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Research Article

Dynamics of Lotka-Volterra Competition Systems with Fokker-Planck Diffusion

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We consider competition systems of two species which have different dispersal strategies and interspecific competitive strengths. One of the dispersal strategies is random dispersal and the other is a Fokker-Planck diffusion whose motility is piecewise constant and jumps up when the resource is not enough. In this paper, first we show the Fokker-Planck diffusion allows ideal free distribution. Next we show the linear stability of semitrivial steady states is determined exactly by a threshold on the interspecific competitive strengths. Some conditions for coexistence and global asymptotic stability are also provided.

1. Introduction and Main Result

In this paper, we study Lotka-Volterra competition systems with Fokker-Planck diffusion of the following form:

$$\begin{aligned} u_t &= \Delta\left(D_1\left(s_1\right)u\right) + u\left[m\left(x\right) - u - b_2 v\right] && \text{in } \Omega, \\ v_t &= \Delta\left(D_2\left(s_2\right)v\right) + v\left[m\left(x\right) - b_1 u - v\right] && \text{in } \Omega, \\ \frac{\partial}{\partial \nu}\left(D_1\left(s_1\right)u\right) &= \frac{\partial}{\partial \nu}\left(D_2\left(s_2\right)v\right) = 0 \quad \text{on } \partial\Omega, \\ u\left(x,0\right) &= u_0\left(x\right) \geq 0, \\ v\left(x,0\right) &= v_0\left(x\right) \geq 0 && \text{in } \Omega, \end{aligned} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbf{R}^n$ and a given resource distribution $m(x)$ is a nonconstant C^2-function satisfying $\min_{\overline{\Omega}} m > 0$. The motility functions D_1, D_2 of Fokker-Planck diffusion are assumed to depend only on

$$\begin{aligned} s_1 &:= \frac{u + b_2 v}{m}, \\ s_2 &:= \frac{v}{m}, \end{aligned} \tag{2}$$

respectively. High values of s_1, s_2 mean a scarcity of resources; the difference between them is that s_1 depends on the competitive strength of other species while s_2 does not. The nonnegative constants b_1, b_2 are interspecific competitive strengths. The intraspecific competitive strengths are assumed to be one.

We consider two types of motility functions. The first one is the random dispersal (RD) where D_i's are constant. The other is based on the following step function:

$$\gamma^{(0)}\left(s\right) := \begin{cases} l, & 0 \leq s \leq 1, \\ h, & 1 < s < \infty, \end{cases} \tag{3}$$

where l and h are positive constants satisfying $l < h$. For a single species case, if we let $s = u/m$, the motility function

$\gamma^{(0)}$ models the situation where organisms drastically change their departing probability in random walk from l to h exactly when the resource becomes insufficient. This motility function is called *starvation-driven diffusion* (SDD) [1]. Unfortunately, due to the jump discontinuity, the Laplacian of $\gamma^{(0)}(s)$ is not well-defined. So we consider a regularized function $\gamma^{(\epsilon)}$ which is a *nondecreasing* smooth function such that $l \le \gamma^{(\epsilon)} \le h$ and

$$\gamma^{(\epsilon)}(s) = \begin{cases} l, & 0 \le s \le 1, \\ h, & 1+\epsilon < s < \infty \end{cases} \tag{4}$$

for given $\epsilon > 0$. For example, we may choose $\gamma^{(\epsilon)} := \gamma^{(0)} * \eta^{(\epsilon)}$ where $\eta^{(\epsilon)}$ is a standard mollifier supported in $(0, \epsilon)$.

The second motility function is given by this $\gamma^{(\epsilon)}$. In such cases we may choose $D_1(s_1) := \gamma^{(\epsilon)}(s_1)$ or $D_2(s_2) := \gamma^{(\epsilon)}(s_2)$. For simplicity we will call the first one SDD_1 and the second one SDD_2.

Our first main result shows why the SDD is worthy of notice; it is an efficient dispersal strategy helping the steady state be the *ideal free distribution* (IFD).

