Research Article

Boundedness and Continuity of Several Integral Operators with Rough Kernels in $W\mathcal{F}_\beta(S^{n-1})$ on Triebel-Lizorkin Spaces

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A systematic treatment is given of singular integrals and Marcinkiewicz integrals associated with surfaces generated by polynomial compound mappings as well as related maximal functions with rough kernels in $W\mathcal{F}_\beta(S^{n-1})$, which relates to the Grafakos-Stefanov function class. Certain boundedness and continuity for these operators on Triebel-Lizorkin spaces and Besov spaces are proved by applying some criterions of bounds and continuity for several operators on the above function spaces.

1. Introduction

Let $\mathbb{R}^n$ ($n \geq 2$) be the $n$-dimensional Euclidean space and $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Assume that $\Omega \in L^1(S^{n-1})$ is a homogeneous function of degree zero and satisfies

$$\int_{S^{n-1}} \Omega (u) d\sigma (u) = 0. \quad (1)$$

For a suitable function $h$ defined on $\mathbb{R}^n$, $\mathbb{C} = (\zeta, \tau \in \mathbb{R}$ with $\zeta > 0)$, and a suitable mapping $\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we consider the singular integral operators $T_{h,\Omega,\Gamma}$ and parametric Marcinkiewicz integral operators $M_{h,\Omega,\Gamma,\rho}$ in $\mathbb{R}^n$ by

$$T_{h,\Omega,\Gamma} f (x) = \text{p.v.} \int_{\mathbb{R}^n} f \left( x - \Gamma (y) \right) \frac{h (\| y \|)}{\| y \|^n} \, dy; \quad (2)$$

$$M_{h,\Omega,\Gamma,\rho} f (x) = \left( \int_0^{\infty} \left| \frac{1}{t^\rho} \right| \, dt \right) \cdot \left( \int_{\| y \| \leq t} f \left( x - \Gamma (y) \right) \frac{h (\| y \|)}{\| y \|^{n-\rho}} \, dy \right)^{1/2} \quad (3)$$

Define the related maximal operators $\delta_{\Omega,\Gamma}$ and $M_{h,\Omega,\Gamma,\rho}$ by

$$\delta_{\Omega,\Gamma} f (x) = \sup_{h \in \mathcal{H}_1} \left| T_{h,\Omega,\Gamma} f (x) \right|, \quad (4)$$

$$M_{h,\Omega,\Gamma,\rho} f (x) = \sup_{h \in \mathcal{H}_1} \left| M_{h,\Omega,\Gamma,\rho} f (x) \right|, \quad (5)$$

where $\mathcal{H}_1$ is the set of all measurable functions $h : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\| h \|_{L^1(\mathbb{R}^n, d\sigma)} \leq 1$.

The primary purpose of this paper is to study the bounds and continuity of the singular integral operators and Marcinkiewicz integral operators associated with surfaces generated by polynomial compound mappings as well as related maximal functions with rough kernels in $W\mathcal{F}_\beta(S^{n-1})$ on the Triebel-Lizorkin spaces and Besov spaces. Before stating our main results, let us recall some pertinent definitions, notations, and backgrounds.

Definition 1 (function class $W\mathcal{F}_\beta(S^{n-1})$). For $\beta > 0$, the function class $W\mathcal{F}_\beta(S^{n-1})$ is the set of all $L^1(S^{n-1})$ functions $\Omega$ which satisfy

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1} \times S^{n-1}} \left| \Omega (\theta) \Omega (\theta') \right|^\beta \left( \frac{2e}{\| \theta - \theta' \| \cdot \xi} \right) d\sigma (\theta) d\sigma (\theta') < \infty. \quad (6)$$
The function class $W\mathcal{F}_\beta(S^{n-1})$ was originally introduced by Fan and Sato [1]. It is closely related to the Grafakos-Stefanov function class $\mathcal{F}_\beta(S^{n-1})$, which was first introduced in [2] and is given by

$$\mathcal{F}_\beta(S^{n-1}) = \left\{ \Omega \in L^1(S^{n-1}) : \sup_{\varepsilon \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \left[ \log \frac{2}{|\varepsilon \cdot y'|} \right] d\sigma(y') < \infty, \beta > 0 \right\}.$$  

It was shown in [1, 3] that

$$W\mathcal{F}_\beta(S^1), W\mathcal{F}_{2\beta}(S^{n-1}) \setminus \mathcal{F}_\beta(S^{n-1}) \neq \emptyset,$$

$$\bigcup_{r \geq 1} L^r(S^{n-1}) \subset \mathcal{F}_{\beta_1}(S^{n-1}) \subset \mathcal{F}_\beta(S^{n-1}),$$

$$0 < \beta_1 < \beta_2 < \infty;$$

$$\bigcup_{r \geq 1} L^r(S^{n-1}) \subset W\mathcal{F}_{2\beta}(S^{n-1}) \subset W\mathcal{F}_{\beta_1}(S^{n-1}),$$

$$0 < \beta_1 < \beta_2 < \infty.$$  

To introduce some known results, we need to recall one more function space $\Delta_\gamma(R_+)$.  

**Definition 2** (function class $\Delta_\gamma(R_+)$). For $1 \leq \gamma \leq \infty$, the function class $\Delta_\gamma(R_+)$ is the set of all measurable functions $h : R_+ \rightarrow R$ satisfying

$$\|h\|_{\Delta_\gamma(R_+)} = \sup_{r > 0} \left( R^1 \int_0^R |h(t)|^r \, dt \right)^{1/r} < \infty.$$  

It is clear that $\Delta_{\gamma_1}(R_+) \subseteq \Delta_{\gamma_2}(R_+)$ for $1 \leq \gamma_1 < \gamma_2 \leq \infty$ and $\Delta_\infty(R_+) = L_{\infty}(R_+)$.  

When $\Gamma(y) = y$, the operators defined in (2) reduce to the classical Calderón-Zygmund operator

$$T_{h,\Omega} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) \frac{h(|y|) \Omega(y)}{|y|^n} \, dy,$$

which was originally studied by Calderón and Zygmund [4] and later investigated by many authors (see [1, 2, 5, 6], etc.). In 2009, Fan and Sato [1] first studied the $L^p$ bounds for $T_{h,\Omega}$ with $\Omega$ which belongs to $W\mathcal{F}_\beta(S^{n-1})$. More precisely, the above authors established the $L^p$ bounds for $T_{h,\Omega}$ with $|1/p - 1/2| < 1/\max(2, \gamma') - 1/\beta$ if $h \in \Delta_\gamma(R_+)$ for $\gamma > 1$ and $\Omega \in W\mathcal{F}_\beta(S^{n-1})$ for some $\beta > \max(\gamma', 2)$. Recently, Liu and Wu [7] extended the result of [1] to the singular integrals along polynomial compound curves in the mixed homogeneity setting.

Let us recall the definitions of Triebel-Lizorkin spaces and Besov spaces.  

**Definition 3** (Triebel-Lizorkin spaces and Besov spaces). Let $\delta'(\mathbb{R}^n)$ be the tempered distribution class on $\mathbb{R}^n$. For $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty (p \neq \infty)$, the homogeneous Triebel-Lizorkin spaces $\mathring{F}^p_q(\mathbb{R}^n)$ and Besov spaces $B^p_q(\mathbb{R}^n)$ are defined by

$$\mathring{F}^p_q(\mathbb{R}^n) = \left\{ f \in \delta'(\mathbb{R}^n) : \|f\|_{\mathring{F}^p_q(\mathbb{R}^n)} < \infty \right\},$$

$$B^p_q(\mathbb{R}^n) = \left\{ f \in \delta'(\mathbb{R}^n) : \|f\|_{B^p_q(\mathbb{R}^n)} < \infty \right\},$$

where $\mathring{F}^p_q(\mathbb{R}^n)$ and $B^p_q(\mathbb{R}^n)$ are obtained by adding the term $\|\hat{\Theta} \ast f\|_{L^p(\mathbb{R}^n)}$ to the right hand side of (11) or (12) with $\sum_{i \geq 1} Z$ replaced by $\sum_{i \geq 1}$, where $\Theta \in \delta'(\mathbb{R}^n)$ (the Schwartz class), $\supp(\Theta) \subset \{ \xi : |\xi| \leq 2 \}$, $\Theta(x) > c > 0$ if $|x| \leq 5/3$.

The following properties are well-known (see [8, 9] for more details):

$$\mathring{F}^p_q(\mathbb{R}^n) = L^p(\mathbb{R}^n), \quad 1 < p < \infty;$$

$$\mathring{F}^p_q(\mathbb{R}^n) = B^p_q(\mathbb{R}^n), \quad \alpha \in \mathbb{R}, \quad 1 < p < \infty;$$

$$\|f\|_{\mathring{F}^p_q(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}, \quad \alpha > 0;$$

$$\|f\|_{B^p_q(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)}, \quad \alpha > 0.$$  

Recently, the investigation of the bounds for singular integrals with rough kernels in $W\mathcal{F}_\beta(S^{n-1})$ on Triebel-Lizorkin spaces and Besov spaces has received some attention of many authors (see [3, 10, 11]). Particularly, Liu et al. [10] obtained the following result.

**Theorem A** (see [10]). Let $\Gamma(y) = P_N(|y|) y'$, where $P_N$ is a real-valued polynomial with $P_N(0) = 0$ and $\deg(P_N) = N$. 
Suppose that \( h \in \Delta_{\gamma}(\mathbb{R}_+) \) for some \( \gamma > 1 \) and \( \Omega \in W\mathcal{F}_\beta(S^{-1}) \) for some \( \beta > \max(2, \gamma') \) satisfying (1). Then

(i) \( T_{h,\Omega,\Gamma} \) is bounded on \( L^{q}_{\gamma}(\mathbb{R}^n) \) for \( \alpha \in \mathbb{R} \) and \( \max\{|1/p - 1/2|, |1/q - 1/2|\} < 1/\max(2, \gamma') - 1/\beta; \)

(ii) \( T_{h,\Omega,\Gamma} \) is bounded on \( B^{pq}_{\alpha}(\mathbb{R}^n) \) for \( \alpha \in \mathbb{R}, q \in (1, \infty), \) and \( |1/p - 1/2| < 1/\max(2, \gamma') - 1/\beta. \)

It should be pointed out that there is a gap in the proof of part (i) in Theorem A. To the best of my knowledge, it is unknown whether Theorem A(i) holds. However, we can obtain the following result.

**Theorem 4.** Let \( \Gamma(\gamma) = (P_i(|y|)y_1', \ldots, P_n(|y|)y_n') \) with each \( P_i \) being a real-valued polynomial on \( \mathbb{R} \) satisfying \( P_i(0) = 0. \) Suppose that \( h \in \Delta_{\gamma}(\mathbb{R}_+) \) for some \( \gamma > 1 \) and \( \Omega \in W\mathcal{F}_\beta(S^{-1}) \) for some \( \beta > \max(2, \gamma') \) satisfying (1).

(i) Then, for \( \alpha \in \mathbb{R} \) and \( (1/p, 1/q) \in \mathcal{R}_{\gamma,\beta} \), there exists a constant \( C > 0 \) such that

\[
\|T_{h,\Omega,\Gamma}f\|_{L^{q}_{\gamma}(\mathbb{R}^n)} \leq C \|f\|_{L^{q}_{\gamma}(\mathbb{R}^n)}
\]

(17)

for all \( f \in L^{p}_{\gamma}(\mathbb{R}^n) \), where \( C = C_{\nu, p, q, \gamma, \beta, \delta} \) is independent of the coefficients of \( \{P_i\}_{i=1}^n \). Let \( \mathcal{R}_{\gamma,\beta} \) be the set of all integers \( \gamma \in \mathbb{R}_+ \) such that \( \Omega \in W\mathcal{F}_\beta(\mathbb{R}^n) \) for some \( \beta > \max(2, \gamma') \) satisfying (1).

