Hindawi
Journal of Function Spaces
Volume 2018, Article ID 7028597, 6 pages
https://doi.org/10.1155/2018/7028597

Research Article

On the Maximum Term and Central Index of Entire Functions and Their Derivatives

Jianren Long [1,2] **and Zhigao Qin** [1]

[1] *School of Mathematical Sciences, Guizhou Normal University, Guiyang 550001, China*
[2] *School of Computer Sciences and School of Sciences, Beijing University of Posts and Telecommunications, Beijing 100876, China*

Correspondence should be addressed to Jianren Long; longjianren2004@163.com

Received 4 January 2018; Revised 16 March 2018; Accepted 16 April 2018; Published 7 June 2018

Academic Editor: Raúl E. Curto

We shall establish some criteria on entire series with finite logarithmic order in terms of maximum term and central index.

1. Introduction

A function f is called meromorphic, if it is nonconstant and analytic in the complex plane $\mathbb{C}$ except at possible isolated poles. If no poles occur, then f reduces to an entire function. In what follows, we assume that the reader is familiar with the standard notation and fundamental results in Nevanlinna theory of meromorphic functions; see [1, 2] or [3] for more details. We often use the order of growth and the lower order of growth to measure the growth of a meromorphic function. For a meromorphic function f in $\mathbb{C}$, the order of growth and the lower order of growth of f are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \tag{1}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}, \tag{2}$$

respectively. If f is entire function in $\mathbb{C}$, the order and lower order of f are defined also by $\log M(r,f)$; i.e., $T(r,f)$ is replaced with $\log M(r,f)$ in above equalities, where $M(r,f) = \max_{|z|=r}|f(z)|$. By the following inequalities which can be found in [3, p. 10], then the order and lower order are same by definition of $T(r,f)$ and $\log M(r,f)$:

$$T(r,f) \le \log M(r,f) \le \frac{R+r}{R-r} T(R,f) \tag{3}$$

which hold for all $r < R$.

The theory of meromorphic functions of finite positive order is fairly complete as compared to the theory of functions of order zero. Techniques that work well for functions of finite positive order often do not work for functions of order zero. In order to make some progress with functions of order zero, Chern introduced the concept of logarithmic order in [4]. For an entire function f of zero order, the logarithmic order of f is defined by

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r}. \tag{4}$$

For an nonconstant entire function f, we must have $\rho_{\log}(f) \ge 1$, by the usual proof of Liouville's theorem. It is easily seen that if $f(z)$ has logarithmic order ρ then so has the function $f(az+b)$ for $a \ne 0$. Furthermore, the function $f(z)^n$ is again of logarithmic order ρ, while $f(z^n)$ has logarithmic order ρ. It is clear that for a polynomial of degree $k \ge 1$ the logarithmic order is 1. There exists also transcendental entire series $f(z)$

such that its logarithmic order is of one; for each positive number $c > 1$, put $r_n = e^{c^n}$, and set

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{r_1 r_2 \cdots r_n} z^n, \tag{5}$$

by a calculation [5, p. 6] we have $\rho_{\log}(f) = 1$; the example can also be found in [6]. On the other hand, there exists transcendental entire series $f(z)$ such that its logarithmic order is bigger than one. For each positive number $\zeta(>1)$, put $k = \zeta - 1 > 0$ and $r_n = e^{n^{1/k}}$, and set

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{r_1 r_2 \cdots r_n} z^n, \tag{6}$$

and by a direct calculation [5, p. 6] we have $\rho_{\log}(f) = \zeta > 1$; the example can also be found in [6]. Another case is of infinite logarithmic order; let $q \in \mathbb{C}$ and suppose that $0 < |q| < 1$; then

$$f(z) = \sum_{n=1}^{\infty} q^{n(\log n)^2} z^n \tag{7}$$

is of order zero, but its logarithmic order is infinite [7]. More results regarding logarithmic order can be found in [8–11].

Wiman-Valiron theory is one of the important concepts in entire function theory; in the present paper, we study the properties of entire functions by Wiman-Valiron theory. To this end, we also need the following notations. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire series in $\mathbb{C}$. Then the maximum term $\mu(r, f)$ and central index $\nu(r, f)$ of f are denoted as $\mu(r) \equiv \mu(r, f) = \max_{n \geq 0}\{|a_n| r^n\}$ and $\nu(r) \equiv \nu(r, f) = \max\{n : \mu(r, f) = |a_n| r^n\}$.