Theorem 1 (approximate IFD). *Suppose there is a unique positive solution $\theta_{\gamma^{(\epsilon)}}$ of*

$$\begin{aligned} \Delta\left(\gamma^{(\epsilon)}(s)\, u\right) + u\,[m-u] &= 0 \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu}\left(\gamma^{(\epsilon)}(s)\, u\right) &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where

$$s := \frac{u}{m}. \tag{6}$$

If the resource distribution m satisfies

$$\frac{\max_{\overline{\Omega}} m}{\min_{\overline{\Omega}} m} < \frac{h}{l}, \tag{7}$$

then

$$\left\| m - \theta_{\gamma^{(\epsilon)}} \right\|_{L^\infty(\overline{\Omega})} = O\left(\epsilon^{1/(\sqrt{n}+1)^2}\right) \quad \text{as } \epsilon \longrightarrow 0. \tag{8}$$

The uniqueness assumption in Theorem 1 can be replaced by some sufficient conditions. For example, a condition on the resource distribution,

$$\max_{\overline{\Omega}} m \le 2 \min_{\overline{\Omega}} m, \tag{9}$$

guarantees the uniqueness [2, Theorem 3(i)]. There are other sufficient conditions restricting the shape of $\gamma^{(\epsilon)}$ [2, Theorem 2] or the shape of m [2, Theorem 3(ii)].

Condition (7) seems not to be a real restriction because it is necessary even when there is no logistic growth term. In [2, Theorem 5], with no logistic growth term, it was shown that

$$\left\| m - \theta_{\gamma^{(\epsilon)}} \right\|_{L^\infty(\overline{\Omega})} = O(\epsilon) \quad \text{as } \epsilon \longrightarrow 0 \tag{10}$$

under assumption (7) although their choice of $\gamma^{(\epsilon)}$ is explicit and slightly different from ours; for example, they assume $\gamma^{(\epsilon)}(1) = (h+l)/2$ but we assume $\gamma^{(\epsilon)}(1) = l$. Furthermore it was suggested that condition (7) seems almost optimal [2, Remark 3].

With the logistic term the same IFD property (10) is proved in [2, Theorem 6]. But it needs a restrictive condition on m [2, equation (62)] and relies on the explicit form of $\gamma^{(\epsilon)}$. Although our decay order in (8) is worse than the order in (10), our theorem does not have such restrictions.

In the sense of Theorem 1, the motility function $\gamma^{(\epsilon)}$ gives a best possible dispersal strategy in single species case; the steady state population distribution $\theta_{\gamma^{(\epsilon)}}$ is very close to the resource distribution and furthermore the steady state is globally asymptotically stable [2, Theorem 2].

Next we consider competitions between two dispersal strategies among RD, SDD_1, and SDD_2. For instance, to observe the competition between SDD_1 and RD, we let $D_1 := \gamma^{(\epsilon)}(s_1)$ and $D_2 := d$ for some positive constant d in (1). Then system (1) can have several steady states such as the trivial steady state ($u \equiv 0, v \equiv 0$) and semitrivial steady states $(\theta_{\gamma^{(\epsilon)}}, 0)$ and $(0, \theta_d)$. Here $\theta_{\gamma^{(\epsilon)}}$ is a positive solution of (5) and θ_d is a unique positive solution of

$$\begin{aligned} d\Delta\theta_d + \theta_d\left[m - \theta_d\right] &= 0 \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu}\theta_d &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{11}$$

Inspecting the linear stability of each steady state, we can understand which dispersal strategy is more advantageous.

In the competition between SDD_1 and RD, the linear stability of the semitrivial steady state $(0, \theta_d)$, which means RD prevails and SDD_1 goes extinct, changes exactly across a threshold $b_2^* > 1$ on RD's interspecific competitive strength. Such a threshold exists for the linear stability of $(\theta_{\gamma^{(\epsilon)}}, 0)$ but it is one so less than b_2^*. This suggests that SDD_1 gains an advantage over RD since the RD needs more interspecific competitive strength to prevail. In the competition between RDs, the RD could prevail with smaller interspecific competitive strength $b_2 < 1$ [3, Theorem 1.1].