(ii) Then, for \( \alpha \in \mathbb{R} \) and \( (1/p, 1/q) \in \mathcal{R}_{\gamma,\beta} \), there exists a constant \( C > 0 \) such that

\[
\|T_{h,\Omega,\Gamma}f\|_{B^{pq}_{\alpha}(\mathbb{R}^n)} \leq C \|f\|_{B^{pq}_{\alpha}(\mathbb{R}^n)}
\]

(18)

for all \( f \in B^{pq}_{\alpha}(\mathbb{R}^n) \), where \( C = C_{\nu, p, q, \gamma, \beta, \delta} \) is independent of the coefficients of \( \{P_i\}_{i=1}^n \). Applying a switched method following from [12], Theo-

rem 4 yields the following more general result.

**Theorem 5.** Let \( \Gamma(\gamma) = (P_i(|y|)y_1', \ldots, P_n(|y|)y_n') \) with each \( P_i \) being a real-valued polynomial on \( \mathbb{R} \) with \( P_i(0) = 0 \) and \( \phi' \in \mathcal{F}_\beta(\mathbb{R}^n) \). Here \( \mathcal{F}_\beta \) is the set of all nonnegative (or non-
popitive) and monotonic \( \mathcal{C}^{1}(\mathbb{R}_+) \) functions \( \phi \) satisfying \( \Gamma(\gamma) := \phi(t)/\phi'(t) \) for \( |\phi(t)| \leq |\phi'(t)| \) \( \gamma \in \mathbb{R}_+ \), where \( C > 0 \) depends only on \( \gamma \). Suppose that \( h \in \Delta_{\gamma}(\mathbb{R}_+) \) for some \( \gamma > 1 \) and \( \Omega \in W\mathcal{F}_\beta(S^{-1}) \) for some \( \beta > \max(2, \gamma') \) satisfying (1). Let \( \mathcal{R}_{\gamma,\beta} \) be given as in Theorem 4.

(i) Then, for \( \alpha \in \mathbb{R} \) and \( (1/p, 1/q) \in \mathcal{R}_{\gamma,\beta} \), there exists a constant \( C > 0 \) such that

\[
\|T_{h,\Omega,\Gamma}f\|_{L^{q}_{\gamma}(\mathbb{R}^n)} \leq C \|f\|_{L^{q}_{\gamma}(\mathbb{R}^n)}
\]

(19)

for all \( f \in L^{p}_{\gamma}(\mathbb{R}^n) \), where \( C = C_{\nu, p, q, \gamma, \beta, \delta} \) is independent of the coefficients of \( \{P_i\}_{i=1}^n \).

(ii) Then, for \( \alpha \in \mathbb{R}, q \in (1, \infty), \) and \( |1/p - 1/2| < 1/\max(2, \gamma') - 1/\beta \), there exists a constant \( C > 0 \) such that

\[
\|T_{h,\Omega,\Gamma}f\|_{B^{pq}_{\alpha}(\mathbb{R}^n)} \leq C \|f\|_{B^{pq}_{\alpha}(\mathbb{R}^n)}
\]

(20)

for all \( f \in B^{pq}_{\alpha}(\mathbb{R}^n) \), where \( C = C_{\nu, p, q, \gamma, \beta, \delta} \) is independent of the coefficients of \( \{P_i\}_{i=1}^n \).
Theorem 11. Let \( \Omega \in W \mathcal{F}_\beta(S^{n-1}) \) for some \( \beta > \max\{2, \gamma\} \) satisfying (I). Let \( \mathcal{R}_{\gamma, \beta} \) be given as in Theorem 4.

(i) Then for \( \alpha \in (0,1) \) and \( (1/p, 1/q) \in \mathcal{R}_{\alpha, \beta} \), there exists a constant \( C > 0 \) such that
\[
\|\mathcal{M}_{\alpha, \Omega, \Gamma, \rho} f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)}
\]
for all \( f \in L^{p, q}(\mathbb{R}^n) \), where \( C = C_{n, \alpha, p, q, \gamma, \beta} \) is independent of the coefficients of \( \{P_j\}_{j=1}^n \).

(ii) \( \mathcal{M}_{h, \Omega, \Gamma, \rho} \) is continuous from \( L^{p, q}(\mathbb{R}^n) \) to \( L^{p, q}(\mathbb{R}^n) \) for \( \alpha \in (0,1) \) and \( (1/p, 1/q) \in \mathcal{R}_{\alpha, \beta} \).

(iii) Then, for \( \alpha \in (0,1), q \in (1, \infty) \), and \( (1/p, 1/q) \in \mathcal{R}_{\alpha, \beta} \), there exists a constant \( C > 0 \) such that
\[
\|\mathcal{M}_{h, \Omega, \Gamma, \rho} f\|_{L^{p, q}(\mathbb{R}^n)} \leq C \|f\|_{L^{p, q}(\mathbb{R}^n)}
\]
for all \( f \in L^{p, q}(\mathbb{R}^n) \), where \( C = C_{n, \alpha, p, q, \gamma, \beta} \) is independent of the coefficients of \( \{P_j\}_{j=1}^n \).

(iv) \( \mathcal{M}_{h, \Omega, \Gamma, \rho} \) is continuous from \( B^{p, q}_\alpha(\mathbb{R}^n) \) to \( B^{p, q}_\alpha(\mathbb{R}^n) \) for \( \alpha \in (0,1), q \in (1, \infty) \), and \( (1/p, 1/q) \in \mathcal{R}_{\alpha, \beta} \).

Remark 10. Parts (i) and (iii) in Theorem 8 extend Theorem B, which corresponds to the case \( P_j(\gamma) = P_j(\gamma) \) \( = \cdots = P_j(\gamma) = t \). Comparing with the singular integral operators, the continuity of the singular integral operators on the Triebel-Lizorkin spaces and Besov spaces can be obtained automatically by the corresponding boundedness since the singular integral operators are linear. However, the continuity of the Marcinkiewicz integral operators on the above function spaces is nontrivial. The reason for this is twofold. First, the Marcinkiewicz integral operators are not linear. Second, \( f \leq g \) cannot imply \( \|f\|_{L^{p, q}(\mathbb{R}^n)} \leq \|g\|_{L^{p, q}(\mathbb{R}^n)} \) and \( \|f\|_{B^{p, q}_\alpha(\mathbb{R}^n)} \leq \|g\|_{B^{p, q}_\alpha(\mathbb{R}^n)} \).

Remark 11. Under the same conditions of Theorems 4, 5, 8, and 11, Remarks 6, 9, 10, and 12, we can obtain the following result immediately.

Theorem 13. Let \( n = 2 \) and \( \Gamma(y) = (P_1(\varphi(y))) \gamma_1, \ldots, P_n(\varphi(y)) \gamma_n \) with each \( P_j \) being a real-valued polynomial on \( \mathbb{R} \) satisfying \( P_j(0) = 0 \) and \( \varphi \in \mathcal{F}_\beta \).

(i) Then, for \( \alpha \in (0, 1) \) and \( (1/p, 1/q) \in \mathcal{G}_\beta \), there exists a constant \( C > 0 \) such that
\[
\|\tilde{\delta}_{\Omega, \Gamma} f\|_{L^{p, q}(\mathbb{R}^n)} + \|\tilde{\mathcal{M}}_{\Omega, \Gamma, \rho} f\|_{L^{p, q}(\mathbb{R}^n)} \leq C \|f\|_{L^{p, q}(\mathbb{R}^n)}
\]
for all \( f \in L^{p, q}(\mathbb{R}^n) \), where \( C = C_{n, \alpha, p, q, \gamma, \beta} \) is independent of the coefficients of \( \{P_j\}_{j=1}^n \). Here \( \mathcal{G}_\beta \) is the set of all interiors of the convex hull of two squares \((1/\beta, 1/2)^2 \) and \((1/2, 1-1/\beta)^2 \).

(ii) \( \delta_{\Omega, \Gamma} \) and \( \mathcal{M}_{\Omega, \Gamma, \rho} \) are continuous from \( F^{p, q}_\alpha(\mathbb{R}^n) \) to \( F^{p, q}_\alpha(\mathbb{R}^n) \) for \( \alpha \in (0,1), p \in [2, \beta], \) and \( (1/q, 1/p + 1/2 - 1/\beta) \).}

(iii) Then, for \( \alpha \in (0,1), p \in [2, \beta], \) and \( q \in (1, \infty) \), there exists a constant \( C > 0 \) such that
\[
\|\delta_{\Omega, \Gamma} f\|_{B^{p, q}_\alpha(\mathbb{R}^n)} + \|\mathcal{M}_{\Omega, \Gamma, \rho} f\|_{B^{p, q}_\alpha(\mathbb{R}^n)} \leq C \|f\|_{B^{p, q}_\alpha(\mathbb{R}^n)}
\]
for all \( f \in B^{p, q}_\alpha(\mathbb{R}^n) \), where \( C = C_{n, \alpha, p, q, \gamma, \beta} \) is independent of the coefficients of \( \{P_j\}_{j=1}^n \).

(iv) \( \delta_{\Omega, \Gamma} \) and \( \mathcal{M}_{\Omega, \Gamma, \rho} \) are continuous from \( B^{p, q}_\alpha(\mathbb{R}^n) \) to \( B^{p, q}_\alpha(\mathbb{R}^n) \) for \( \alpha \in (0,1), p \in [2, \beta], \) and \( q \in (1, \infty) \).
The paper is organized as follows. Section 2 contains some known results, which play key roles in the proofs of main results. In Section 3, we will present some criterions on the boundedness and continuity of several operators on Triebel-Lizorkin spaces and Besov spaces, which are the main ingredients of our proofs. The proofs of main results will be given in Section 4. We remark that the methods employed in this paper follow from a combination of ideas and arguments in [10, 12, 17, 18, 23, 24, 26, 27], among others. It should be also point out that our methods can be used to deal with other integral operators, such as singular integrals, Marcinkiewicz integrals, and related maximal functions associated with other surfaces with other rough kernels.

Throughout the paper, we denote \( p' \) by the conjugate index of \( p \), which satisfies \( 1/p + 1/p' = 1 \). The letter \( C \) or \( c \), sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence but are independent of the essential variables. In what follows, we set \( \mathfrak{R}_n = \{ \xi \in \mathbb{R}^n : 1/2 < |\xi| \leq 1 \} \), among others. Its should be given in Section 4. We remark that the methods employed in this paper follow from a combination of ideas and arguments in [10, 12, 17, 18, 23, 24, 26, 27], among others. It should be also point out that our methods can be used to deal with other integral operators, such as singular integrals, Marcinkiewicz integrals, and related maximal functions associated with other surfaces with other rough kernels.

2. Preliminary Lemmas

This section is devoted to recalling some known lemmas, which plays key roles in the proofs of main theorems. Let us begin with the following lemma of van der Corput type.

Lemma 15 (see [28]). Let \( \Phi(t) = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_n} \) and \( \Psi \in C^1([0,1]) \), where \( \alpha_1, \ldots, \alpha_n \) are real parameters, and \( \alpha_1, \ldots, \alpha_n \) are distinct positive (not necessarily integer) exponents. Then, for \( \lambda \neq 0 \), the following holds:

\[
\left| \int_a^b \exp (i \lambda \Phi(t)) \Psi(t) \, dt \right| \leq C |\lambda|^{-\epsilon} \left( \sup_{a \leq t \leq b} |\Psi(t)| + \int_a^b |\Psi'(t)| \, dt \right),
\]

where \( \epsilon = \min \{ 1/\alpha_1, 1/\alpha_n \} \) and \( C \) does not depend on \( \mu_2, \ldots, \mu_n \) as long as \( 0 < a < b \leq 1 \).