In [6], Chern and Kim consider some criteria conditions of logarithmic order with terms of maximum term and central index and proved the following consequence.

Theorem A. *Let f be a transcendental entire series with finite logarithmic order; then the following statements are equivalent.*

(1) *$T(r, f)$ has logarithmic order ρ.*

(2) *$\log M(r, f)$ has logarithmic order ρ.*

(3) *$\log \mu(r, f)$ has logarithmic order ρ.*

(4) *$\nu(r, f)$ has logarithmic order $\rho - 1$.*

Although, for any given entire series f of positive finite order, $\log \mu(r, f)$ and $\nu(r, f)$ both have the same order, the proof can be found in [12] or [2], but the situation is different for function of finite logarithmic order; from the Theorem A, we have

$$\begin{aligned} \rho_{\log}(f) - 1 &= \limsup_{r \to \infty} \frac{\log \log \mu(r, f)}{\log \log r} - 1 \\ &= \limsup_{r \to \infty} \frac{\log \nu(r, f)}{\log \log r}. \end{aligned} \tag{8}$$

In [7], Berg and Pedersen described the logarithmic order by using Taylor coefficient of entire function and obtained the following result.

Theorem B. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire series with finite logarithmic order; then its logarithmic order satisfies*

$$\rho_{\log}(f) = 1 + \limsup_{n \to \infty} \frac{\log n}{\log \log \left(1/\sqrt[n]{|a_n|}\right)}. \tag{9}$$

Now the purpose of the paper is that the logarithmic order is described by using other forms in terms of maximum term and central index. To the end, we also need the following notations. Let $\mu_k(r) = \mu(r, f^{(k)})$ and $\nu_k(r) = \nu(r, f^{(k)})$, where $f^{(k)}$, $k = 1, 2, \ldots$, denotes the kth-derivative of f. We reckon $\nu_k(r)$, $k = 0, 1, 2, \cdots$, from the first term of the series of f. For the uniformity in the notation we write $\mu_0(r) = \mu(r)$ and $\nu_0(r) = \nu(r)$. We denote the kth-derivative of $\mu(r)$ by $\mu^{(k)}(r)$ at the point of its existence in $(0, \infty)$. It is easily seen that the functions $\mu_k(r)$ and $\nu_k(r)$, $k = 1, 2, \ldots$, are positive, nondecreasing, and unbounded functions of r, having only ordinary discontinuities and $\nu_k(r) \geq \nu(r)$. For the entire function f, in the present paper, we find a precise measure of the rates of growth of $\mu_k(r)/\mu(r)$, $\mu^{(k)}(r)/\mu(r)$, and $\nu_k(r) - \nu(r)$ as $r \to \infty$ in terms of the parameters defined in (4). These results will be shown in Section 2.

This paper is organized as follows. In Section 2, we will state main results and prove them. In Section 3, we will discuss some further results.

2. Main Results and Proofs

In the proof of our theorems, the growth relationship between meromorphic function f and its kth-derivative is needed. We prove the following result by using similar way in [[8], Theorem 3] and then omit the proof of details.

Lemma 1. *Let f be a transcendental meromorphic function in $\mathbb{C}$ with zero order; then f and $f^{(k)}$, $k = 1, 2, \ldots$, have the same logarithmic order.*

The following two consequences are due to G. Valiron [13], which can also be found in [12] or [2].

Lemma 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function in $\mathbb{C}$. Then, for any $0 < r_0 < r$,*

$$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{\nu(t)}{t} dt. \tag{10}$$

Lemma 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function in $\mathbb{C}$. Then, for all $r < R$,*

$$M(r, f) < \mu(r) \left(\nu(R) + \frac{R}{R - r}\right). \tag{11}$$

The first result is stated as follows.