Theorem 2 (SDD_1 versus RD). *Assume the hypotheses in Theorem 1 and let $D_1 := \gamma^{(\epsilon)}(s_1)$ with $s_1 := (u + b_2 v)/m$ and $D_2 := d > 0$. Then we obtain the following:*

(i) *Assume ϵ is sufficiently small. There exists a constant b_2^* such that the semitrivial steady state $(0, \theta_d)$ is linearly stable if $b_2 > b_2^*$ and is linearly unstable if $b_2 < b_2^*$. Furthermore,*

$$1 < b_2^* \le \frac{\max_{\overline{\Omega}} m}{\min_{\overline{\Omega}} m}. \tag{12}$$

(ii) *Assume ϵ is sufficiently small. If $b_1 > 1$, then $(\theta_{\gamma^{(\epsilon)}}, 0)$ is linearly stable. If $b_1 < 1$, then $(\theta_{\gamma^{(\epsilon)}}, 0)$ is linearly unstable.*

(iii) *When $b_1 \ge 1 \ge b_2$, $b_1 > b_2$, and $l \le d$, there is no positive coexistence steady state for any ϵ.*

In the competition between RDs, the one with smaller diffusivity always prevails [4, 5] if $b_1 = b_2 = 1$. Also even if b_2 is slightly less than one (v has weak strength), a lower diffusivity of v (v has strong dispersal strategy) can overcome the disadvantage and make v prevail [3].

But in the competition between SDD_1 and RD, Theorem 2 (i) shows that RD cannot prevail no matter how small diffusivity it has if $b_2 < b_2^*$. When $b_1 = b_2 = 1$, the same observation was given in [6, Theorem 1]. Furthermore Theorem 2 (ii) claims that SDD_1 loses its advantage as soon as b_1 is smaller than one. This means that low interspecific competitive strength of SDD_1 cannot be overcome by the dispersal strategy, contrary to the RD case. This is unexpected since the SDD_1 seems to be more competitive than RD; the SDD_1 can build up the ideal free distribution while the RD cannot.

In the competition between RD and SDD_2, the dynamics is similar to the case of Theorem 2 but SDD_2 is less competitive than SDD_1.

Theorem 3 (RD versus SDD_2). *Assume the hypotheses in Theorem 1 and let $D_1 := d > 0$ and $D_2 := \gamma^{(\epsilon)}(s_2)$ with $s_2 := v/m$. Then we obtain the following:*

(i) *Assume ϵ is sufficiently small. There exists a constant b_1^* such that the semitrivial steady state $(\theta_d, 0)$ is linearly stable if $b_1 > b_1^*$ and is linearly unstable if $b_1 < b_1^*$. Furthermore,*

$$\frac{\int_\Omega m}{\int_\Omega \theta_d} \le b_1^* \le \frac{\max_{\overline{\Omega}} m}{\min_{\overline{\Omega}} m}. \tag{13}$$

(ii) *Assume ϵ is sufficiently small. If $b_2 > 1$, then $(0, \theta_{\gamma^{(\epsilon)}})$ is linearly stable. If $b_2 < 1$, then $(0, \theta_{\gamma^{(\epsilon)}})$ is linearly unstable.*

(iii) *With the b_1^* in (i), suppose either* (a) *$b_1 < b_1^*$ and $b_2 < 1$ or* (b) *$b_1 > b_1^*$ and $b_2 > 1$. Then there is a coexistence steady state for all sufficiently small ϵ.*

(iv) *If $b_1 < 1 < b_2$ and $l \le d$, then $(0, \theta_{\gamma^{(\epsilon)}})$ is globally asymptotically stable for all sufficiently small ϵ.*

Since $\int_\Omega m / \int_\Omega \theta_d < 1$ for any $d > 0$ [7, Theorem 1.2], b_1^* is not necessarily bigger than one. So smaller interspecific competitive strength is enough for RD to prevail against SDD_2 than the strength required against SDD_1 is.

