lambda}\right) \|f\|_{L^1(\omega)}. \end{aligned} \tag{21}$$

□

By the boundedness of M_ω and the following covering lemma, we obtain some properties for $A_p(\phi)$.

Lemma 14 (see [8]). *There exists a sequence of points* x_j, $j \ge 1$ *in* $\mathbb{R}^n$ *so that the family of cubes* $\{Q_j\}_j$ *where* $Q_j := B(x_j, 1)$, $j \ge 1$, *satisfies the following:*

(a) $\cup_j Q_j = \mathbb{R}^n$.

(b) *There exists a constant C such that for any* $\sigma > 1$, $\sum_j \chi_{\sigma Q_j} \le C\sigma^n$.

Proposition 15. *The following statements hold:*

(i) *If* $1 \le p_1 < p_2 < \infty$, *then* $A_{p_1}^\theta(\phi) \subset A_{p_2}^\theta(\phi)$.

(ii) $\omega \in A_p^\theta(\phi)$ *if and only if* $\omega^{1-p'} \in A_{p'}^\theta(\phi)$, *where* $(1/p) + (1/p') = 1$.

(iii) *If* $\omega_1, \omega_2 \in A_p^\theta(\phi)$, $p \ge 1$, *then* $\omega_1^\alpha \omega_2^{1-\alpha} \in A_p^\theta(\phi)$ *for any* $0 < \alpha < 1$.

(iv) *If* $\omega \in A_p^\theta(\phi)$ *for* $1 \le p < \infty$, *then*

$$\begin{aligned} &\frac{1}{[\phi(|Q|)]^\theta |Q|} \int_Q |f(y)|\, dy \\ &\quad \le C_\theta \left(\frac{1}{\omega(5Q)} \int_{5Q} |f(y)|^p\, \omega(y)\, dy\right)^{1/p}. \end{aligned} \tag{22}$$

Particularly, let $f = \chi_E$ *for any measurable set* $E \subset Q$,

$$\frac{|E|}{[\phi(|Q|)]^\theta |Q|} \le C_\theta \left(\frac{\omega(E)}{\omega(2Q)}\right)^{1/p}. \tag{23}$$

(v) *If* $\omega \in A_p^\theta(\phi)$ *with* $p \ge 1$, *then there exist positive numbers* δ, η, *and C so that for all cubes Q*

$$\begin{aligned} &\left(\frac{1}{|Q|} \int_Q \omega(x)^{1+\delta}\, dx\right)^{1/(1+\delta)} \\ &\quad \le C \left(\frac{1}{|Q|} \int_Q \omega(x)\, dx\right) [\phi(|Q|)]^\eta. \end{aligned} \tag{24}$$

(vi) *If* $\omega \in A_p^\infty(\phi)$ *with* $p > 1$ *then there exists* $\epsilon > 0$ *such that* $\omega \in A_{p-\epsilon}^\infty(\phi)$.

Proof. (i), (ii), and (iii) are easy from the definition. We only prove (iv). In fact, note that $\phi(|5Q|) \le 5^{n\beta}\phi(|Q|)$; by Hölder's inequality and $A_p^\theta(\phi)$ condition we then have

$$\begin{aligned} &\frac{1}{[\phi(|Q|)]^\theta |Q|} \int_Q |f(y)|\, dy \\ &\quad \le \left(\frac{1}{[\phi(|Q|)]^\theta |Q|} \int_Q |f(y)|^p\, \omega(y)\, dy\right)^{1/p} \\ &\quad \cdot \left(\frac{1}{[\phi(|Q|)]^\theta |Q|} \int_Q \omega(y)^{1-p'}\, dy\right)^{1/p'} \\ &\quad \le C \left(\frac{1}{[\phi(|Q|)]^\theta |Q|} \int_Q |f(y)|^p\, \omega(y)\, dy\right)^{1/p} \\ &\quad \cdot \left(\frac{1}{[\phi(|5Q|)]^\theta |5Q|} \int_{5Q} \omega(y)\, dy\right)^{-1/p} \\ &\quad \le C \left(\frac{1}{\omega(5Q)} \int_{5Q} |f(y)|^p\, \omega(y)\, dy\right)^{1/p}, \end{aligned} \tag{25}$$

for all $p > 1$. For $p = 1$, we note that

$$\sup_{y\in 2Q} \omega(y)^{-1} \le \left(\frac{1}{[\phi(|5Q|)]^{\theta} |5Q|} \int_{5Q} \omega(y)\, dy \right)^{-1}; \quad (26)$$

thus it concludes (iv).