Theorem 4. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a transcendental entire series with finite logarithmic order ρ. Then*

$$\rho-1=\limsup_{r\to\infty}\frac{\log\left(r\{\mu_k(r)/\mu(r)\}^{1/k}\right)}{\log\log r},\qquad k=1,2,\ldots. \tag{12}$$

Proof. By Lemma 1, we know that the logarithmic order of a function and its derivative are the same. In view of (8), we have

$$\rho-1=\limsup_{r\to\infty}\frac{\log\nu_k(r,f)}{\log\log r},\quad k=0,1,2,\ldots. \tag{13}$$

Now, for $k=0,1,2,\ldots$, let $f^{(k)}(z)=\sum_{n=k}^{\infty}A_nz^{n-k}$, where $A_n=n(n-1)\cdots(n-k+1)a_n$, and let $\nu_k(r)=N$ and $\nu_{k+1}(r)=N_1$; then

$$\begin{aligned}\mu_{k+1}(r)&=(N_1-k)\left|A_{N_1}\right|r^{N_1-k-1}\\&=\frac{N_1-k}{r}\left|A_{N_1}\right|r^{N_1-k}\le\frac{\nu_{k+1}(r)-k}{r}\mu_k(r).\end{aligned} \tag{14}$$

It follows that

$$r\frac{\mu_{k+1}(r)}{\mu_k(r)}\le\nu_{k+1}(r)-k,\quad k=0,1,2,\ldots. \tag{15}$$

On the other hand, for $k=0,1,2,\ldots$, we have

$$\begin{aligned}\mu_k(r)&=\left|A_N\right|r^{N-k}=\frac{r}{N-k}(N-k)\left|A_N\right|r^{N-k-1}\\&\le\frac{r}{\nu_k(r)-k}\mu_{k+1}(r).\end{aligned} \tag{16}$$

This implies

$$r\frac{\mu_{k+1}(r)}{\mu_k(r)}\ge\nu_k(r)-k,\quad k=0,1,2,\ldots. \tag{17}$$

By using (15) and (17), we get

$$\nu_k(r)-k\le r\frac{\mu_{k+1}(r)}{\mu_k(r)}\le\nu_{k+1}(r)-k,\quad k=0,1,2,\ldots. \tag{18}$$

By (18) and a simple calculation, we have

$$\frac{\left[\nu_0(r)-(k-1)\right]^k}{r^k}\le\frac{\mu_k(r)}{\mu_0(r)}\le\frac{\nu_k(r)^k}{r^k},\quad k=1,2,\ldots. \tag{19}$$

This implies

$$\nu(r)-(k-1)\le r\left(\frac{\mu_k(r)}{\mu(r)}\right)^{1/k}\le\nu_k(r),\qquad k=1,2,\ldots. \tag{20}$$

From (20), we get

$$\begin{aligned}\frac{\log\left[\nu(r)-(k-1)\right]}{\log\log r}&\le\frac{\log\left(r\{\mu_k(r)/\mu(r)\}^{1/k}\right)}{\log\log r}\\&\le\frac{\log\nu_k(r)}{\log\log r},\quad k=1,2,\ldots\end{aligned} \tag{21}$$

Combining the inequality above and (13), we obtain

$$\limsup_{r\to\infty}\frac{\log\left(r\{\mu_k(r)/\mu(r)\}^{1/k}\right)}{\log\log r}=\rho-1,\qquad k=1,2,\ldots. \tag{22}$$

The proof of Theorem 4 is completed. □

Theorem 5. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a transcendental entire series with finite logarithmic order ρ. Let $\phi(r,k)=\nu_k(r)-\nu(r)$, $k=1,2,\ldots$. Then, for $0<r_0<r$, one has*

$$\rho-1=\limsup_{r\to\infty}\left\{\frac{1}{k}(\log\log r)^{-1}\int_{r_0}^{r}\frac{\phi(t,k)}{t}dt\right\},\qquad k=1,2,\ldots. \tag{23}$$

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$, then, for $k=1,2,\ldots$, we have

$$f^{(k)}(z)=\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)a_nz^{n-k} \tag{24}$$

and

$$\begin{aligned}&\mu_k(r)\\&=\nu_k(r)(\nu_k(r)-1)\cdots(\nu_k(r)-k+1)\left|a_{\nu_k(r)}\right|r^{\nu_k(r)-k}.\end{aligned} \tag{25}$$