Applying Lemma 15 and the arguments similar to those used in the proof of [16, Lemma 2.2], we can obtain the following result.

Lemma 16. Let \( \Phi(t) = t^{\alpha_1} + t^{\alpha_2} + \cdots + t^{\alpha_n} \), where \( \alpha_1, \ldots, \alpha_n \) are real parameters, and \( \alpha_1, \ldots, \alpha_n \) are distinct positive (not necessarily integer) exponents. Suppose that \( \varphi \in \mathfrak{R} \) satisfying \( t^\delta \varphi'(t) \) is monotonic on \( \mathfrak{R}_+ \) for some \( \delta \in \mathbb{R} \). Then, for any \( r > 0 \) and \( \lambda \neq 0 \), the following holds:

\[
\left| \int_{r/2}^r \exp (i \lambda \Phi(\varphi(t))) \frac{dt}{t} \right| \leq C |\lambda \varphi(r)^{\alpha_n}|^{-\epsilon},
\]

with \( \epsilon = \min \{ 1/\alpha_1, 1/\alpha_n \} \), where \( C \) is independent of \( \mu_2, \ldots, \mu_n \), but may depend on \( \varphi, \varphi, \text{and} \delta \).

Proof. By the change of variables, we have

\[
\int_{r/2}^r \exp (i \lambda \Phi(\varphi(t))) \frac{dt}{t} = \int_{\varphi(r/2)}^{\varphi(r)} \exp (i \lambda \Phi(t)) \frac{dt}{\varphi'(\varphi^{-1}(t)) \varphi^{-1}(t)}
\]

\[
= \int_{\varphi(r/2)}^{\varphi(r)} \exp (i \lambda \Phi(t)) \left( \varphi^{-1}(t) \right)^{\delta-1} \frac{dt}{\varphi^{-1}(t) \varphi'(\varphi^{-1}(t))} = \varphi(r)
\]

\[
\cdot \int_{\varphi(r/2)}^{\varphi(r)} \exp (i \lambda \Phi(\varphi(r)t)) \left( \varphi^{-1}(\varphi(r)t) \right)^{\delta-1} \frac{dt}{\varphi^{-1}(\varphi(r)t) \varphi'(\varphi(r)t)}
\]

\[
= \varphi(r) \int_{\zeta}^{1} \exp (i \lambda \Phi(\varphi(r)t)) \varphi_{\varphi}(t) \psi(t) \, dt,
\]

where \( \zeta = \varphi(r/2)/\varphi(r) \), and \( \varphi_{\varphi}(t) = 1/\left( \varphi^{-1}(\varphi(r)t) \right)^{\delta} \varphi'(\varphi(r)t) \). and \( \psi(t) = \left( \varphi^{-1}(\varphi(r)t) \right)^{\delta-1} \).

We can also write

\[
\int_{r/2}^r \exp (i \lambda \Phi(\varphi(t))) \frac{dt}{t} = \varphi(r) \int_{\zeta}^{1} \psi(t) \, dJ(t),
\]

where

\[
J(t) = \int_{\zeta}^{t} \exp (i \lambda \Phi(\varphi(r)s)) \varphi_{\varphi}(s) \, ds, \quad \zeta \leq t \leq 1.
\]
\[ \cdot \left( |J(1)\psi(1)| + \int_{\mathbb{R}^n} |J(t)| \psi'(t) dt \right) \leq C \varphi(r) \]
\[ \cdot |\lambda \varphi(r)|^\eta \left( \left( r^\theta (t^\theta)^{-1} + \left( \frac{r}{2} \right)^\theta (t^\theta)^{-1} \right)^{-1} \right) \]
\[ \cdot \left( r^{\delta-1} + \left( \frac{r}{2} \right)^{\delta-1} \right) \leq C (\varphi, \delta) |\lambda \varphi(r)|^\eta \cdot \varepsilon. \]

(32)

This proves Lemma 16.

Lemma 17 (see [26]). Let \( \Gamma(y) = (P_1(|y|)a_1(y/|y|), \ldots, P_n(|y|)a_n(y/|y|)) \), where \( P_1, \ldots, P_n \) are real-valued polynomials defined on \( \mathbb{R} \) and \( a_1, \ldots, a_n \) are arbitrary functions defined on \( S^{n-1} \). Suppose that \( \Omega \in L^1(S^{n-1}) \) is a homogeneous function of degree zero and \( h \in \Delta_\gamma(\mathbb{R}_n) \) for some \( \gamma > 1 \). Define the measures \( |\sigma_{\lambda,\Gamma}|_{\mathbb{R}_n} \) by

\[ \int_{\mathbb{R}^n} f(x) d|\sigma_{\lambda,\Gamma}|(x) = \int_{S^{n-1}} f(\Gamma(x)) \frac{|h(|x|)| \Omega(x)}{|x|^n} \frac{dx}{|x|^n}. \]

(33)

If \( (1/p, 1/q) \) belongs to the interior of the convex hull of three squares \( (1/2, 1/2 + 1/ \max(2, \gamma)^2) \), \((1/2 - 1/ \max(2, \gamma)^2, 1/2) \), and \((1/2, 1 - 1/ \gamma)^2 \), then, for arbitrary functions \( |g_{k,j}|_{k,j} \in L^p(\mathbb{E}(\mathbb{E}^2), \mathbb{R}^n) \), there exists \( C > 0 \) independent of \( \gamma \) such that

\[ \left\| \left( \sum_{k,j} \left\| g_{k,j} \right\|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq C \|\Omega\|_{L^1(S^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_n)}. \]

(34)

The constant \( C \) is independent of \( \Omega \) and the coefficients of \( |P_j|_{j=1}^n \).

Lemma 18 (see [17]). Let \( \Gamma(y) = (P_1(|y|)a_1(y/|y|), \ldots, P_n(|y|)a_n(y/|y|)) \), where \( \varphi \in \mathcal{F} \), \( P_1, \ldots, P_n \) are real-valued polynomials on \( \mathbb{R} \) and \( a_1(y), \ldots, a_n(y) \) are arbitrary functions independent of \( |y| \). Define the family of measures \( |\sigma_{\lambda,\Gamma}|_{\mathbb{R}_n} \) on \( \mathbb{R}^n \) by

\[ \int_{\mathbb{R}^n} f(x) d|\sigma_{\lambda,\Gamma}|(x) = \frac{1}{p'} \int_{|x|<\lambda} f(\Gamma(x)) \left| \frac{h(|x|)}{|x|^n} \Omega(x) \right| \frac{dx}{|x|^n}. \]

(35)

Suppose that \( h \in \Delta_\gamma(\mathbb{R}_n) \) for some \( \gamma > 1 \) and \( \Omega \in L^1(S^{n-1}) \). If \( (1/p, 1/q, 1/r) \) belongs to the interior of the convex hull of three cubes \( (1/2, 1/2 + 1/ \max(2, \gamma)^2) \), \((1/2 - 1/ \max(2, \gamma)^2, 1/2) \), and \((1/2, 1 - 1/ \gamma)^2 \), then, for arbitrary functions \( |g_{j,k}|_{j,k} \in L^p(\mathbb{E}(\mathbb{E}^2), \mathbb{R}^n) \), there exists \( C > 0 \) such that

\[ \left\| \left( \sum_{j,k} \left\| g_{j,k} \right\|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq C \|\Omega\|_{L^1(S^{n-1})} \left\| \left( \sum_{j,k} \left\| g_{j,k} \right\|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}. \]

(36)

The constant \( C > 0 \) is independent of \( \Omega \) and the coefficients of \( |P_j|_{j=1}^n \).

Lemma 19 (see [24]). Let \( \Gamma(y) = (P_1(|y|)a_1(y/|y|), \ldots, P_n(|y|)a_n(y/|y|)) \), where \( \varphi \in \mathcal{F} \) and \( P_1, P_2, \ldots, P_n \) are real-valued polynomials on \( \mathbb{R} \), and \( a_1, a_2, \ldots, a_n \) are arbitrary functions defined on \( S^{n-1} \). Suppose that \( \Omega \in L^1(S^{n-1}) \). Define the measures \( |\sigma_{\lambda,\Gamma}|_{\mathbb{R}_n} \) by

\[ |\sigma_{\lambda,\Gamma}|(x) = \int_{S^{n-1}} e^{-2\pi i \Gamma(x) y} |\Omega(y)| \frac{dy}{|y|^n}. \]

(37)

If \( (1/p, 1/q, 1/r) \) belongs to the interior of the convex hull of two cubes \( (0, 1/2)^3 \) and \((1/2, 1)^3 \), then, for arbitrary functions \( |g_{j,k}|_{j,k} \in L^p(\mathbb{E}(\mathbb{E}^2), \mathbb{R}^n) \), there exists \( C > 0 \) independent of \( \Gamma \) and the coefficients of \( |P_j|_{j=1}^n \) such that

\[ \left\| \left( \sum_{j,k} \left\| g_{j,k} \right\|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq C \|\Omega\|_{L^1(S^{n-1})} \left\| \left( \sum_{j,k} \left\| g_{j,k} \right\|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}. \]

(38)

Below is the vector-valued inequality of the Hardy-Littlewood maximal functions, which is one of the main ingredients of our proofs.

Lemma 20 (see [16]). Let \( M_{\Omega_0} \) be the Hardy-Littlewood maximal operator on \( \mathbb{R}^n \). Then

\[ \left\| \left( \sum_{j,k} \left\| M_{\Omega_0} g_{j,k} \right\|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left( \sum_{j,k} \left\| g_{j,k} \right\|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}. \]

(39)

for any \( 1 < p, q, r < \infty \).

In order to deal with Marcinkiewicz integrals and maximal functions, we need a useful characterization of Triebel-Lizorkin spaces and Besov spaces.

Lemma 21 (see [18]). Let \( 0 < \alpha < 1 \).
(i) If $1 < p < \infty$, $1 < q \leq \infty$, and $1 \leq r < \min(p, q)$, then
\[
\| f \|_{L^q_p(\mathbb{R}^n)} 
\sim \left( \sum_{k \in Z} 2^{kn} \left( \int_{\mathbb{R}^n} \left| \Delta_k \chi f \right|^r \, dx \right)^{\frac{q}{r}} \right)^{\frac{1}{q}}.
\]
(40)

(ii) If $1 \leq p < \infty$, $1 \leq q < \infty$, and $1 \leq r \leq p$, then
\[
\| f \|_{L^q_p(\mathbb{R}^n)} 
\sim \left( \sum_{k \in Z} 2^{kn} \left( \int_{\mathbb{R}^n} \left| \Delta_k \chi f \right|^r \, dx \right)^{\frac{q}{r}} \right)^{\frac{1}{q}}.
\]
(41)

To prove Theorem 5, we need the following results.

**Lemma 22** (see [12]). Let $Y, \varphi$ be given as in Theorem 5. Suppose that $h \in \Delta_r(\mathbb{R}_+)$ for some $r > 0$, and $h(\varphi^{-1})(\varphi^{-1}) \in \Delta_r(\mathbb{R}_+)$. The following lemma is a key switched result about singular integrals associated with compound surfaces.