For (v), by Lemma 2.4 [1], there exist positive constants $C > 0$ and $\delta > 0$, such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x)^{1+\delta}\, dx \right)^{1/(1+\delta)} \le C \frac{1}{|Q|} \int_Q \omega(x)\, dx \quad (27)$$

for any $Q = Q(x_0, r) \subset \mathbb{R}^n$ with $r < 1$. On the other hand, let $Q = Q(x, r)$, with $r > 1$ and $\mathcal{F} = \{j : Q_j \cap B \neq \emptyset\}$, where $Q_j = Q(x_j, 1)$ and $\{x_j\}_j$ is the sequence of Lemma 14. It is easy to see that $\cup_j Q_j \subset 5Q$. Now let δ be the constant in (27). By using Lemma 14, $A_p^{\theta}(\phi)$ condition, and Hölder's inequality,

$$\begin{aligned}
&\left(\int_Q \omega(x)^{1+\delta}\, dx \right)^{1/(1+\delta)} \\
&\quad \le \sum_{j\in\mathcal{F}} \left(\int_{Q_j} \omega(x)^{1+\delta}\, dx \right)^{1/(1+\delta)} \\
&\quad \le C \sum_{j\in\mathcal{F}} \omega(Q_j) |Q_j|^{-\delta/(1+\delta)} \le C\omega(5Q) \\
&\quad \le C |Q|^p \phi(|5Q|)^p \left(\int_Q \omega(x)^{1-p'}\, dx \right)^{-1/p'} \\
&\quad \le C\omega(Q) [\phi(|5Q|)]^p \\
&\quad \le C\omega(|Q|) [\phi(|Q|)]^{p-(\delta/(1+\delta))(1/\theta\beta)} |Q|^{\delta/(1+\delta)} \phi(1),
\end{aligned} \quad (28)$$

we obtain the desired inequality with $\eta = p - \delta/(1+\delta)\theta\beta$; here β is sufficiently big. And (vi) is a consequence of (v). □

Then we also define maximal functions associated with ϕ and $\eta > 0$ by

$$M_{\phi,\eta} f(x) = \sup_{Q\ni x} \frac{1}{[\phi(|Q|)]^{\eta} |Q|} \int_Q |f(y)|\, dy. \quad (29)$$

From (iv) in Proposition 15, we have that

$$M_{\phi,\theta} f(x) \le \left(M_{\omega}\left(|f|^p \right)(x) \right)^{1/p}, \quad \text{a.e. } x \in \mathbb{R}^n, \quad (30)$$

for $\omega \in A_p^{\theta}(\phi)$. From this and using (18) and (19), we can get the following result.

Proposition 16. *Let $1 \le p_1 < \infty$ and suppose that $\omega \in A_{p_1}^{\theta}(\phi)$. If $p_1 < p < \infty$, then the equality*

$$\int_{\mathbb{R}^n} |M_{\phi,\theta} f(x)|^p \omega(x)\, dx \le C_p \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\, dx. \quad (31)$$

Furthermore, let $1 \le p < \infty$ and $\omega \in A_p^{\theta}(\phi)$ if and only if

$$\begin{aligned}
&\omega\left(\left\{ x \in \mathbb{R}^n : M_{\phi,\theta} f(x) > \lambda \right\} \right) \\
&\quad \le \frac{C_p}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\, dx.
\end{aligned} \quad (32)$$

Proposition 17. *Let $1 < p < \infty$ and $p' = p/(p-1)$ and suppose that $\omega \in A_p^{\theta}(\phi)$. There exists a constant $C > 0$ such that*

$$\left\| M_{\phi,p'\theta} f \right\|_{L^p(\omega)} \le C \|f\|_{L^p(\omega)}. \quad (33)$$

As a consequence of Proposition 17, we have the following.

Corollary 18. *Let $1 \le p < \infty$ and $\omega \in A_p^{\theta}(\phi)$. Let ψ be a radial, positive function with compact support and total integral 1. Set $\psi_t(x) = t^{-n}\psi(x/t)$. Then*

(i) $\sup_{0<t<1} |f * \psi_t(x)| \le C_\eta M_{\phi,\eta} f(x)$ *for $f \in L^p(\omega)$ and $0 < \eta < \infty$;*

(ii) $f * \psi_t(x) \to f(x)$, *as $t \to 0$, almost every for $f \in L^p(\omega)$;*

(iii) $\|f * \psi_t - f\|_{L^p(\omega)} \to 0$, *as $t \to 0$, almost every for $f \in L^p(\omega)$.*

Next we define the dyadic maximal function and sharp maximal function $M_{\phi,\eta}^{\triangle}$ and $M_{\phi,\eta}^{\sharp,\triangle}$ by for $\eta > 0$

$$\begin{aligned}
M_{\phi,\eta}^{\triangle}(f)(x) &= \sup_{\substack{Q\ni x \\ Q \text{ dyadic}}} \frac{1}{[\phi(|Q|)]^{\eta} |Q|} \int_Q |f(y)|\, dy; \\
M_{\phi,\eta}^{\sharp,\triangle}(f)(x) &= \sup_{\substack{Q\ni x, l(Q)<1 \\ Q \text{ dyadic}}} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy \\
&\quad + \sup_{\substack{Q\ni x, l(Q)\ge 1 \\ Q \text{ dyadic}}} \frac{1}{[\phi(|Q|)]^{\eta} |Q|} \int_Q |f(y)|\, dy \\
&= \sup_{\substack{Q\ni x, l(Q)<1 \\ Q \text{ dyadic}}} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c|\, dy \\
&\quad + \sup_{\substack{Q\ni x, l(Q)\ge 1 \\ Q \text{ dyadic}}} \frac{1}{[\phi(|Q|)]^{\eta} |Q|} \int_Q |f(y)|\, dy
\end{aligned} \quad (34)$$

Then we have the following.