The functions $\nu_k(r)$ and $|a_{\nu_k(r)}|$ are constants in intervals and have at most an enumerable number of discontinuities and so their derivatives vanish almost everywhere except possibly at a set of measure zero. Taking logarithmic order of both the sides of (25), differentiating with respect to r, and denoting the derivative of $\mu_k(r)$ by $\mu_k'(r)$ at the point of its existence, we have for almost all values of $r>r_0>0$

$$\frac{\mu_k'(r)}{\mu_k(r)}=\frac{\nu_k(r)-k}{r},\quad k=1,2,\ldots. \tag{26}$$

By (26), for sufficiently large r, we get

$$\log\mu_k(r)=\log\mu_k(r_0)+\int_{r_0}^{r}\frac{\nu_k(t)-k}{t}dt,\qquad k=1,2,\ldots. \tag{27}$$

By using Lemma 2, for $r>r_0>0$, we have

$$\log\mu(r)=\log\mu(r_0)+\int_{r_0}^{r}\frac{\nu(t)}{t}dt. \tag{28}$$

Combining (27) and (28), for sufficiently large r, we have

$$\log\frac{\mu_k(r)}{\mu(r)}=O(1)+\int_{r_0}^{r}\frac{\nu_k(t)-k-\nu(t)}{t}dt,\qquad k=1,2,\ldots; \tag{29}$$

i.e.,

$$\log\left(r\left\{\frac{\mu_k(r)}{\mu(r)}\right\}^{1/k}\right)=O(1)+\frac{1}{k}\int_{r_0}^{r}\frac{\nu_k(t)-\nu(t)}{t}dt,\qquad k=1,2,\ldots. \tag{30}$$

Thus, for $k = 1, 2, \ldots$,

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\log\left(r\{\mu_k(r)/\mu(r)\}^{1/k}\right)}{\log\log r} \\
&= \limsup_{r\to\infty} \frac{(1/k)\int_{r_0}^{r}((\nu_k(t)-\nu(t))/t)\,dt}{\log\log r} \\
&= \limsup_{r\to\infty} \left\{\frac{1}{k}(\log\log r)^{-1}\int_{r_0}^{r}\frac{\phi(t,k)}{t}dt\right\}.
\end{aligned} \tag{31}$$

In view of Theorem 4, (31) and (12) yield (23). This completes the proof of Theorem 5. □

Theorem 6. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire series with finite logarithmic order ρ. Then, for $k = 1, 2, \ldots$ and for almost all values of r satisfying $0 < r_0 < r$,*

$$r\frac{\mu^{(k)}(r)}{\mu^{(k-1)}(r)} = \nu(r) - k + 1 \tag{32}$$

and

$$\rho - 1 = \limsup_{r\to\infty, r\notin E} \frac{\log\left(r\{\mu^{(k)}(r)/\mu(r)\}^{1/k}\right)}{\log\log r}, \tag{33}$$

where E is set of r with zero measure which $\mu^{(k)}(r)$ does not exist.

Proof. Since $\nu(r)$ is step function with r, so $\mu(r) = |a_{\nu(r)}|r^{\nu(r)}$ is differential everywhere except at an enumerable set of points of discontinuities of $|a_{\nu(r)}|$ and $\nu(r)$. Hence, we have, at the points of existence of $\mu'(r)$,

$$\begin{aligned}
\mu'(r) &= \nu(r)\left|a_{\nu(r)}\right|r^{\nu(r)-1} = \frac{\nu(r)}{r}\left|a_{\nu(r)}\right|r^{\nu(r)} \\
&= \frac{\nu(r)}{r}\mu(r),
\end{aligned} \tag{34}$$

for the derivatives of $|a_{\nu(r)}|$ and $\nu(r)$ vanish almost everywhere.

This implies

$$r\frac{\mu'(r)}{\mu(r)} = \nu(r). \tag{35}$$

By differentiating (35) at the points of existence of $\mu'(r)$ and $\mu''(r)$, we get

$$\mu(r)\left[\mu'(r) + r\mu''(r)\right] - r\left(\mu'(r)\right)^2 = 0. \tag{36}$$

Combining (35) and (36), we have

$$r\frac{\mu''(r)}{\mu'(r)} = \nu(r) - 1. \tag{37}$$

On repeating the differentiation i times, we have

$$r\frac{\mu^{(i)}(r)}{\mu^{(i-1)}(r)} = \nu(r) - i + 1. \tag{38}$$

This proves (32).