**Lemma 23** (see [29]). Let $\varphi \in \mathcal{S}$ and $Y$ be given as in Theorem 5. Let $T_{h,\Omega,\Gamma}$ be defined as in (2) and $\Omega(y) = \Omega(-y)$. Define the operator $T_{h,\Omega,\Gamma,\varphi}$ by
\[
T_{h,\Omega,\Gamma,\varphi} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(y - \Gamma(\varphi(|y|)y')) \frac{h(|y|)\Omega(y)}{|y|^n} \, dy.
\]
(42)

(i) If $\varphi$ is nonnegative and increasing, then $T_{h,\Omega,\Gamma,\varphi} f = T_{h(\varphi^{-1})(\varphi^{-1}),Y,\Omega,\Gamma} f$.
(ii) If $\varphi$ is nonnegative and decreasing, then $T_{h,\Omega,\Gamma,\varphi} f = -T_{h(\varphi^{-1})(\varphi^{-1}),Y,\Omega,\Gamma} f$.
(iii) If $\varphi$ is nonpositive and decreasing, then $T_{h,\Omega,\Gamma,\varphi} f = T_{h(\varphi^{-1})(\varphi^{-1}),Y,\Omega,\Gamma} f$.
(iv) If $\varphi$ is nonpositive and increasing, then $T_{h,\Omega,\Gamma,\varphi} f = -T_{h(\varphi^{-1})(\varphi^{-1}),Y,\Omega,\Gamma} f$.

3. Some Criteria

To prove Theorem 4, we need the following criterion on the boundedness of the convolution operators on Triebel-Lizorkin spaces, which is a variant of [15, Theorem 1.10]. This can be proved by making some minor modifications in the proof of [15, Theorem 1.10]. We omit the details.

**Proposition 24.** Let $l \in \mathbb{N} \setminus \{0\}$ and $\{\sigma_{t,l} : 0 \leq s \leq l \text{ and } k \in Z\}$ be a family of measures on $\mathbb{R}^n$. For $1 \leq s \leq l$, let $\{a_{k,s}\}_{k \in Z}$ be some sequences of positive real numbers with satisfying
\[
\inf_{k \in Z} \frac{a_{k+1,s}}{a_{k,s}} \geq \eta^l
\]
(43)
for some $\eta > 1$. For $1 \leq s \leq l$, let $\xi_s \in \mathbb{N} \setminus \{0\}$ and $L_{s} : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Suppose that there exist constants $C > 0$ and $\beta > 1$ such that

(i) $\sigma_{t,k} = 0$ for every $k \in Z$;

(ii) $\| \sigma_{t,k} \| \leq C$ for every $k \in Z$ and $1 \leq s \leq l$;

(iii) $|\sigma_{t,k}(\xi)| \leq c \log(1 + \langle a_{k,s}, L_s(\xi) \rangle)^{\beta}$ if $|a_{k,s}, L_s(\xi)| > 1$ for $\xi \in \mathbb{R}^p$, $k \in Z$, and $1 \leq s \leq l$;

(iv) $|\sigma_{t,k}(\xi) - \sigma_{t,-k}(\xi)| \leq c|a_{k,s}, L_s(\xi)|$ for $\xi \in \mathbb{R}^n$, $k \in Z$, and $1 \leq s \leq l$;

(v) for any $1 \leq s \leq 1$ and arbitrary functions $\{g_{k,j}\}_{k,j} \in L_p(\mathcal{E}^p, \mathbb{R}^m)$, there exists a positive constant $C$ which is independent of $|L_s|_{s=1}$ such that
\[
\left\| \left( \sum_{j \in Z} \left( \sum_{k \in Z} |g_{k,j}|^2 \right)^{2/q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left( \sum_{j \in Z} \left( \sum_{k \in Z} |g_{k,j}|^2 \right)^{2/q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}
\]
(44)
for some $p_0, q_0 \in (1, \infty)$ with $p_0 \neq 2$ and $q_0 \neq 2$.

Let $P_1, P_2$ be the line segment from $P_1$ to $P_2$ with $P_1 = (1/2, 1/2)$ and $P_2 = (1/2, 1/2) + (1/p_0)(1 - 1/\beta)$, $1/2 + (1/q_0)(1 - 1/\beta)$. Then there exists a positive constant $C$ such that
\[
\left\| \sum_{j \in Z} \sigma_{t,k} * f \right\|_{L^p_{\mathbb{R}^n}} \leq C \| f \|_{L^p_{\mathbb{R}^n}}
\]
(45)
holds for any $\alpha \in \mathbb{R}$ and $(1/p, 1/q) \in P_1, P_2 \setminus \{P_1\}$.

To establish the Triebel-Lizorkin space boundedness parts in Theorems 8 and 11, we will give the following lemma, which is the heart of the proofs of Theorems 8 and 11.

**Proposition 25.** Let $\Lambda \in \mathbb{N} \setminus \{0\}$ and $\{\sigma_{t,l} : t > 0, 1 \leq s \leq \Lambda\}$ be a family of Borel measures on $\mathbb{R}^n$. Let $\{a_{k,s}\}$ be the total variation of $\sigma_{t,s}$. For $1 \leq s \leq \Lambda$, let $\{a_{k,s}\}_{k \in Z}$ be some sequences of positive real numbers with satisfying
\[
\delta_s \geq \inf_{k \in Z} \frac{a_{k+1,s}}{a_{k,s}} \geq \eta_s > 1
\]
(46)
for some $\eta_s, \delta_s > 1$. For $1 \leq s \leq \Lambda$, let $\eta_s \in \mathbb{N} \setminus \{0\}$ and $L_s : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Suppose that there exist $p_0, q_0 > 1$, $1 < r_0 \leq \min(p_0, q_0)$, $\beta > 1$, and $C > 0$ such that the following conditions hold for $1 \leq s \leq \Lambda$, $t > 0$, $\xi \in \mathbb{R}^n$, and $\{g_{k,j}\}_{k,j} \in L^p_{\mathcal{E}^p(\mathcal{E}^p, \mathbb{R}^m)}$:

(i) $\sigma_{t,k} = 0$;

(ii) $\left( \int_{R^n} |\sigma_{t,k}(\xi) - \sigma_{t,-k}(\xi)|^2 (dt/t)^{1/2} \right)^{1/2} \leq C \min\{1, |a_{k+1,s}, L_s(\xi)|\}$;

(iii) $\left( \int_{R^n} |\sigma_{t,k}(\xi) - \sigma_{t,-k}(\xi)|^2 (dt/t)^{1/2} \right)^{1/2} \leq C(\log|a_{k,s}, L_s(\xi)|)^{\beta}$ if $|a_{k,s}, L_s(\xi)| > 1$.
Then there exists a positive constant $C$ such that

$$
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}} \left| \sigma_{\Lambda,k} \ast \Delta_{2^{-k}} f \right|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right) \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} \left| \gamma_{\Lambda,k} \right|^2 \right) \right\|_{L^q(\mathbb{R}^n)}^{1/q} \left\| f \right\|_{L^p(\mathbb{R}^n)}^{1/q}. \tag{47}
$$

Let $P_1, P_2$ be the line segment from $P_1$ to $P_2$, with $P_1 = (1/2, 1/2)$ and $P_2 = (1/2 + (1/p_0)(1 - 1/\beta), 1/2 + (1/q_0)(1 - 1/\beta))$. Then there is a positive constant $C$ such that

$$
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}} \left| \sigma_{\Lambda,k} \ast \Delta_{2^{-k}} f \right|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right) \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} \left| \gamma_{\Lambda,k} \right|^2 \right) \right\|_{L^q(\mathbb{R}^n)}^{1/q} \left\| f \right\|_{L^p(\mathbb{R}^n)}^{1/q}. \tag{48}
$$

holds for any $\alpha \in (0, 1)$ and $(1/p, 1/q) \in P_1, P_2 \setminus \{P_2\}$.

**Proof.** For any $1 \leq s \leq \Lambda$, let $l_s = \text{rank}(L_s) \leq \min[n, \ell_s]$. By [5, Lemma 6.1], there are two nonsingular linear transformations $\mathcal{H}_s : \mathbb{R}^l \rightarrow \mathbb{R}^l$ and $\mathcal{G}_s : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$
\left| \mathcal{H}_s \pi^m_s \mathcal{G}_s \xi \right| \leq \left| L_s(\xi) \right| \leq M_s \left| \mathcal{H}_s \pi^m_s \mathcal{G}_s \xi \right|, \quad \xi \in \mathbb{R}^n. \tag{49}
$$

Therefore, to prove (48), it suffices to show that there exists $C > 0$ such that

$$
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}} \left| \sigma_{\Lambda,k} \ast \Delta_{2^{-k}} f \right|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right) \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^n)} \tag{53}
$$

for any $1 \leq s \leq \Lambda, \alpha \in (0, 1)$, and $(1/p, 1/q) \in P_1, P_2 \setminus \{P_2\}$.

We now prove (53). By our assumptions (ii)-(iii), we have

$$
\left( \int_{2^k}^{2^{k+1}} \left| \sigma_{\Lambda,k} \ast \Delta_{2^{-k}} f \right|^2 \frac{dt}{t} \right)^{1/2} \leq C \min \{1, |a_{k+1,s} L_s(\xi)|\};
$$

$$
\left( \int_{2^k}^{2^{k+1}} \left| \sigma_{\Lambda,k} \ast \Delta_{2^{-k}} f \right|^2 \frac{dt}{t} \right)^{1/2} \leq C (\log |a_{k,s} L_s(\xi)|)^{\beta} \tag{54}
$$

if $|a_{k,s} L_s(\xi)| > M_s$.

Let $\{\psi_{k,s}\}_{k \in \mathbb{Z}}$ be a collection of $C^\infty_0$ functions on $(0, \infty)$ with the following properties:

$$
\text{supp} \left( \psi_{k,s} \right) \subset \left[ a_{k+1,s}, a_{k-1,s}^{-1} \right];
$$

$$
0 \leq \psi_{k,s}(t) \leq 1; \sum_{k \in \mathbb{Z}} \psi_{k,s}(t) = 1. \tag{55}
$$

Define the multiplier operator $S_{k,s}$ on $\mathbb{R}^n$ by

$$
\widehat{S_{k,s} f}(\xi) = \psi_{k,s} \left( \mathcal{H}_s \pi^m_s \mathcal{G}_s \xi \right) \hat{f}(\xi). \tag{56}
$$

Note that $\delta_s \geq a_{k+1,s}/a_{k,s} \geq \eta_s > 1$. By [16, Lemma 2.5] we obtain

$$
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}} \left| S_{k,s} f \right|^2 \frac{dt}{t} \right)^{1/2} \right) \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} \left| f \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \tag{57}
$$

Therefore, to prove (48), it suffices to show that there exists $C > 0$ such that

$$
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}} \left| \sigma_{\Lambda,k} \ast \Delta_{2^{-k}} f \right|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right) \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^n)} \tag{53}
$$

for any $1 \leq s \leq \Lambda, \alpha \in (0, 1)$, and $(1/p, 1/q) \in P_1, P_2 \setminus \{P_2\}$.

We now prove (53). By our assumptions (ii)-(iii), we have

$$
\left( \int_{2^k}^{2^{k+1}} \left| \sigma_{\Lambda,k} \ast \Delta_{2^{-k}} f \right|^2 \frac{dt}{t} \right)^{1/2} \leq C \min \{1, |a_{k+1,s} L_s(\xi)|\};
$$

$$
\left( \int_{2^k}^{2^{k+1}} \left| \sigma_{\Lambda,k} \ast \Delta_{2^{-k}} f \right|^2 \frac{dt}{t} \right)^{1/2} \leq C (\log |a_{k,s} L_s(\xi)|)^{\beta} \tag{54}
$$

if $|a_{k,s} L_s(\xi)| > M_s$. 