Lemma 19. *Let f be a locally integrable function on $\mathbb{R}^n$, η, $\lambda > 0$, and $\Omega_\lambda = \{x \in \mathbb{R}^n : M_{\phi,\eta} f(x) > \lambda\}$. Then Ω_λ may be written as a disjoint union of maximal dyadic cubes $\{Q_j\}$ with*

$$\lambda < \frac{1}{[\phi(|Q_j|)]^{\eta} |Q_j|} \int_{Q_j} |f(x)|\, dx < c_0 2^{n(1+\beta\eta)} \lambda. \quad (35)$$

Moreover, we have that $|f(x)| \le \lambda$ for almost all $x \in \mathbb{R}^n \backslash \cup_j Q_j$ and $|\Omega_\lambda| \le \lambda^{-1} \|f\|_{L^1}$.

Proof. The proof is standard; see Lemma 1 on P. 150 of [9]. □

Lemma 20. *Let $\omega \in A_p^\theta(\phi)$ and $0 < \eta < \infty$. For a locally integrable function f, and for b and γ positive $\gamma < b < b_0 = 1/\phi(2^n)$, then there exist positive constants C and δ_1 such that we have the following inequality:*

$$\begin{aligned}\omega\left(\left\{x \in \mathbb{R}^n : M_{\phi,\eta}^{\triangle} f(x) > \lambda, M_{\phi,\eta}^{\sharp,\triangle} f(x) < \gamma\lambda\right\}\right) \\ \le C a^{\delta_1} \omega\left(\left\{x \in \mathbb{R}^n : M_{\phi,\eta}^{\triangle} f(x) > b\lambda\right\}\right),\end{aligned} \tag{36}$$

for all $\lambda > 0$, where $a = 2^n\gamma/(1 - b/b_0)$.

Proof. The proof is very similar to that of Lemma 2.6 [1]. □

Corollary 21. *Let $1 < p < \infty$, $\omega \in A_\infty^\theta(\phi)$, and $0 < \theta < \infty$. Then there exists a constant $C > 0$ such that*

$$\left\|M_{\phi,\theta}^{\triangle} f\right\|_{L^p(\omega)} \le C \left\|M_{\phi,\theta}^{\sharp,\triangle} f\right\|_{L^p(\omega)}. \tag{37}$$

Note that $|f(x)| \le M_{\phi,\theta} f(x)$ a.e. $x \in \mathbb{R}^n$ for any $\theta > 0$. By Corollary 21, we have the following.

Corollary 22. *Let $1 < p < \infty$, $\omega \in A_p^\theta(\phi)$, $0 < \theta < \infty$, and $f \in L^p(\omega)$, and then*

$$\|f\|_{L^p(\omega)} \le \left\|M_{\phi,\theta}^{\triangle} f\right\|_{L^p(\omega)} \le C \left\|M_{\phi,\theta}^{\sharp,\triangle} f\right\|_{L^p(\omega)}. \tag{38}$$

For $\eta > 0$ and the Young function $\Phi(t) = t(1 + \log^+ t)$, we define the maximal function $M_{L\log L,\phi,\eta}$ by

$$M_{L\log L,\phi,\eta}(f)(x) = \sup_{Q \ni x} [\phi(|Q|)]^{-\eta} \|f\|_{\Phi,Q}. \tag{39}$$

Lemma 23. *Let $0 < \eta < \infty$ and $M_{\phi,\eta/2} f$ be locally integrable. Then there exist positive constants C_1 and C_2 independent of f and x such that*

$$\begin{aligned}C_2 M_{\phi,\eta} M_{\phi,\eta} f(x) &\le M_{L\log L,\phi,\eta} f(x) \\ &\le C_1 M_{\phi,\eta/2} M_{\phi,\eta/2} f(x).\end{aligned} \tag{40}$$

Proof. On the one hand, we take f with $\|f\|_{L\log L,Q} = 1$ which implies

$$\begin{aligned}\|f\|_{L\log L,Q} = 1 &\le \frac{C}{|Q|} \\ &\cdot \int_Q |f(y)| \left(1 + \log^+(|f(y)|)\right) dy \le \frac{C}{|Q|} \\ &\cdot \int_Q |f(y)| \int_0^{|f(y)|+e} \frac{dt}{t} dy \le \frac{C}{|Q|} \int_Q |f(y)| \\ &\cdot \int_0^{|f(y)|+e} \frac{dt}{t} dy \le \frac{C}{|Q|} \int_1^e \int_Q |f(y)| \, dy \frac{dt}{t} + \frac{C}{|Q|} \\ &\cdot \int_e^\infty \int_{\{y \in Q : |f(y)| > t-e\}} |f(y)| \, dy \frac{dt}{t} \le \frac{C}{|Q|} \\ &\cdot \int_Q M(f\chi_Q)(x) \, dx \le \frac{C\phi(|Q|)^{\eta/2}}{|Q|} \\ &\cdot \int_Q M_{\phi,\eta/2}(f)(x) \, dx \le C\phi(|Q|)^\eta \\ &\cdot M_{\phi,\eta/2} M_{\phi,\eta/2} f(x).\end{aligned} \tag{41}$$