Now, using (38), for $i = 1, 2, \ldots, k$, and then multiplying the k-inequalities thus obtained give

$$\begin{aligned}
r^k\frac{\mu^{(k)}(r)}{\mu(r)} &= \nu(r)(\nu(r)-1)\cdots(\nu(r)-k+1) \\
&= \nu(r)^k(1-o(1)).
\end{aligned} \tag{39}$$

Thus,

$$\frac{\log\left(r\{\mu^{(k)}(r)/\mu(r)\}^{1/k}\right)}{\log\log r} + o(1) = \frac{\log\nu(r)}{\log\log r}. \tag{40}$$

Combining (40) and (13), we get (33), on proceeding to limits as $r \to \infty$ and $r \notin E$. This completes the proof of Theorem 6. □

By the proof of Theorem 6, we can get the following.

Corollary 7. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire series. Then, for $k = 1, 2, \ldots$,*

$$\frac{\mu^{(k)}(r)}{\mu^{(k-1)}(r)} \sim \frac{\nu(r)}{r} \tag{41}$$

and

$$\frac{\mu^{(k)}(r)}{\mu(r)} \sim \left(\frac{\nu(r)}{r}\right)^k \tag{42}$$

as $r \to \infty$ and $r \notin E$.

3. Further Discussion

In [4], Chern introduced the definition of logarithmic order of meromorphic function; however, there are not discussions of the lower logarithmic order. Hence, by using similar definition of lower order, we can define the lower logarithmic order and discuss some properties of entire functions in terms of lower logarithmic order.

Definition 8. Let f be an entire function with zero order. Then its lower logarithmic order is defined by

$$\lambda_{\log}(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r}, \tag{43}$$

where $M(r,f) = \max_{|z|=r}|f(z)|$.

In Theorem A, Chern obtained the growth relationship of logarithmic order by using maximum modulus, maximum term, and central index. In this section, we will try to find the growth relationship of lower logarithmic order by using maximum modulus, maximum term, and central index. To this end, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire series and set

$$l_1 = \liminf_{r\to\infty} \frac{\log\log\mu(r,f)}{\log\log r}, \tag{44}$$

and

$$l_2 = \liminf_{r\to\infty} \frac{\log \nu(r, f)}{\log\log r}. \tag{45}$$

Then we have the following consequences.

Theorem 9. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire series with finite logarithmic order. Then $\lambda_{\log}(f) = l_1$.*

Proof. By using Cauchy inequality, we have $\mu(r, f) \le M(r, f)$ for all $r > 0$. Hence, $l_1 \le \lambda_{\log}(f)$. Thus we just only prove $l_1 \ge \lambda_{\log}(f)$. By Lemma 2, there exists $r_0 > 0$, such that, for all $r > r_0$,

$$\nu(r)\log 2 = \nu(r)\int_r^{2r} \frac{1}{t}dt \le \int_r^{2r} \frac{\nu(t)}{t}dt \le \log \mu(2r). \tag{46}$$

By Lemma 3 for $R = 2r$, we get

$$M(r, f) < \mu(r)(\nu(2r) + 2). \tag{47}$$

Combining (46) and (47), we have

$$\log M(r, f) < \log \mu(r) + \log\log \mu(4r) + C, \tag{48}$$

where C is positive constant. By (43) and (48), we have $l_1 \ge \lambda_{\log}(f)$. □

Theorem 10. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire series with finite logarithmic order. Then $l_1 - 1 \le l_2 \le l_1$.*

Proof. By using similar way of Theorem 9, we get (46); hence $l_2 \le l_1$ is obvious.

On the other hand, set $a = \max_n\{|a_n|\}$; then, we get $\mu(r) \le ar^{\nu(r)}$. Hence,

$$\log\log \mu(r) \le \log \nu(r) + \log\log r + C, \tag{49}$$

where C is positive constant number. This implies that $l_1 - 1 \le l_2$. So, we have $l_1 - 1 \le l_2 \le l_1$. □

Under Theorems 9 and 10, we will consider similar results of Section 2 with respect to lower logarithmic order. To this end, let L^* denote the class of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of transcendental entire function in $\mathbb{C}$, satisfying

$$\lambda_{\log}(f) - 1 = \liminf_{r\to\infty} \frac{\log \nu(r)}{\log\log r}. \tag{50}$$

By using similar way to Section 2, we can also prove the following consequences.