For $t > 0$ and $1 \leq s \leq \Lambda$, we define the family of measures $\{\tau_{s,t}\}_{t>0}$ by

$$
\tau_{s,t}(\xi) = \int_0^t \psi_n \left( \int_0^1 \left( \sum_{j=s+1}^\Lambda \left( \int_0^\infty \left| \sigma_{\Lambda,j} \right|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right)^q \right)^{1/q} \frac{dt}{t} \tag{50}
$$

where $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(t) \equiv 1$ for $|t| \leq 1/2$ and $\psi(t) \equiv 0$ for $|t| > 1$. Then (50) together with assumption (i) implies that

$$
\sigma_{\Lambda,j} = \sum_{s=1}^\Lambda \tau_{s,t} \tag{51}
$$

It follows that

$$
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}} \left| \sigma_{\Lambda,k} \ast \Delta_{2^{-k}} f \right|^2 \frac{dt}{t} \right)^{1/2} d\zeta \right) \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^n)} \tag{52}
$$

for $1 \leq s \leq \Lambda, \alpha \in (0, 1)$, and $(1/p, 1/q) \in P_1, P_2 \setminus \{P_2\}$.
By Minkowski’s inequality we have

\[
\left\| \left( \sum_{\ell \in \mathbb{Z}} 2^{\ell q_s} \left( \int_{\mathbb{R}_n} \left( \int_0^\infty \left| r_{s,\ell} \ast \Delta_{2^{-\gamma}} f \right|^2 \frac{dt}{t} \right)^{1/2} \right)^{q} \right\|_{L^p(\mathbb{R}^n)}^{1/q} \leq \sum_{j \in \mathbb{Z}} \left\| \left( \sum_{\ell \in \mathbb{Z}} 2^{\ell q_s} \left( \int_{\mathbb{R}_n} \left( \int_0^\infty \left| r_{s,\ell} \ast \Delta_{2^{-\gamma}} f \right|^2 \frac{dt}{t} \right)^{1/2} \right)^{q} \right\|_{L^p(\mathbb{R}^n)}^{1/q}.
\]

(58)

Define the mixed norm \( \| \cdot \|_{E_{\mathbb{R}^n}^s} \) for measurable functions on \( \mathbb{R}^n \times \mathbb{R}_n \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}_+ \) by

\[
\|g\|_{E_{\mathbb{R}^n}^s} := \left\| \left( \sum_{\ell \in \mathbb{Z}} 2^{\ell q_s} \left( \int_{\mathbb{R}_n} \left( \int_0^\infty \left| g(x, \zeta, l, k, t) \right|^2 \frac{dt}{t} \right)^{1/2} \right)^{q} \right\|_{L^p(\mathbb{R}^n)}^{1/q}.
\]

(59)

For any \( j \in \mathbb{Z} \), let

\[
V_{j,s} (f) (x, \zeta, l, k, t)
= r_{s,j} \ast S_{j-k,s} \Delta_{2^{-\zeta}} f (x) \chi_{[2^{j-2},2^{j-1}]} (t).
\]

(60)

Then we have

\[
\left\| V_{j,s} (f) \right\|_{E_{\mathbb{R}^n}^s} \leq \sum_{j \in \mathbb{Z}} \left\| V_{j,s} (f) \right\|_{E_{\mathbb{R}^n}^s}.
\]

(61)

By (54), Hölder’s inequality, Minkowski’s inequality, Fubini’s theorem, Plancherel’s theorem, and Lemma 21(ii), we have

\[
\left\| V_{j,s} (f) \right\|_{E_{\mathbb{R}^n}^s}^2 = \left\| \left( \sum_{\ell \in \mathbb{Z}} 2^{2\ell} \left( \int_{\mathbb{R}_n} \left( \int_0^\infty \left| r_{s,\ell} \ast S_{j-k,s} \Delta_{2^{-\gamma}} f \right|^2 \frac{dt}{t} \right)^{1/2} \right)^{q} \right\|_{L^p(\mathbb{R}^n)}^{2} \leq C \sum_{\ell \in \mathbb{Z}} 2^{2\ell} \left\| \left( \int_{\mathbb{R}_n} \left( \int_0^\infty \left| r_{s,\ell} \ast S_{j-k,s} \Delta_{2^{-\gamma}} f \right|^2 \frac{dt}{t} \right)^{1/2} \right)^{q} \right\|_{L^p(\mathbb{R}^n)}^{2} \leq C B_{j,s}^2 \left\| f \right\|_{L^2(\mathbb{R}^n)}^2.
\]

(62)
where \( B_{j,k,s} = M_j r_j^{-1} X_{[j,k]}(j) + |j|^{-2} X_{[j-k]}(j) \) with \( k = \max \{ k \in \mathbb{Z} : k < -1 - \log p, M_j \} \) and
\[
E_{j,k,s} = \{ x \in \mathbb{R}^n : a_j^{-1} R_{-1}^1 \leq |L_{j}(x)| \leq M_j a_j^{-1} \}.
\]
It follows from (62) and (14) that
\[
\|V_{j,s}(f)\|_{L_j^2(\mathbb{R}^n)} \leq CB_{j,s} \|f\|_{L_j^2(\mathbb{R}^n)}.
\]
We now prove
\[
\|V_{j,s}(f)\|_{L_j^p(\mathbb{R}^n)} \leq C \|f\|_{L_j^p(\mathbb{R}^n)}.
\]
For \( 1 \leq s \leq \Lambda \), let \( \Phi \) be a radial function in \( \mathcal{S}(\mathbb{R}^l) \) defined by \( \Phi(x) = \psi(|x|) \), where \( x \in \mathbb{R}^l \) and \( \psi \) is given as in (50). Define \( J_s \) and \( \chi_s \) by
\[
J_s = \left( \sum_{k \in \mathbb{Z}} \left| \chi_{s} g_{k,l,k} \right|^2 \right)^{1/2} \| \chi_{s} g_{k,l,k} \|_{L_j(\mathbb{R}^n)}
\]
and
\[
\chi_{s} g_{k,l,k} = \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \left[ J_{s}^{-1} \circ (M_{l,j} \otimes I_{\mathbb{R}^n}) \circ J_s \right] \left| g_{k,l,k} \right|^2 \right)^{1/2} \| g_{k,l,k} \|_{L_j(\mathbb{R}^n)}
\]
for any \( 1 \leq s \leq \Lambda \) and \( 1 < p, q, r < \infty \). Define \( X_{k,s} f = X_{s} \circ X_{s+1} \circ \cdots \circ X_{s+1} f \) for \( 1 \leq s \leq \Lambda \). We get from (67) that, for any \( 1 \leq s \leq \Lambda \) and \( 1 < p, q, r < \infty \),
\[
\left\| \left( \sum_{k \in \mathbb{Z}} \left| X_{s} g_{k,l,k} \right|^2 \right)^{1/2} \| X_{s} g_{k,l,k} \|_{L_j(\mathbb{R}^n)} \right\|_{L_j^p(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} \left| g_{k,l,k} \right|^2 \right)^{1/2} \| g_{k,l,k} \|_{L_j(\mathbb{R}^n)} \right\|_{L_j^p(\mathbb{R}^n)}.
\]
On the other hand, by the definition of \( X_{k,s} \), we have
\[
\tau_{s} \ast f = \sigma_{s} \ast (X_{k,s+1} \ast X_{k,s+2} \ast \cdots \ast X_{k,s} f)
\]
and
\[
- \sigma_{s-1} \ast (X_{k,s-1} \ast X_{k,s-2} \ast \cdots \ast X_{k,s} f).
\]
\[ \|V_{j,\alpha}(f)\|_{\dot{B}^\alpha_p} = \left\| \sum_{l \in \mathbb{Z}} 2^{\lambda l / \alpha} \left( \int_{\mathbb{R}^n} \left( \sum_{k \in 2^l l_k^2} |t_{k,j} * S_{j-k,x} \Delta_2^{-\gamma} f| \frac{dt}{t} \right)^{1/2} \right)^{q_0} \right\|_{L^p(\mathbb{R}^n)}^{1/q_0} \]
\[ \leq C \left\| \sum_{l \in \mathbb{Z}} 2^{\lambda l / \alpha} \left( \sum_{k \in 2^l l_k^2} |S_{j-k,x} \Delta_2^{-\gamma} f|^2 \right)^{q_0 / 2} \right\|_{L^p(\mathbb{R}^n)}^{1/q_0} \leq C \left\| \sum_{l \in \mathbb{Z}} 2^{\lambda l / \alpha} |\Delta_2^{-\gamma} f|^{q_0} \right\|_{L^p(\mathbb{R}^n)}^{1/q_0} \]  \( \tag{73} \)
\[ \leq C \left( \frac{B^\alpha_p}{B^\alpha_q - 1} \right)^{n/2} \|f\|_{\dot{B}^\alpha_p} \]  .

This proves (65).

By the interpolation between (64) and (65), we obtain that, for \( \alpha \in (0,1) \) and \((1/p,1/q) \in P_1 \setminus P_2\), there exists \( \theta \in (1/\beta,1) \) such that
\[ \|V_{j,\alpha}(f)\|_{\dot{B}^\alpha_p} \leq CB^\beta_{j,\alpha} \|f\|_{\dot{B}^\alpha_q(\mathbb{R}^n)} \]  \( \tag{74} \)
Then, (74) together with (61) yields (48) and completes the proof of Proposition 25.

The following result is a criterion on the boundedness and continuity of several operators on Besov spaces, which can be used to prove the boundedness and continuity result on Besov spaces in Theorems 8 and 11.

**Proposition 26** (see [23]). Assume that \( T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \) for some \( p \in (1, \infty) \). If
\[ |\Delta_2^{-\gamma} (Tf)(x)| \leq |T(\Delta_2^{-\gamma} f)(x)| \]  \( \tag{75} \)
for any \( x, \zeta \in \mathbb{R}^n \). Then \( T \) is bounded on \( \dot{B}^\alpha_p(\mathbb{R}^n) \) for any \( s \in (0,1) \) and \( q \in (1, \infty) \). Particularly, if \( T \) satisfies
\[ |Tf - Tg| \leq |T(f - g)| \]  \( \tag{76} \)
for arbitrary functions \( f, g \) defined on \( \mathbb{R}^n \), then \( T \) is continuous from \( \dot{B}^\alpha_p(\mathbb{R}^d) \) to \( \dot{B}^\alpha_p(\mathbb{R}^d) \) for any \( s \in (0,1) \) and \( q \in (1, \infty) \).

To establish the Triebel-Lizorkin space continuity parts in Theorems 8 and 11, we will give the following criterion of continuity for several sublinear operators on Triebel-Lizorkin spaces.

**Proposition 27.** Assume that \( T \) is a sublinear operator and the following conditions hold.
(i) \( T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) for some \( p \in (1, \infty) \).
(ii) For all \( x, \zeta \in \mathbb{R}^n \), the following holds:
\[ |\Delta_2^{-\gamma} (Tf)(x)| \leq |T(\Delta_2^{-\gamma} f)(x)| \]  \( \tag{77} \)
(iii) For arbitrary functions \( f, g \) defined on \( \mathbb{R}^n \), the following holds:
\[ |Tf - Tg| \leq |T(f - g)| \]  \( \tag{78} \)
(iv) There exist \( \alpha \in (0,1) \) and \( q \in (1, \infty) \) such that
\[ \left\| \sum_{l \in \mathbb{Z}} 2^{\lambda l / \alpha} \left( \int_{\mathbb{R}^n} \left| T \left( \Delta_2^{-\gamma} f \right) \left( \zeta \right) \right|^q \frac{d\zeta}{q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \]
\[ \leq C \|f\|_{\dot{B}^\alpha_p(\mathbb{R}^n)} \]  \( \tag{79} \)

Then \( T \) is continuous from \( F^p_q(\mathbb{R}^n) \) to \( F^p_q(\mathbb{R}^n) \).