By Theorem 3 (iv), we can see that $b_1^* \ge 1$ when $l \le d$. When $b_1 = b_2 = 1$, it is known that if $l < d$, $(0, \theta_{\gamma^{(\epsilon)}})$ is globally asymptotically stable; if $l > d$, $(\theta_d, 0)$ is linearly stable [8, Theorem 2.1].

Finally, we consider the competition between SDD_1 and SDD_2.

Theorem 4 (SDD with s_1 versus SDD with s_2). *Assume the hypotheses in Theorem 1 and let $D_1 := \gamma^{(\epsilon)}(s_1) > 0$ and $D_2 := \gamma^{(\epsilon)}(s_2)$ where $s_1 := (u + b_2 v)/m$; $s_2 := v/m$. Then for all sufficiently small ϵ, we obtain the following:*

(i) *If $b_1 > 1$, then $(\theta_{\gamma^{(\epsilon)}}, 0)$ is linearly stable. If $b_1 < 1$, then $(\theta_{\gamma^{(\epsilon)}}, 0)$ is linearly unstable.*

(ii) *If $b_2 > 1$, then $(0, \theta_{\gamma^{(\epsilon)}})$ is linearly stable. If $b_2 < 1$, then $(0, \theta_{\gamma^{(\epsilon)}})$ is linearly unstable.*

(iii) *If $b_1 \ge 1 > b_2$, then there is no coexistence steady state.*

Although we expect SDD_1 to be more competitive than SDD_2, our linear stability analysis does not indicate it. We solve system (1) numerically to observe it in the next section (see Figure 3).

The proofs of the theorems are given in the subsequent sections.

2. Numerical Simulation and Conjecture

In this section, by numerically solving the competition system (1), we observe how competitive the dispersal strategies RD, SDD_1, and SDD_2 are.

We fix the following parameters:

$$\begin{aligned} \Omega &= (0, \pi) \subset \mathbf{R}, \\ m(x) &= 2\sin x + 1, \\ l &= 1, \\ h &= 4, \\ \epsilon &= 0.1. \end{aligned} \tag{14}$$

To construct the motility function $\gamma^{(\epsilon)}$, we choose the following mollifier whose support is $[0, 1]$:

$$\eta(y) := \begin{cases} \exp \dfrac{\left[-1/\left(1-(1-2y)^2\right)\right]}{I} & \text{if } 0 < y < 1, \\ 0 & \text{otherwise}, \end{cases} \tag{15}$$

where $I \simeq 0.221997$ is the normalization constant which makes $\int \eta = 1$. Let $\eta^{(\epsilon)}(y) := \eta(y/\epsilon)/\epsilon$ and $a(s) := \min\{(s-1)_+/\epsilon, 1\}$. Then our motility function is given by