Theorem 11. *Suppose that f belongs to L^*. Then, for $k = 1, 2, \ldots$,*

$$\lambda_{\log}(f) - 1 = \liminf_{r\to\infty} \frac{\log\left(r\{\mu_k(r)/\mu(r)\}^{1/k}\right)}{\log\log r}. \tag{51}$$

Theorem 12. *Suppose that f belong to L^*. Then, for $0 < r_0 < r$ and $k = 1, 2, \ldots$,*

$$\begin{aligned}\lambda_{\log}(f) - 1 \\ = \liminf_{r\to\infty}\left\{\frac{1}{k}(\log\log r)^{-1}\int_{r_0}^{r} \frac{\phi(t,k)}{t}dt\right\}.\end{aligned} \tag{52}$$

Theorem 13. *Suppose that f belongs to L^*. Then, for $k = 1, 2, \ldots$,*

$$\lambda_{\log}(f) - 1 = \liminf_{r\to\infty, r\notin E} \frac{\log\left(r\{\mu^{(k)}(r)/\mu(r)\}^{1/k}\right)}{\log\log r}, \tag{53}$$

where E is set of r with zero measure where $\mu^{(k)}(r)$ does not exist.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The first author wrote the main part of the manuscript, and the second author pointed out many valuable ideas to modify the present manuscript. All authors read and approved the final manuscript.

Acknowledgments

This research work is supported by the National Natural Science Foundation of China (Grant no. 11501142), the Foundation of Science and Technology of Guizhou Province of China (Grant no. [2015]2112), the Foundation of Doctoral Research Program of Guizhou Normal University 2016, and the Foundation of Qian Ceng Ci Innovative Talents of Guizhou Province 2016.

References

[1] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, UK, 1964.

[2] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter De Gruyter, Berlin, Germany, 1993.

[3] L. Yang, *Value Distribution Theory*, Springer, Berlin, Germany, 1993.

[4] P. T. Chern, "On meromorphic functions with finite logarithmic order," *Transactions of the American Mathematical Society*, vol. 358, no. 2, pp. 473–489, 2006.

[5] G. Pólya and G. Szegö, *Problems and Theorem*, Springer-Verlag, 1972.

[6] P. T. Chern and I.-S. Kim, "Criteria of an entire series with finite logarithmic order," *Taiwanese Journal of Mathematics*, vol. 15, no. 1, pp. 331–336, 2011.

[7] C. Berg and H. L. Pedersen, "Logarithmic order and type of indeterminate moment problems," in *Difference equations, special functions and orthogonal polynomials*, pp. 51–79, World Sci. Publ., Hackensack, NJ, 2007.

[8] T. P. Chern, "Common Borel directions of a meromorphic function with zero order and its derivative," *Proceedings of the*

American Mathematical Society, vol. 132, no. 4, pp. 1171–1175, 2004.