**Proof.** Let \( f_j \to f \) in \( F^p_q(\mathbb{R}^d) \) as \( j \to \infty \). By (15) we see that \( f_j \to f \) in \( F^p_q(\mathbb{R}^n) \) and in \( L^p(\mathbb{R}^n) \) as \( j \to \infty \). Since \( \|f_j - f\|_{L^p(\mathbb{R}^n)} \to 0 \) as \( j \to \infty \), by assumptions (i) and (iii) we obtain that \( Tf_j \to Tf \) in \( L^p(\mathbb{R}^n) \) as \( j \to \infty \). Therefore, it suffices to show that \( Tf_j \to Tf \) in \( F^p_q(\mathbb{R}^n) \) as \( j \to \infty \).

We shall prove this claim by contradiction. Without loss of generality we may assume that there exists \( c > 0 \) such that
\[ \|Tf_j - T\|^q_{F^p_q(\mathbb{R}^n)} = c \]  \( \tag{80} \)
for every \( j \). Since \( Tf_j \to Tf \) in \( L^p(\mathbb{R}^n) \) as \( j \to \infty \), by extracting a subsequence we may assume that \( \|Tf_j(x) - Tf(x)\| \to 0 \) as \( j \to \infty \) for almost every \( x \in \mathbb{R}^n \). It follows that \( \Delta_2^{-\gamma} (Tf_j - Tf_j)(x) \to 0 \) as \( j \to \infty \) for every \( (l, \zeta) \in \mathbb{Z} \times \mathbb{R}^n \) and almost every \( x \in \mathbb{R}^n \). We get from assumption (iv) and the sublinearity of \( T \) that
\[ \left| \Delta_2^{-\gamma} (Tf_j - Tf)(x) \right| \leq 2T \left( \Delta_2^{-\gamma} f \right)(x) + T \left( \Delta_2^{-\gamma} (f_j - f) \right)(x) \]  \( \tag{81} \)
for \( x, l, \zeta \in \mathbb{R}^n \times \mathbb{Z} \times \mathbb{R}^n \). For convenience, we set
\[ \|g\|_{p,q,\alpha} = \left\| \sum_{l \in \mathbb{Z}} 2^{\lambda l / \alpha} \left( \int_{\mathbb{R}^n} \left| g(x, l, \zeta) \right|^q \frac{d\zeta}{q} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \]  \( \tag{82} \)
for \( \alpha \in \mathbb{R} \) and \( (p,q) \in (1, \infty)^2 \). It follows from Lemma 21(i) that \( \|f\|_{\dot{B}^\alpha_p(\mathbb{R}^n)} \sim \|\Delta_2^{-\gamma} f\|_{p,q,\alpha} \) for \( \alpha \in (0,1) \) and \( (p,q) \in (1, \infty)^2 \). By assumption (iv) we obtain
\[ \left\| T \left( \Delta_2^{-\gamma} f \right) \right\|_{p,q,\alpha} \]
and the dominated convergence theorem that

\[ \lim_{j \to \infty} \left\| \Delta_{2^{-j}}(f - f_J) \right\|_{p,q,a} = 0 \]

as \( j \to \infty \). One can extract a subsequence such that

\[ \left\| \Delta_{2^{-j}}(f - f_J) \right\|_{p,q,a} \leq C \]

for almost every \((x, l, \zeta) \in \mathbb{R}^n \times Z \times \mathbb{R}_n^*\). One can easily check that \(\|G\|_{p,q,a} < \infty\) and

\[ \left| \Delta_{2^{-j}}(f - f_J)(x) \right| \leq G(x, l, \zeta) \]

for almost every \((x, l, \zeta) \in \mathbb{R}^n \times Z \times \mathbb{R}_n^*\).

Since \(\|G\|_{p,q,a} < \infty\), we have that \(\int_{\mathbb{R}^n} G(x, l, \zeta) d\zeta < \infty\) for every \(k \in Z\) and almost every \(x \in \mathbb{R}^n\). (85) together with the dominated convergence theorem leads to

\[ \int_{\mathbb{R}^n} \left| \Delta_{2^{-j}}(f - f_J)(x) \right| d\zeta \to 0 \text{ as } j \to \infty \]

for every \(l \in Z\) and almost every \(x \in \mathbb{R}^n\). By the fact \(\|G\|_{p,q,a} < \infty\) again, we have

\[ \left( \sum_{l \in Z} 2^{b_{ij}} \left( \int_{\mathbb{R}^n} G(x, l, \zeta) d\zeta \right)^q \right)^{1/q} < \infty \]

for almost every \(x \in \mathbb{R}^n\). Using (85) we obtain

\[ \int_{\mathbb{R}^n} \left| \Delta_{2^{-j}}(f - f_J)(x) \right| d\zeta \leq \int_{\mathbb{R}^n} G(x, l, \zeta) d\zeta \]

for almost every \(x \in \mathbb{R}^n\) and \(l \in Z\). It follows from (86)–(88) and the dominated convergence theorem that

\[ \left( \sum_{l \in Z} 2^{b_{ij}} \left( \int_{\mathbb{R}^n} \left| \Delta_{2^{-j}}(f - f_J)(x) \right| d\zeta \right)^q \right)^{1/q} \to 0 \]

as \( j \to \infty \)

for almost every \(x \in \mathbb{R}^n\). By (85) again, it holds that

\[ \left( \sum_{l \in Z} 2^{b_{ij}} \left( \int_{\mathbb{R}^n} \left| \Delta_{2^{-j}}(f - f_J)(x) \right| d\zeta \right)^q \right)^{1/q} \leq \left( \sum_{l \in Z} 2^{b_{ij}} \left( \int_{\mathbb{R}_n} |G(x, l, \zeta)| d\zeta \right)^q \right)^{1/q} \]

for almost every \(x \in \mathbb{R}^n\). By (89)-(90), the fact \(\|G\|_{p,q,a} < \infty\), and the dominated convergence theorem, we obtain

\[ \left( \Delta_{2^{-j}}(f - f_J) \right) \to 0 \text{ as } j \to \infty. \]

This yields that \(\|T f - T f_J\|_{p,q,a} \to 0\) as \( j \to \infty \) and gives a contradiction. □

4. Proofs of Theorems 4, 5, and 11

In this section we shall prove Theorems 4–11. In what follows, let \(\deg(P) = \max_{1 \leq j \leq \deg(P)} \). For \(1 \leq j \leq n\), we set \(P(t) = \sum_{i=1}^{\deg(P)} a_i t_i\). Then there are integers \(0 < l_1 < l_2 < \cdots < l_{\Lambda} \leq \deg(P)\) such that \(P(t) = \sum_{i=1}^{\Lambda} a_i t_i\) for any \(1 \leq j \leq n\) and \((a_{i_1}, a_{i_2}, \ldots, a_{i_\Lambda}) \neq (0, 0, \ldots, 0) \in \mathbb{R}_n^*\) for all \(1 \leq i \leq \Lambda\). For \(1 \leq j \leq n\) and \(0 \leq s \leq \Lambda\), we set \(P_j(t) = \sum_{i=1}^{j} a_i t_i\) for \(1 \leq s \leq \Lambda\) and \(P_j(t) = (0, \ldots, 0)\). For \(1 \leq s \leq \Lambda\), we define the linear transformation \(L_j : \mathbb{R}_n^* \to \mathbb{R}_n^*\) by

\[ L_j(\underline{\xi}) = (a_1 \xi_1, a_2 \xi_2, \ldots, a_n \xi_n). \]

We now turn to prove Theorems 4, 5, and 11

Proof of Theorem 4. Define \(\{\Phi_j\}_{j=0}^\Lambda\) by

\[ \Phi_j(\xi) = \left( P_j(\xi) (|y| y_1', \ldots, |y| y_n') \right), \]

\[0 \leq s \leq \Lambda.\]

It is clear to see that

\[ \Phi_j(\xi) = \sum_{j=1}^{\Lambda} P_j(t) (|y| y_1', \ldots, |y| y_n') \]

for any \(x, \xi \in \mathbb{R}^n\) and \(1 \leq s \leq \Lambda\). For \(0 \leq s \leq \Lambda\) and \(\xi \in \mathbb{R}^n\), we define the measures \(\sigma_{k,s}(\xi) \in \mathbb{R}_n^*\) by

\[ \sigma_{k,s}(\xi) = \int_{\mathbb{R}_n^*} e^{-2n \frac{|y| y_1'}{|y|^2} \Omega(\frac{y'}{|y|^2})} \frac{\Omega(\frac{y'}{|y|^2})}{|y|^2} dy. \]

It is clear that

\[ T_{h,\Omega, \lambda, \frac{f}{k}, \frac{f}{k}} = \sum_{k \in Z} \sigma_{k,s} \ast f. \]

By the change of the variables, we have

\[ |\sigma_{k,s}(\xi) - \sigma_{k-1,s}(\xi)| = \int_{\mathbb{R}_n^*} \int_{\mathbb{R}_n^*} \Omega(\frac{y'}{|y|^2}) \cdot \left( e^{-2n \frac{|y| y_1'}{|y|^2} \Omega(\frac{y'}{|y|^2})} - e^{-2n \frac{|y| y_1'}{|y|^2} \Omega(\frac{y'}{|y|^2})} \right) dy'. \]

\[ : h(t) \frac{dt}{t} \leq C \|\Omega\|_{L^1(S^{n-1})} \|h\|_{\Delta_c(\mathbb{R}_n^*)} \]

\[ \cdot |2^{(k+1)s} L_s(\xi)|. \]

On the other hand, it is easy to check that

\[ \|\sigma_{k,s}\| \leq C \|\Omega\|_{L^1(S^{n-1})} \|h\|_{\Delta_c(\mathbb{R}_n^*)}. \]
\[ \sigma_{k,0} = 0. \quad (98) \]

By the change of the variables and Hölder's inequality, we have

\[
\left| \tilde{\sigma}_{k,\xi} (\xi) \right| = \int_{Z}^{\infty} \int_{\mathbb{R}^{n}} \Omega (y') e^{-2\pi i \Phi_{k}(t) y' \cdot \xi} d\sigma (y') h (t) \frac{dt}{t} \leq C \| h \|_{\Delta, \mathbb{R}} \]

\[
\cdot \left( \int_{Z}^{\infty} \int_{\mathbb{R}^{n}} \Omega (y') e^{-2\pi i \Phi_{k}(t) y' \cdot \xi} d\sigma (y') \right)^{1/\gamma} \leq C \left( \int_{Z}^{\infty} \int_{\mathbb{R}^{n}} \Omega (y') e^{-2\pi i \Phi_{k}(t) y' \cdot \xi} d\sigma (y') \right)^{2 \gamma} \frac{dt}{t} \]

\[
\cdot \left( \int_{Z}^{\infty} \int_{\mathbb{R}^{n}} \Omega (y') e^{-2\pi i \Phi_{k}(t) y' \cdot \xi} d\sigma (y') \right)^{1/\max (2, \gamma')} \leq C \left( \int_{Z}^{\infty} \int_{\mathbb{R}^{n}} \Omega (y') e^{-2\pi i \Phi_{k}(t) y' \cdot \xi} d\sigma (y') \right)^{1/\max (2, \gamma')} \]

where

\[ H_k (\xi, y', \theta) = \int_{Z}^{\infty} e^{-2\pi i \Phi_{k}(t) y' \cdot \xi} dt. \quad (100) \]

By the Van der Corput lemma, there exists a constant \( C > 0 \), such that

\[ |H_k (\xi, y', \theta)| \leq C \min \left\{ 1, \left( 2^{k+1} \right)^{r} L_{s} (\xi) \cdot \left( y' - \theta \right) \right\} \]

When \( 2^{(k+1) r} L_{s} (\xi) > 1 \), since \( t/(\log t)^{\beta} \) is increasing in \( (e^{\theta}, \infty) \), we have

\[ |H_k (\xi, y', \theta)| \leq C \frac{\log 2 e^{\theta} \left( \eta \cdot (y' - \theta) \right)^{-1/\gamma}}{(\log 2^{(k+1)} L_{s} (\xi))^{\beta}}, \quad (102) \]

where \( \eta = \frac{L_{s} (\xi)}{L_{s} (\xi)} \). Combining (99), (102) with the fact that \( \Omega \in W \mathcal{F}_{\beta} (S^{n-1}) \) yields that

\[ \left| \tilde{\sigma}_{k,\xi} (\xi) \right| \leq C \left( \log 2^{(k+1)} L_{s} (\xi) \right)^{-\beta/\max (2, \gamma')} \]

\[ \text{if } 2^{(k+1) r} L_{s} (\xi) > 1. \quad (103) \]

On the other hand, Lemma 17 yields that

\[ \left\| \sum_{k,\xi} \left( \sum_{j \in Z} \left| g_{k,j} \right| \right)^{q/2} \right\|_{L^{p} (\mathbb{R}^{n})} \]

\[ \leq C \left\| \left( \sum_{j \in Z} \left( \sum_{k,\xi} \left| g_{k,j} \right|^{2} \right)^{q/2} \right)^{1/2} \right\|_{L^{p} (\mathbb{R}^{n})} \]

\[ \leq \frac{1}{t^{\rho} \int_{1/2 < |x| \leq 2} f (\Phi_{s} (x)) \frac{h (|x|) \Omega (x)}{|x|^{n-p}} dx}, \quad (108) \]

for \((1/p, 1/q)\) belonging to the interior of the convex hull of three squares \((1/2, 1/2 + 1/ \max (2, y'))^{2}, (1/2 - 1/ \max (2, y'))^{2}, (1/2, 1/2)^{2}\). Here \( C > 0 \) is independent of the coefficients of \( \{ p_{j} \}_{j=1}^{n} \).