On the other hand, for any $x \in \mathbb{R}^n$ and any fixed cube $Q \ni x$, write $f = f_1 + f_2$, with $f_1 = f\chi_{3Q}$. Thus

$$\begin{aligned}&\frac{1}{[\phi(|Q|)]^\theta |Q|} \int_Q M_{\phi,\eta} f(y) \, dy \\ &\quad \le \frac{1}{[\phi(|Q|)]^\theta |Q|} \int_Q M_{\phi,\eta} f_1(y) \, dy \\ &\qquad + \frac{1}{[\phi(|Q|)]^\theta |Q|} \int_Q M_{\phi,\eta} f_2(y) \, dy := I + II.\end{aligned} \tag{42}$$

Since $\|M(g)\chi_Q\|_{L^1} \le C|Q|\|g\|_{L\log L,Q}$ for any g with supp $g \subset Q$, we have

$$\begin{aligned}I &\le \frac{1}{[\phi(|Q|)]^\theta |3Q|} \int_{3Q} M_{\phi,\eta} f_1(y) \, dy \\ &\le C M_{L\log L,\phi,\eta}(f)(x).\end{aligned} \tag{43}$$

Noting that $M_{\phi,\eta} f_2(y) \le CM(f)(z)$ for any two $y, z \in Q$, then

$$\begin{aligned}II &\le \frac{C}{[\phi(|Q|)]^\theta |Q|} \int_Q \inf_{z \in Q} M(f)(z) \, dy \\ &\le C M_{\phi,\eta}(f)(x) \le C M_{L\log L,\phi,\eta}(f)(x).\end{aligned} \tag{44}$$

□

The following proposition is a technical result. Its proof copies almost verbatim the proof of Proposition 3 [10] and is omitted.

Proposition 24. *Let $q \ge 1$. If $b \in BMO^\theta(\phi)$ then for all cube Q*

$$\begin{aligned}\left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^q \, dx\right)^{1/q} &\le C[\phi(|Q|)]^\theta, \\ \left(\frac{1}{|2^k Q|} \int_{2^k Q} |f(x) - f_Q|^q \, dx\right)^{1/q} &\le Ck\left[\phi\left(|2^k Q|\right)\right]^\theta\end{aligned} \tag{45}$$

for all $k \in \mathbb{N}$.

Proposition 25. *Supposing that f is in $BMO^\theta(\phi)$, there exist positive constants c_1 and c_2 such that*

$$\sup_Q \frac{1}{|Q|} \int_Q \exp\left\{\frac{c_1 |f(x) - f_Q|}{\|f\|_{BMO^\theta(\phi)} [\phi(|Q|)]^\theta}\right\} dx \le c_2. \tag{46}$$

Proof. The proof is classical; we refer to Proposition 4.3 [11]. □

As we know that $\Psi(t) = e^t - 1$ is also a Young function, the corresponding average is denoted by

$$\|f\|_{\Psi,Q} = \|f\|_{\exp L,Q}. \tag{47}$$

Then there is a generalized Hölder inequality

$$\frac{1}{|Q|}\int_Q |f(x)\,g(x)|\,dx \le 2\|f\|_{\exp L,Q}\|g\|_{L\log L,Q}. \tag{48}$$

By Proposition 25 and (48), we can get

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,|g(x)|\,dx \\ \le 2\|f\|_{BMO^\theta(\phi)}\|g\|_{L\log L,Q}[\phi(|Q|)]^\theta.\end{aligned} \tag{49}$$

3. Proofs of Theorems

The proof of Theorem 6 needs the following technical lemma.

Lemma 26. *Assume that T satisfies (3), (4), (5), and (6). Let $0<\epsilon<1$ and $\eta>0$. Then there exists a constant $C>0$ such that for all $x\in\mathbb{R}^n$ and all $f\in\mathcal{S}$*

$$M^{\sharp,\triangle}_{\phi,\epsilon,\eta}(Tf)(x) \le CM_{\phi,\eta}f(x), \tag{50}$$

where $M^{\sharp,\triangle}_{\phi,\epsilon,\eta}$ is a variant of dyadic sharp maximal operator

$$M^{\sharp,\triangle}_{\phi,\epsilon,\eta}(f) = M^{\sharp,\triangle}_{\epsilon,\eta}\left(|f|^\epsilon\right)^{1/\epsilon}. \tag{51}$$

Proof. Fix $x\in\mathbb{R}^n$ and let a dyadic cube $Q=Q(x_0,r)\ni x$, and we write $f(x)=f\chi_{4Q}(x)+f\chi_{\mathbb{R}^n\setminus 4Q}(x)=f_1(x)+f_2(x)$. When $0<r\le 1$, since $0<\epsilon<1$, we have the inequality

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \left||Tf(x)|^\epsilon-|c|^\epsilon\right|dx\right)^{1/\epsilon}\\ &\le\left(\frac{1}{|Q|}\int_Q |Tf_1(x)|^\epsilon dx\right)^{1/\epsilon}\\ &\quad+\left(\frac{1}{|Q|}\int_Q |Tf_2(x)-c|^\epsilon dx\right)^{1/\epsilon}:=I+II.\end{aligned} \tag{52}$$

Applying Kolmogorov's inequality to the term I and the weak $(1,1)$ boundedness of T, we have

$$\begin{aligned}I &\le C\|Tf_1\|_{L^{1,\infty}(Q,dx/|Q|)} \le C\frac{1}{|4Q|}\int_{4Q}|f(x)|\,dx\\ &\le C\frac{1}{[\phi(|4Q|)]^\eta|4Q|}\int_{4Q}|f(x)|\,dx\\ &\le CM_{\phi,\eta}(f)(x).\end{aligned} \tag{53}$$