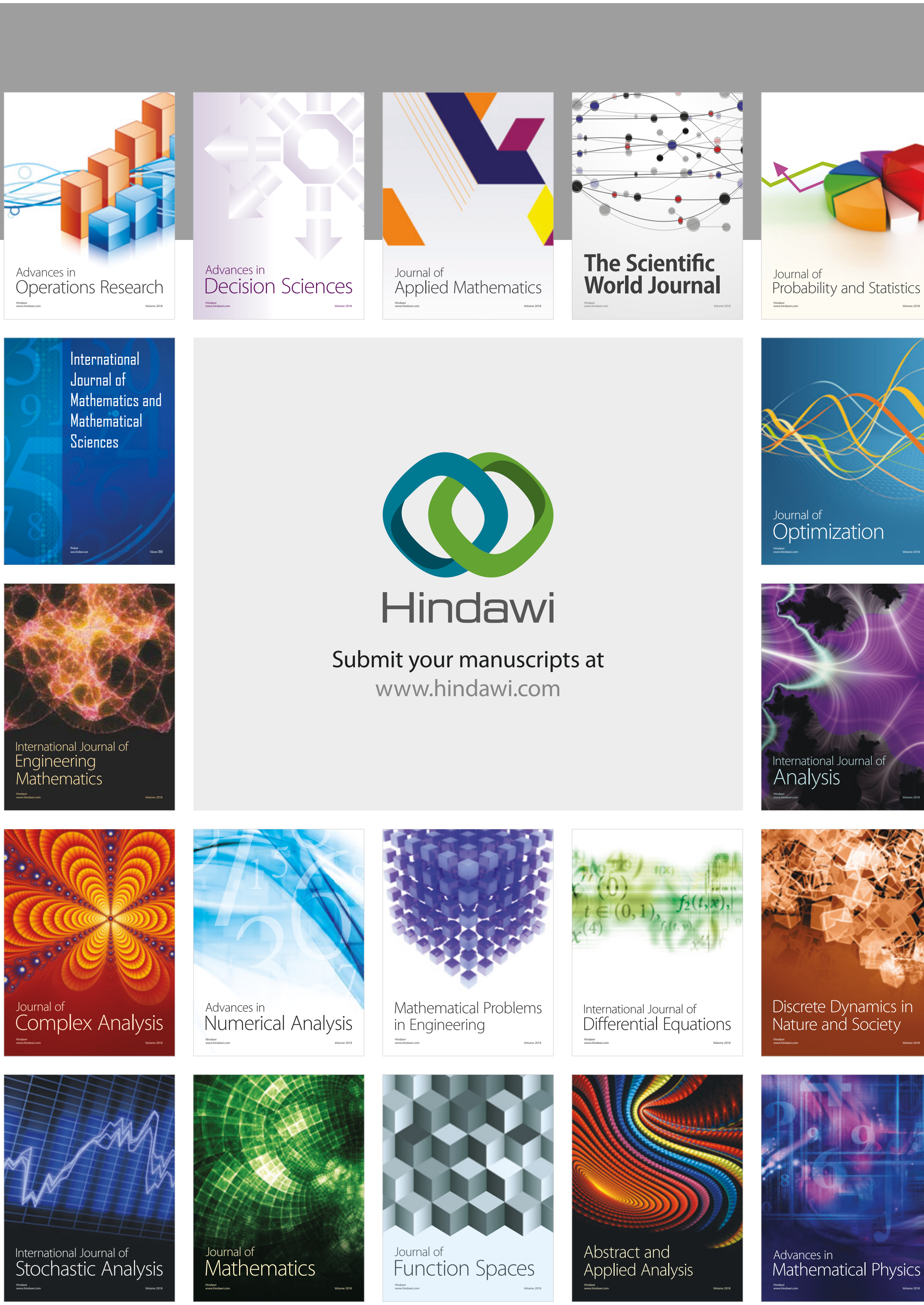
[9] H. L. Pedersen, "Logarithmic order and type of indeterminate moment problems," *Journal of Computational and Applied Mathematics*, vol. 233, no. 3, pp. 808–814, 2009.

[10] S. M. Shah and M. Ishaq, "On the maximum modulus and the coefficients of an entire series," *J. Indian Math. Sot*, vol. 16, pp. 177–182, 1952.

[11] G. S. Srivastava, "Approximation of entire functions of zero order," *Journal of Mathematical Analysis and Applications*, vol. 129, no. 1, pp. 142–152, 1988.

[12] Y. Z. He and X. Z. Xiao, *Algebroid Function and Ordinary Differential Equation*, Science Press, Beijing, 1988.

[13] G. Valiron, "Lectures on the General Theory of Integral Functions, Edouard privat," *Lectures on the General Theory of Integral Functions, Edouard privat*, 1923.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
The Scientific
World Journal
Journal of
Probability and Statistics
International
Journal of
Mathematics and
Mathematical
Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of
Optimization
International Journal of
Engineering
Mathematics
International Journal of
Analysis
Journal of
Complex Analysis
Advances in
Numerical Analysis
Mathematical Problems
in Engineering
International Journal of
Differential Equations
Discrete Dynamics in
Nature and Society
International Journal of
Stochastic Analysis
Journal of
Mathematics
Journal of
Function Spaces
Abstract and
Applied Analysis
Advances in
Mathematical Physics