Take \( \delta_{ks} = 2^{k} \). By (96)–(98), (103)–(104) and Proposition 24, we obtain

\[ \| T_{h, \Omega, \alpha, \beta} f \|_{L^{p} (\mathbb{R}^{n})} \leq C \| f \|_{L^{p} (\mathbb{R}^{n})} \quad (105) \]

for \( \beta > \max (2, \gamma'') \), \( \alpha \in \mathbb{R} \), and all \((1/p, 1/q) \in \mathcal{R}_{\gamma', \beta} \) where \( \mathcal{R}_{\gamma', \beta} \) is given as in Theorem 4. This proves Theorem 4(i).

On the other hand, it follows from Theorem 4(ii) and (13) that \( T_{h, \Omega, \alpha} f \) is bounded on \( L^{p} (\mathbb{R}^{n}) \) for \( \beta > \max (2, \gamma') \) and \( |1/p - 1/2| < 1/ \max (2, \gamma') - 1/\beta \). This together with the arguments similar to those used in deriving [30, Theorem 1.2] yields Theorem 4(ii). \( \square \)

**Proof of Theorem 5.** Theorem 5 follows from Theorem 4 and Lemmas 22 and 23. \( \square \)

**Proof of Theorem 8.** Define \( \Phi_{0}, \Phi_{1}, \ldots, \Phi_{\Lambda} \) by

\[ \Phi_{s} (y) = \left( \sum_{j=1}^{n} p_{j} (\varphi (|y|)) y^{(s)} \right), \quad 0 \leq s \leq \Lambda. \]

Clearly,

\[ \Phi_{s} (x) \cdot \psi = \sum_{j=1}^{n} p_{j} (\varphi (|x|)) x^{(s)} \psi, \]

\[ = \sum_{i=1}^{n} \varphi (|x|) y^{(s)} \left( L_{i} (\xi) \cdot x' \right), \]

for any \( x, \xi \in \mathbb{R}^{n} \) and \( 1 \leq s \leq \Lambda \). For \( 0 \leq s \leq \Lambda \), define the family of measures \( \{ g_{s} \}_{s} \) by

\[ \int_{\mathbb{R}^{n}} f (x) d\sigma_{s} (x), \]

\[ = \frac{1}{t^{\rho} \int_{1/2 < |x| \leq 2} f (\Phi_{s} (x)) \frac{h (|x|) \Omega (x)}{|x|^{n-p}} dx}, \]
where $|\sigma_{l,s}|$ is defined in the same way as $\sigma_{l,s}$, but with $h$ and $Ω$ replaced by $h$ and $|Ω|$, respectively. By the change of variables and Minkowski’s inequality, we have

$$M_{n,Ω,Γ,p,f}(x) = \left( \int_{-\infty}^{0} \left( \sum_{k=-\infty}^{0} 2^k \sigma_{l,s}^{k} * f(x) \right)^2 \frac{dt}{t} \right)^{1/2} \leq \sum_{k=-\infty}^{0} 2^k \left( \int_{0}^{\infty} |\sigma_{l,s}^{k} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \leq \frac{1}{1 - 2^{-n}} \left( \int_{0}^{\infty} |\sigma_{l,s} * f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$  

(109)

By Lemma 18, we obtain

$$\left\| \left( \sum_{j \in \mathbb{Z}} \left( \int_{\mathbb{R}_n} \left( \sum_{k \in \mathbb{Z}} \left| \sigma_{l,s}^{k} * g_j \right|^2 \right)^{1/2} \frac{dt}{t} \right)^{1/2} \right) \right\|_{L^p(\mathbb{R}^n)} \leq C \|\Omega\|_{L^{1/(p-1)}} \left\| \left( \sum_{j \in \mathbb{Z}} \left| g_j \right|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}_n)}^{1/q}.$$  

(110)

for $(1/p, 1/q, 1/r)$ belonging to the interior of the convex hull of three cubes $(1/2, 1/2 + 1/\max\{2, \gamma\}^2, (1/2 - 1/\max\{2, \gamma\}'', 1/2)^3$, and $(1/2\gamma, 1 - 1/2\gamma)^3$. Here $C > 0$ is independent of the coefficients of $\{\rho_j\}_{j=1}^n$.

One can easily check that

$$\sigma_{l,0} = 0; \quad \sigma_{l,s}(\xi) - \sigma_{l,s}^{s-1}(\xi) \leq C \min \{1, \left| \varphi(t)^l \right| L_s(\xi) \}.$$  

(112)

By a change of variable, we have

$$\left| \sigma_{l,s}^{k}(\xi) \right| = \left| \frac{1}{r^p} \int_{t/2}^{t} \int_{S^{n-1}} \left[ -2\pi i \sum_{j=1}^{s} (L_j(\xi) \cdot y') \varphi(r^j) \right] \Omega(y') d\sigma(y') h(r) \frac{dr}{r^{1-p}} \right|$$

$$\leq \int_{t/2}^{t} \int_{S^{n-1}} \left[ -2\pi i \sum_{j=1}^{s} (L_j(\xi) \cdot y') \varphi(r^j) \right] \Omega(y') d\sigma(y') \left| h(r) \right| \frac{dr}{r} \leq C \|h\|_{L^{1}(\mathbb{R}^n)} \|\Omega\|_{L^{p}(S^{n-1})}^{\max\{0,1-2/\gamma\}''}$$

$$\times \left( \int_{t/2}^{t} \int_{S^{n-1}} \left[ -2\pi i \sum_{j=1}^{s} (L_j(\xi) \cdot y') \varphi(r^j) \right] \Omega(y') d\sigma(y') \right)^{2} \frac{dr}{r}.$$  

(114)

Since

$$\int_{t/2}^{t} \int_{S^{n-1}} \left[ -2\pi i \sum_{j=1}^{s} (L_j(\xi) \cdot y') \varphi(r^j) \right] \Omega(y') d\sigma(y') \frac{dr}{r} \leq \int_{t/2}^{t} \int \int_{S^{n-1} \times S^{n-1}} \left[ -2\pi i \sum_{j=1}^{s} (L_j(\xi) \cdot (y' - u')) \varphi(r^j) \right] \Omega(y') \Omega(u') d\sigma(y') d\sigma(u') \frac{dr}{r}$$
by Lemma 16, we have
\[
\left| \int_{t/2}^{t} \exp \left( -2\pi i \sum_{j=1}^{s} (L_j(\xi) \cdot (y' - u')) \varphi(r^j) \right) \frac{dr}{r} \right| 
\leq C \min \left\{ 1, \| \varphi(t)^j \| L_\infty(\xi) \cdot (y' - u') \|^{-1/2} \right\} .
\]
(116)

For $\| \varphi(t)^j \| L_\infty(\xi) > 1$, since $r/(\log r)^\beta$ is increasing in $(e^\beta, \infty)$, we have
\[
\left| \int_{t/2}^{t} \exp \left( -2\pi i \sum_{j=1}^{s} (L_j(\xi) \cdot (y' - u')) \varphi(r^j) \right) \frac{dr}{r} \right| 
\leq C \left( \frac{\log 2e^\beta \eta \cdot (y' - \theta)^{-1}}{\log |\varphi(t)^j| L_\infty(\xi) \beta} \right)^\gamma,
\]
(117)

where $\eta = \frac{L_j(\xi)}{L_\infty(\xi)}$. Combining (114), (115), and (117) with the fact that $\Omega \in W_{p,q}(S^{n-1})$ yields that
\[
\| \tilde{\sigma}_{\alpha,\gamma}(\xi) \| \leq C \left( \| \varphi(t)^j \| L_\infty(\xi) \right)^{-\beta/\max(2\gamma, 1)}.
\]
when $\| \varphi(t)^j \| L_\infty(\xi) > 1$. It follows from (118) that
\[
\left( \int_{2^j}^{2^{j+1}} \| \tilde{\sigma}_{\alpha,\gamma}(\xi) \|^2 \frac{dt}{t} \right)^{1/2} 
\leq C \left( \frac{\log \| \varphi(t)^j \| L_\infty(\xi) \beta}{\max(2\gamma, 1)} \right)^{-\beta/\max(2\gamma, 1)}
\]
(119)

when $\| \varphi(t)^j \| L_\infty(\xi) > 1$.

Take $\alpha_{k,\beta} = \varphi(t)^j$. By Remark 7 we have that $c_\beta ^j \geq \alpha_{k+1,\beta} / \alpha_{k,\beta} \geq B_k^\beta > 1$. It follows from (110)-(111), (113), (119), and Proposition 25 that
\[
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \left| \sigma_{\alpha,\gamma} \cdot \Delta_{\gamma,\xi} f \right| \frac{dt}{t} \right)^{1/2} \right)^q \right) \right\|_{L^q(\mathbb{R}^n)} 
\leq C \left\| f \right\|_{L^{\alpha,\gamma}(\mathbb{R}^n)}
\]
(120)

holds for $\beta > \max(2, \gamma')$, any $\alpha \in (0, 1)$, and all $(1/p, 1/q) \in \mathcal{R}_{p,q}^\beta$. Thus (120) together with (109) yields that
\[
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}^n} \left| \mathcal{M}_{\alpha,\gamma,\xi}(\Delta_{\gamma,\xi} f) \right| \frac{d\xi}{\gamma} \right)^q \right) \right\|_{L^q(\mathbb{R}^n)} 
\leq C \left\| f \right\|_{L^{\alpha,\gamma}(\mathbb{R}^n)}
\]
(121)

for any $x, \zeta \in \mathbb{R}^n$ and $\mathcal{M}_{\alpha,\gamma,\xi}(f) = \mathcal{M}_{\alpha,\gamma,\xi}(f - g)$

for arbitrary functions $f$ and $g$ defined on $\mathbb{R}^n$. By Lemma 21(i), we have
\[
\left\| \mathcal{M}_{\alpha,\gamma,\xi}(f - g) \right\|_{L^{\alpha,\gamma}(\mathbb{R}^n)} 
\leq C \left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}^n} \left| \Delta_{\gamma,\xi} \left( \mathcal{M}_{\alpha,\gamma,\xi}(f) \right) \right| \frac{d\xi}{\gamma} \right)^q \right) \right\|_{L^q(\mathbb{R}^n)}
\]
(124)

for all $\alpha \in (0, 1)$ and $(p, q) \in (1, \infty)^2$. Here $C = C_{n,\alpha,\gamma,\beta}$ is independent of the coefficients of $(P_j)_{j=1}^n$. On the other hand, one can easily check that
\[
\left| \Delta_{\zeta} \left( \mathcal{M}_{\alpha,\gamma,\xi}(f) \right)(x) \right| 
\leq \left| \mathcal{M}_{\alpha,\gamma,\xi}(\Delta_{\zeta}(f))(x) \right|
\]
(122)

for any $x, \zeta \in \mathbb{R}^n$ and
\[
\left| \mathcal{M}_{\alpha,\gamma,\xi}(f - g) \right| 
\leq \left| \mathcal{M}_{\alpha,\gamma,\xi}(f \cdot g) \right|
\]
(123)

for theorem 8(ii). Theorem 8(iii) and (iv) follow from the bounds for $\mathcal{M}_{\alpha,\gamma,\xi}$ and $\mathcal{M}_{\alpha,\gamma,\xi}(f)$. By Remark 7, we have that $\Delta_{\gamma,\xi} \left( \mathcal{M}_{\alpha,\gamma,\xi}(f) \right) = \mathcal{M}_{\alpha,\gamma,\xi}(\Delta_{\zeta}(f))$. The facts together with (121)-(123) yield Theorem 8(ii). Theorem 8(iii) and (iv) follow from the $L^p$ bounds for $\mu_{\alpha,\gamma,\xi}$ and (122)-(123).