Set $c=T(f)(x_0)$, for $N=\eta$ in the regularity in (4) and (5); we have

$$\begin{aligned}|T(f_2)(x)-T(f_2)(x_0)| &\le C\int_{\mathbb{R}^n\setminus 4Q}\frac{|x-x_0|^\delta}{(|x-y|+|x_0-y|)^{n+\delta}}\frac{|f(y)|}{\left[\phi\left((|x-y|+|x_0-y|)^n\right)\right]^\eta}dy\\ &\le C\sum_{k=1}^\infty\frac{4^{-k(\delta/n)}}{|4^{k+1}Q|}\frac{1}{[\phi(|4^{k+1}Q|)]^\eta}\int_{4^{k+1}Q}|f(y)|\,dy \le CM_{\phi,\eta}(f)(x).\end{aligned} \tag{54}$$

Thus $II\le CM_{\phi,\eta}(f)(x)$.

When $r>1$, by Hölder's inequality with $0<\epsilon<1$, we have

$$\begin{aligned}&\left(\frac{1}{[\phi(|Q|)]^\eta|Q|}\int_Q|Tf(x)|^\epsilon dx\right)^{1/\epsilon}\\ &\le\frac{1}{[\phi(|Q|)]^\eta|Q|}\int_Q|Tf(x)|\,dx\\ &\le\frac{1}{[\phi(|Q|)]^\eta|Q|}\int_Q|Tf_1(x)|\,dx\\ &\quad+\frac{1}{|Q|}\int_Q|Tf_2(x)|\,dx:=III+IV.\end{aligned} \tag{55}$$

Similar to I, it follows that $III\le CM_{\phi,\eta}(f)(x)$ with Kolmogorov's inequality to the term I and the weak (1,1) boundedness of T. For IV, taking $N=\eta+1$ in the size condition (3), we have

$$\begin{aligned}IV &\le C\int_{\mathbb{R}^n\setminus 4Q}\frac{1}{(|x-y|+|x_0-y|)^n}\\ &\quad\cdot\frac{|f(y)|}{\left[\phi\left((|x-y|+|x_0-y|)^n\right)\right]^{\eta+1}}dy\\ &\le C\sum_{k=1}^\infty\frac{1}{|4^{k+1}Q|}\frac{4^{-nk}}{[\phi(|4^{k+1}Q|)]^\eta}\int_{4^{k+1}Q\setminus 4^kQ}|f(y)|\,dy\\ &\le CM_{\phi,\eta}(f)(x).\end{aligned} \tag{56}$$

The proof is complete. □

Proof of Theorem 6. From Proposition 17 and Lemmas 20 and 26, the standard statement gets the proof of Theorem 6. □

To prove Theorem 10, we establish the following pointwise estimate firstly.

Lemma 27. *Assume that T satisfies (3), (4), (5), and (6). Let $b \in BMO^{\theta}(\phi)$ for $\theta > 0$, $0 < \epsilon < \gamma < 1$, and $\eta > \theta\epsilon/(1-\epsilon)$. Then for almost all $x \in \mathbb{R}^n$*

$$M^{\sharp,\triangle}_{\phi,\epsilon,\eta}([b,T]f)(x) \le C\|b\|_{BMO^{\theta}(\phi)} \cdot \left[M^{\triangle}_{\phi,\gamma,\eta}(Tf)(x) + M_{L\log L,\phi,\eta}(f)(x)\right], \tag{57}$$

where $M^{\triangle}_{\phi,\gamma,\eta}$ is a variant of dyadic maximal operator

$$M^{\triangle}_{\phi,\gamma,\eta}(f) = M^{\triangle}_{\phi,\eta}\left(|f|^{\gamma}\right)^{1/\gamma}. \tag{58}$$

Proof. Fix $x \in \mathbb{R}^n$ and a dyadic cube $Q = Q(x_0, r) \ni x$, we decompose the function f into $f = f_1 + f_2$ with $f_1 = f\chi_{\widetilde{Q}}$, where $\widetilde{Q} = 4Q$. For $b \in BMO^{\theta}(\phi)$, we can write

$$[b,T](f)(x) = \left(b - b_{\widetilde{Q}}\right)T(f)(x) - T\left(\left(b - b_{\widetilde{Q}}\right)f\right)(x). \tag{59}$$

When $0 < r < 1$, Since $0 < \epsilon < 1$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|\left|[b,T]f(x)\right|^{\epsilon} - |c|^{\epsilon}\right| dx\right)^{1/\epsilon} \\
&\le \left(\frac{1}{|Q|}\int_Q \left|[b,T]f(x) - c\right|^{\epsilon} dx\right)^{1/\epsilon} \\
&\le \left(\frac{1}{|Q|}\int_Q \left|\left(b(x) - b_{\widetilde{Q}}\right)Tf(x)\right|^{\epsilon} dx\right)^{1/\epsilon} \\
&\quad + \left(\frac{1}{|Q|}\int_Q \left|T\left(\left(b - b_{\widetilde{Q}}\right)f_1\right)(x)\right|^{\epsilon} dx\right)^{1/\epsilon} \\
&\quad + \left(\frac{1}{|Q|}\int_Q \left|T\left(\left(b - b_{\widetilde{Q}}\right)f_2\right)(x) - c\right|^{\epsilon} dx\right)^{1/\epsilon} \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{60}$$

Choosing any $q \in (1, \gamma/\epsilon)$, by the Hölder inequality and Proposition 24, it estimates that

$$\begin{aligned}
I_1 &\le \left(\frac{1}{|Q|}\int_Q \left|\left(b(x) - b_{\widetilde{Q}}\right)\right|^{q'\epsilon} dx\right)^{1/q'\epsilon} \cdot \left(\frac{1}{|Q|}\int_Q |Tf(x)|^{q\epsilon} dx\right)^{1/q\epsilon} \\
&\le \left(\frac{1}{|Q|}\int_Q \left|\left(b(x) - b_{\widetilde{Q}}\right)\right|^{q'\epsilon} dx\right)^{1/q'\epsilon} \cdot \left(\frac{1}{[\phi(|Q|)]^{\eta}|Q|}\int_Q |Tf(x)|^{\gamma} dx\right)^{1/\gamma} \\
&\le C\|b\|_{BMO^{\theta}(\phi)} M_{\gamma,\phi,\eta}(Tf)(x).
\end{aligned} \tag{61}$$

For I_2, we recall that T is a Calderón-Zygmund operator and is of weak type $(1,1)$. By Kolmogorov's inequality, (49), and Proposition 24, we then have

$$\begin{aligned}
I_2 &\le C\left\|T\left(\left(b - b_{\widetilde{Q}}\right)f_1\right)\right\|_{L^{1,\infty}(Q,dx/|Q|)} \\
&\le C\frac{1}{\left|\widetilde{Q}\right|}\int_{\widetilde{Q}} \left|\left(b(x) - b_{\widetilde{Q}}\right)f(x)\right| dx \\
&\le C\|b\|_{BMO^{\theta}(\phi)}\|f\|_{L\log L,Q}[\phi(|Q|)]^{\theta} \\
&\le C\|b\|_{BMO^{\theta}(\phi)} M_{L\log L,\phi,\eta}f(x),
\end{aligned} \tag{62}$$

and the last inequality holds by $1 \le \phi(|Q|) \le \phi(1)$. It turns to the term I_3. Choosing $c = T((b - b_{\widetilde{Q}})f_2)(x_0)$ and using (4), (5), and (49) and Proposition 24, we get that for any $x \in Q$ and $N = \eta + \theta$

$$\begin{aligned}
&\left|T\left(\left(b - b_{\widetilde{Q}}\right)f_2\right)(x) - T\left(\left(b - b_{\widetilde{Q}}\right)f_2\right)(x)\right| = \int_{\mathbb{R}^n\setminus\widetilde{Q}} |K(x,y) - K(x_0,y)|\left|b(y) - b_{\widetilde{Q}}\right||f(y)| dy \\
&\le \sum_{k=1}^{\infty}\int_{\mathbb{R}^n\setminus\widetilde{Q}} \frac{|x - x_0|^{\delta}}{(|x-y| + |x_0 - y|)^{n+\delta}} \frac{|f(y)|}{\left[\phi\left((|x-y| + |x_0-y|)^n\right)\right]^N}\left|b(y) - b_{\widetilde{Q}}\right| dy \\
&\le \sum_{k=1}^{\infty}\frac{4^{-\delta/n}}{\left|4^{k+1}Q\right|\left[\phi\left(\left|4^{k+1}Q\right|\right)\right]^N}\int_{4^{k+1}Q\setminus 4^kQ}\left|b(y) - b_{\widetilde{Q}}\right||f(y)| dy \le \sum_{k=1}^{\infty}\frac{4^{-\delta/n}}{\left[\phi\left(\left|4^{k+1}Q\right|\right)\right]^N}\|b\|_{\exp L,4^{k+1}Q}\|f\|_{L\log L,4^{k+1}Q} \\
&\le \sum_{k=1}^{\infty}\frac{k4^{-\delta/n}}{\left[\phi\left(\left|4^{k+1}Q\right|\right)\right]^N}\left[\phi\left(\left|4^{k+1}Q\right|\right)\right]^{\phi}\|b\|_{BMO^{\theta}(\phi)}\|f\|_{L\log L,4^{k+1}Q} \le C\|b\|_{BMO^{\theta}(\phi)} M_{L\log L,\phi,\eta}f(x).
\end{aligned} \tag{63}$$

Thus $I_3 \le C\|b\|_{BMO^{\theta}(\phi)} M_{L\log L,\phi,\eta}f(x)$.