Proof of Theorem 11. We first consider the operator $\delta_{\alpha,\gamma}$. One can easily check that
\[
\left| \Delta_{\zeta} \left( \delta_{\alpha,\gamma}(f) \right)(x) \right| 
\leq \left| \delta_{\alpha,\gamma}(\Delta_{\zeta}(f))(x) \right|
\]
(125)

for any $x, \zeta \in \mathbb{R}^d$.

By Lemma 21(i) and (125) we have
\[
\left\| \delta_{\alpha,\gamma}(f) \right\|_{L^{\alpha,\gamma}(\mathbb{R}^n)} 
\leq C \left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}^n} \left| \Delta_{\zeta}(\delta_{\alpha,\gamma}(f)) \right| \frac{d\xi}{\gamma} \right)^q \right) \right\|_{L^q(\mathbb{R}^n)}
\]
(126)

for all $\alpha \in (0, 1)$ and $(p, q) \in (1, \infty)^2$. Therefore, to prove Theorem 11(i) for $\delta_{\alpha,\gamma}$, it suffices to show that
\[
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}^n} \left| \delta_{\alpha,\gamma}(\Delta_{\zeta}(f)) \right| \frac{d\xi}{\gamma} \right)^q \right) \right\|_{L^q(\mathbb{R}^n)}
\leq C \left\| f \right\|_{L^{\alpha,\gamma}(\mathbb{R}^n)}
\]
(127)
for $\alpha \in (0,1)$ and $(1/p, 1/q)$ belonging to the set of all interiors of the convex hull of two squares $(1/\beta, 1/2)^2$ and $(1/2, 1-1/\beta)^2$. Here $C = C_{n,\alpha,p,q,\beta} \delta$ is independent of the coefficients of $\{P_{j}^{n}\}_{j=1}^{\infty}$.

Let $\Phi$, $L$, and $\Lambda$ be given as in the proof of Theorem 8. Define the family of measures $\{\sigma_{t,\lambda}\}_{t \in \mathbb{R}}$ and $\{|\sigma_{t,\lambda}|\}_{t \in \mathbb{R}}$ on $\mathbb{R}^n$ by

$$
\sigma_{t,\lambda}(x) = \int_{\mathbb{S}^{n-1}} e^{-2\pi i \phi(t, \xi) \cdot x} \Omega(y') d\sigma(y');
$$

$$
|\sigma_{t,\lambda}|(x) = \int_{\mathbb{S}^{n-1}} e^{-2\pi i \phi(t, \xi) \cdot x} \Omega(y') |d\sigma(y')|.
$$

(128)

By duality we have

$$
\delta_{t,\lambda} f(x) = \left( \int_{0}^{\infty} |\sigma_{t,\lambda} \ast f(x)|^2 \frac{dt}{t} \right) \frac{1}{2}.
$$

(129)

One can easily check that

$$
\alpha_{t,0} = 0,
$$

$$
\frac{1}{2} \left( \int_{2^k}^{2^{k+1}} |\sigma_{t,\lambda}(\xi) - \sigma_{t,\lambda,\beta}^{(k)}(\xi)|^2 \frac{dt}{t} \right)^{1/2}
\leq \frac{1}{2} \left( \int_{2^{k}}^{2^{k+1}} \max \left( 1, |\phi(t) \cdot L(\xi)| \right)^2 \frac{dt}{t} \right)^{1/2}
\leq C \min \left\{ 1, |\phi(2^{k+1}) \cdot L(\xi)| \right\}.
$$

(130)

On the other hand,

$$
|\sigma_{t,\lambda}(\xi)|^2 = \int_{\mathbb{S}^{n-1}} e^{-2\pi i \phi(t, \xi)y} \Omega(y') d\sigma(y')^2
$$

$$
= \left\| \Omega(y') d\sigma(y') \right\|^2.
$$

$$
\left( \sum_{j \in \mathbb{Z}} \left( \int_{2^k}^{2^{k+1}} |\sigma_{t,\lambda}(\xi) \ast g_{j,k}|^2 \frac{dt}{t} \right)^{1/2} \right)^{1/2}
\leq C \| \Omega \|_{L^1(\mathbb{S}^{n-1})}
\left( \sum_{j \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} |g_{j,k}|^2 \right\|_{L^1(\mathbb{R}^n)} \right)^{1/2}.
$$

(136)

It follows that

$$
\frac{1}{2} \left( \int_{2^k}^{2^{k+1}} |\sigma_{t,\lambda}(\xi)|^2 \frac{dt}{t} \right)^{1/2}
\leq \frac{1}{2} \left( \int_{2^{k}}^{2^{k+1}} \max \left( 1, |\phi(t) \cdot L(\xi)| \right)^2 \frac{dt}{t} \right)^{1/2}
\leq C \min \left\{ 1, |\phi(2^{k+1}) \cdot L(\xi)| \right\}.
$$

(133)

(134)

When $|\phi(2^{k+1}) \cdot L(\xi)| > 1$, since $t/(\log t)^{\beta} \delta$ is increasing in $(e^\delta, \infty)$, we have

$$
\left( \int_{2^k}^{2^{k+1}} \exp \left( -2\pi i \sum_{j=1}^{x} \phi(t)^j (L_j(\xi), \cdot \cdot \cdot (y') - \theta) \right) \frac{dt}{t} \right)
\leq C \left( \log 2 e^{\delta \beta} \min \left\{ 1, |\phi(2^{k+1}) \cdot L(\xi)|^{-\beta/2} \right\} \right).
$$

(135)

if $|\phi(2^{k+1}) \cdot L(\xi)| > 1$. By Lemma 19 we have

$$
\sum_{j \in \mathbb{Z}} \left( \int_{2^k}^{2^{k+1}} |\sigma_{t,\lambda}(\xi) \ast g_{j,k}|^2 \frac{dt}{t} \right)^{1/2}
\leq C \| \Omega \|_{L^1(\mathbb{S}^{n-1})}
\left( \sum_{j \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} |g_{j,k}|^2 \right\|_{L^1(\mathbb{R}^n)} \right)^{1/2}.
$$

(136)

Take $\alpha_{k,x} = q(2^{k+1})$. By Remark 7 we have that $c^2_{q} \geq \alpha_{k+1,1}$]

$$
\alpha_{k,x} \geq B_{q}^2 > 1.\text{ By (130), (135)-(136), and Proposition 25 we obtain that}
$$

for $(1/p, 1/q, 1/r)$ belonging to the interior of the convex hull of two cubes $(0, 1/2)^3$ and $(1/2, 1)^3$. Here $C > 0$ is independent of the coefficients of $\{P_{j}^{n}\}_{j=1}^{\infty}$.\]
for \( \beta > 2, \alpha \in (0, 1) \) and \((1/p, 1/q)\) belonging to the interior of the convex hull of two squares \((1/\beta, 1/2)^2 \) and \((1/2, 1 - 1/\beta)^2 \). Here \( C = C_{\alpha, \rho, \beta} \) is independent of the coefficients of \({P_j}\). Equation (137) together with (129) yields (127). By arguments similar to those used in deriving (3.14) and (3.16) in [31], one can obtain

\[
M_{\Omega, \Gamma, \rho, f}(x) \leq C \delta_{\Omega, \Gamma, f}(x) \quad \forall x \in \mathbb{R}^n. \tag{138}
\]

Thus (138) together with (127) yields that

\[
\left\| \left( \sum_{k \in \mathbb{Z}} 2^{kn} \left( \int_{\mathbb{R}^n} |M_{\Omega, \Gamma, \rho, f}(\Delta_{\xi} f)| \, d\xi \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^q(\mathbb{R}^n)} \tag{139}
\]

for \( \alpha \in (0, 1) \) and \((1/p, 1/q)\) belonging to the set of all interiors of the convex hull of two squares \((1/\beta, 1/2)^2 \) and \((1/2, 1 - 1/\beta)^2 \). Here \( C = C_{\alpha, \rho, \beta} \) is independent of the coefficients of \({P_j}\). One can easily check that

\[
|\Delta_{\xi} (M_{\Omega, \Gamma, \rho, f}(f))(x)| \leq |M_{\Omega, \Gamma, \rho, f}(\Delta_{\xi} f)(x)| \quad \forall x, \xi \in \mathbb{R}^n. \tag{140}
\]

By Lemma 21(i) and (140) we have

\[
\left\| M_{\Omega, \Gamma, \rho, f} \right\|_{L^q(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} 2^{kn} \left( \int_{\mathbb{R}^n} |M_{\Omega, \Gamma, \rho, f}(\Delta_{\xi} f)| \, d\xi \right)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \tag{141}
\]

for \( \alpha \in (0, 1) \) and \((p, q)\) \((1, \infty)^2 \). Then Theorem 1(i) follows from (126)-(127), (139), and (141).

It is known that both \( \delta_{\Omega, \Gamma} \) and \( M_{\Omega, \Gamma, \rho, f} \) are sublinear operators. Moreover, one can easily check that

\[
|\delta_{\Omega, \Gamma, f} - \delta_{\Omega, \Gamma, g}| \leq |\delta_{\Omega, \Gamma, f} - f| + |f - g|, \tag{142}
\]

\[
|M_{\Omega, \Gamma, \rho, f} - M_{\Omega, \Gamma, \rho, g}| \leq |M_{\Omega, \Gamma, \rho, f} - f| + |f - g| \tag{143}
\]

for arbitrary functions \( f, g \) defined on \( \mathbb{R}^n \). It follows from Remark 12 that

\[
\max \left\{ \left\| \delta_{\Omega, \Gamma, f} \right\|_{L^p(\mathbb{R}^n)}, \left\| M_{\Omega, \Gamma, \rho, f} \right\|_{L^p(\mathbb{R}^n)} \right\} \leq C \left\| f \right\|_{L^p(\mathbb{R}^n)} \tag{143}
\]

for \( p \in [2, \beta) \). It follows from (142)-(143), (127), (139), and Proposition 27 that Theorem 1(ii) holds. Theorem 1(iii)-(iv) follows from (125), (140), (142)-(143), and Proposition 26.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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