When $r > 1$, since $0 < \epsilon < \gamma < 1$, we have

$$\begin{aligned}
&\left(\frac{1}{[\phi(|Q|)]^{\eta}|Q|}\int_Q |[b,T]f(x)|^{\epsilon}dx\right)^{1/\epsilon} \\
&\quad\le \frac{1}{[\phi(|Q|)]^{\eta/\epsilon}}\left(\frac{1}{|Q|}\right. \\
&\quad\cdot\left.\int_Q \left|\left(b(x)-b_{\widetilde{Q}}\right)Tf(x)\right|^{\epsilon}dx\right)^{1/\epsilon} \\
&\quad+\frac{1}{[\phi(|Q|)]^{\eta/\epsilon}}\left(\frac{1}{|Q|}\right. \\
&\quad\cdot\left.\int_Q \left|T\left(\left(b-b_{\widetilde{Q}}\right)f_1\right)(x)\right|^{\epsilon}dx\right)^{1/\epsilon} \\
&\quad+\frac{1}{[\phi(|Q|)]^{\eta/\epsilon}}\left(\frac{1}{|Q|}\right. \\
&\quad\cdot\left.\int_Q \left|T\left(\left(b-b_{\widetilde{Q}}\right)f_2\right)(x)\right|^{\epsilon}dx\right)^{1/\epsilon} := II_1 + II_2 \\
&\quad+ II_3.
\end{aligned} \tag{64}$$

Choosing any $q \in (1, \gamma/\epsilon)$ and denoting $\eta > \theta\epsilon/(1-\epsilon) > \theta\gamma\epsilon/(\gamma-\epsilon)$, by the Hölder inequality and Proposition 24, the same estimate with I_1 is obtained that

$$\begin{aligned}
II_1 &\le C\frac{1}{[\phi(|Q|)]^{\eta/\epsilon}}\left(\frac{1}{|Q|}\int_Q \left|\left(b(x)-b_{\widetilde{Q}}\right)\right|^{q'\epsilon}dx\right)^{1/q'\epsilon} \\
&\quad\cdot\left(\frac{1}{|Q|}\int_Q |Tf(x)|^{q\epsilon}dx\right)^{1/q\epsilon} \\
&\le C[\phi(|Q|)]^{\theta-\eta/\epsilon+\eta/q\epsilon}\|b\|_{BMO^{\theta}(\phi)}M_{\gamma,\phi,\eta}(Tf)(x) \\
&\le C\|b\|_{BMO^{\theta}(\phi)}M_{\gamma,\phi,\eta}(Tf)(x).
\end{aligned} \tag{65}$$

Also, since $\eta > \theta\epsilon/(1-\epsilon)$, by Kolmogorov's inequality, (49), and Proposition 24, we have

$$\begin{aligned}
II_2 &\le C\frac{1}{[\phi(|Q|)]^{\eta/\epsilon}}\left\|T\left(\left(b-b_{\widetilde{Q}}\right)f_1\right)\right\|_{L^{1,\infty}(Q,dx/|Q|)} \\
&\le C\|b\|_{BMO^{\theta}(\phi)}\|f\|_{L\log L,Q}[\phi(|Q|)]^{\theta-\eta/\epsilon} \\
&\le C\|b\|_{BMO^{\theta}(\phi)}M_{L\log L,\phi,\eta}f(x).
\end{aligned} \tag{66}$$

The last term II_3 remains. Choosing $c = T((b-b_{\widetilde{Q}})f_2)(x_0)$ and using (3) and (49) and Proposition 24, we get that for any $x \in Q$ and $N = \theta + \eta + 2$

$$\begin{aligned}
II_3 &\le \frac{1}{[\phi(|Q|)]^{\eta}|Q|}\int_Q \left|T\left(\left(b-b_{\widetilde{Q}}\right)f\right)(y)\right|dy \le C\frac{1}{|Q|} \\
&\quad\cdot\int_Q\int_{\mathbb{R}^n\setminus 4Q}|K(x,y)|\left|\left(b(y)-b_{\widetilde{Q}}\right)f(y)\right|dy\,dx \le C\frac{1}{|Q|} \\
&\quad\cdot\int_Q\int_{\mathbb{R}^n\setminus 4Q}\frac{1}{|x-y|^n\left[\phi\left(|x-y|^n\right)\right]^{\theta+\eta+2}}\left|\left(b(y)-b_{\widetilde{Q}}\right)\right. \\
&\quad\cdot\left.f(y)\right|dy\,dx \\
&\le C\sum_{k=1}^{\infty}\frac{1}{\left|4^{k+1}Q\right|\left[\phi\left(\left|4^{k+1}Q\right|\right)\right]^{\theta+\eta+1}}\int_{4^{k+1}Q}\left|\left(b(y)-b_{\widetilde{Q}}\right)\right. \\
&\quad\cdot\left.f(y)\right|dy \le C\sum_{k=1}^{\infty}\frac{k\left[\phi\left(\left|4^{k+1}Q\right|\right)\right]^{\theta}}{\left[\phi\left(\left|4^{k+1}Q\right|\right)\right]^{\theta+\eta+2}}\|b\|_{BMO^{\theta}(\phi)} \\
&\quad\cdot\|f\|_{L\log L,4^{k+1}Q} \le C\sum_{k=1}^{\infty}\frac{k}{\left[\phi\left(\left|4^{k+1}Q\right|\right)\right]^{2}}\|b\|_{BMO^{\theta}(\phi)} \\
&\quad\cdot M_{L\log L,\phi,\eta}f(x) \le C\|b\|_{BMO^{\theta}(\phi)}M_{L\log L,\phi,\eta}f(x).
\end{aligned} \tag{67}$$

We complete the proof. □

Proof of Theorem 10. By Corollary 22, Lemma 27, Lemma 23, and Theorem 6, we have

$$\begin{aligned}
&\|[b,T]f\|_{L^p(\omega)} \le C\|b\|_{BMO^{\theta}(\phi)} \\
&\quad\cdot\left(\left\|M^{\triangle}_{\phi,\gamma,\eta}(Tf)\right\|_{L^p(\omega)} + \left\|M_{L\log L,\phi,\eta}(f)\right\|_{L^p(\omega)}\right) \\
&\quad\le C\|b\|_{BMO^{\theta}(\phi)} \\
&\quad\cdot\left(\|Tf\|_{L^p(\omega)} + \left\|M_{\phi,\eta/2}M_{\phi,\eta/2}f\right\|_{L^p(\omega)}\right) \\
&\quad\le C\|b\|_{BMO^{\theta}(\phi)}\|f\|_{L^p(\omega)}.
\end{aligned} \tag{68}$$

□

Lemma 28. *Let $\omega \in A_1^{\theta}(\phi)$ and $\eta \ge 2\theta$. There exists a positive C such that, for any function f and $\lambda > 0$,*

$$\begin{aligned}
&\omega\left(\left\{x \in \mathbb{R}^n : M_{L\log L,\phi,\eta}f(x) > \lambda\right\}\right) \\
&\quad\le C\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)\omega(x)\,dx.
\end{aligned} \tag{69}$$

Proof. Let F be any compact subset in $\{x \in \mathbb{R}^n : M_{L\log L,\phi,\eta}f(x) > \lambda\}$. For $x \in F$, there exists a cube Q with $x \in Q$ such that

$$\|f\|_{\Phi,\phi,\eta,Q} = [\phi(|Q|)]^{-\eta}\|f\|_{\Phi,Q} > \lambda. \tag{70}$$

by Vitali covering lemma, we can choose a finite family of disjoint cubes $\{Q_j\}$ such that $F \subset \cup_j 3Q_j$ and $\{Q_j\}$ satisfies (70). Thus

$$\begin{aligned}
[\phi(|Q|)]^{2\theta}\left|Q_j\right| &\le [\phi(|Q|)]^{\eta}\left|Q_j\right| \\
&\le \int_{Q_j}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx.
\end{aligned} \tag{71}$$

From this, by (iv) in Proposition 15 with $p = 1$ and $E = Q$, we obtain that

$$\begin{aligned}\omega\left(3Q_j\right) &\le C\phi\left(\left|Q_j\right|\right)\omega\left(Q_j\right)\\ &\le C\frac{\omega\left(Q_j\right)}{\phi\left(\left|Q_j\right|\right)\left|Q_j\right|}\int_{Q_j}\Phi\left(\frac{\left|f(x)\right|}{\lambda}\right)dx\\ &\le C\inf_{x\in Q_j}\omega(x)\int_{Q_j}\Phi\left(\frac{\left|f(x)\right|}{\lambda}\right)dx\\ &\le C\int_{Q_j}\Phi\left(\frac{\left|f(x)\right|}{\lambda}\right)\omega(x)\,dx.\end{aligned} \tag{72}$$

Thus, (69) holds. □

Lemma 29. *Let $b \in BMO^{\theta}(\phi)$, $\eta > 2\theta$, and $\omega \in A_1^{\theta}(\phi)$. Then there exists a positive constant C such that for any smooth function f with compact support*

$$\begin{aligned}&\sup_{t>0}\frac{1}{\Phi(t)}\omega\left(\left\{x\in\mathbb{R}^n : \left|[b,T](f)(x)\right| > \lambda\right\}\right)\\ &\quad\le C\Phi\left(\|b\|_{BMO^{\theta}(\phi)}\right)\sup_{t>0}\frac{1}{\Phi(t)}\\ &\quad\cdot\omega\left(\left\{x\in\mathbb{R}^n : \left|M_{L\log L,\phi,\eta}(f)(x)\right| > \lambda\right\}\right).\end{aligned} \tag{73}$$

Proof. Similar to the proof of Lemma 1.5 in [12] or Lemma 4.4 in [1], we can get the desired result by Proposition 16, Lemma 27, and Lemma 23. We omit the details here. □

Proof of Theorem 11. By homogeneity, we need only to show that

$$\begin{aligned}&\omega\left(\left\{x\in\mathbb{R}^n : \left|[b,T](f)\right| > 1\right\}\right)\\ &\quad\le C\int_{\mathbb{R}^n}\Phi\left(\left|f(x)\right|\right)\omega(x)\,dx.\end{aligned} \tag{74}$$

Now, since Φ is submultiplicative, we have by Lemma 28 and Lemma 29

$$\begin{aligned}&\omega\left(\left\{x\in\mathbb{R}^n : \left|[b,T](f)\right| > 1\right\}\right) \le C\sup_{t>0}\frac{1}{\Phi(1/t)}\omega\\ &\quad\cdot\left(\left\{x\in\mathbb{R}^n : \left|[b,T](f)(x)\right| > t\right\}\right)\\ &\quad\le C\Phi\left(\|b\|_{BMO^{\theta}(\phi)}\right)\sup_{t>0}\frac{1}{\Phi(1/t)}\\ &\quad\cdot\omega\left(\left\{x\in\mathbb{R}^n : \left|M_{L\log L,\phi,\eta}(f)(x)\right| > t\right\}\right)\\ &\quad\le C\int_{\mathbb{R}^n}\Phi\left(\left|f(x)\right|\right)\omega(x)\,dx.\end{aligned} \tag{75}$$

□

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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