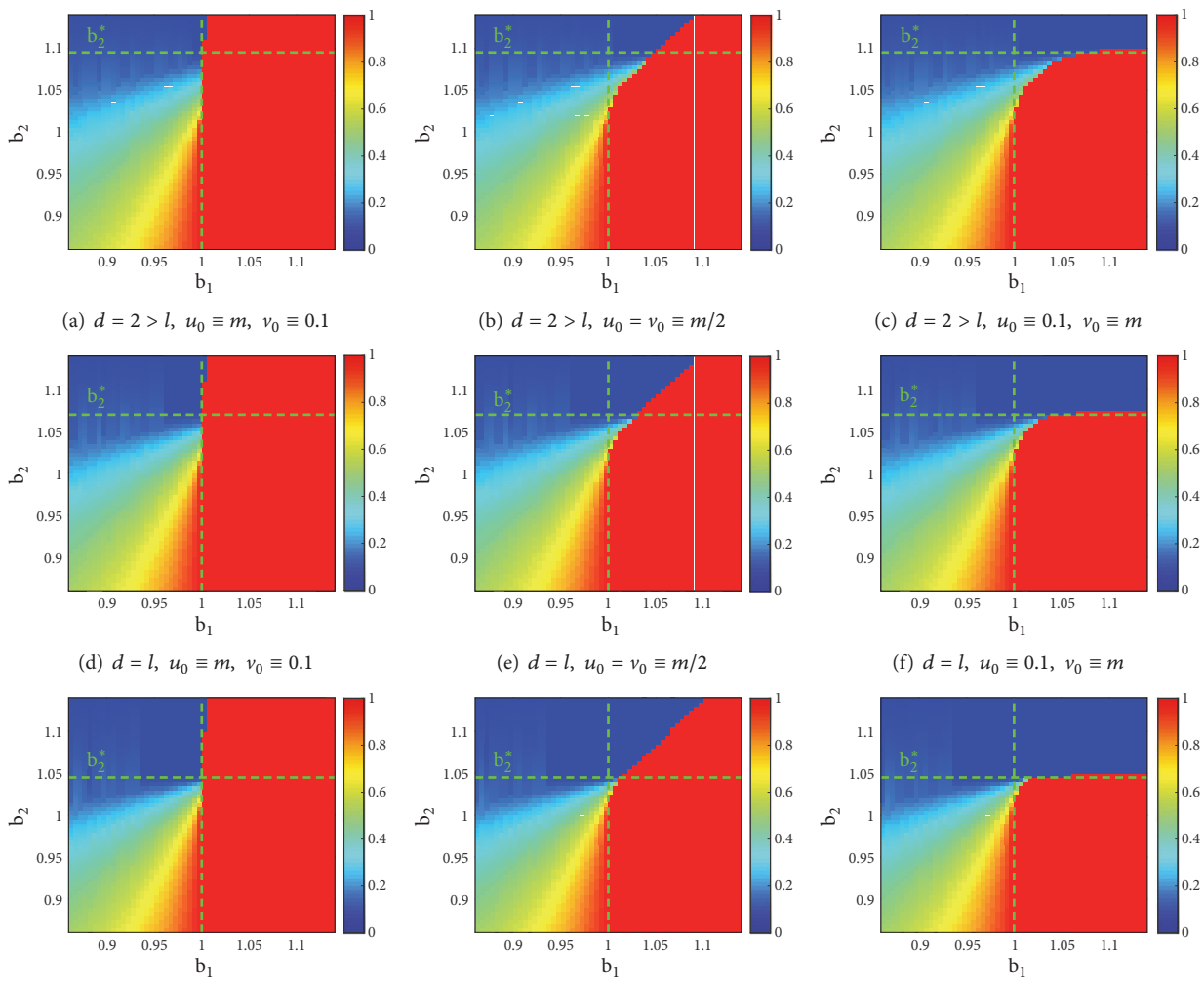
$$\gamma^{(\epsilon)}(s) := \left(\gamma^0 * \eta^{(\epsilon)}\right)(s) = h\int_0^{a(s)} \eta + l\int_{a(s)}^1 \eta \tag{16}$$

and $(d/ds)\gamma^{(\epsilon)}(s) = (h-l)\eta^{(\epsilon)}(s-1)$.

To numerically solve the competition system (1), we use the pdepe function of MATLAB [9] which implements a method of lines [10] with 100 evenly spaced spatial grid points. The mass ratio of u, $\int u(t)dx / \int (u+v)(t)dx$, at $t = 1000$ is plotted in Figures 1, 2, and 3 for the situations of Theorems 2, 3, and 4, respectively.

When the interspecific competitive strengths are small ($b_1, b_2 \ll 1$), the opponents are relatively weak so two species can coexist (see [3, 7, 11, 12] for other models). Numerical simulation shows in our case the theoretically claimed constants b_2^*, b_1^* or one play as the "smallness" thresholds of the interspecific competitive strengths for coexistence. For example, in Figure 1, two species coexist if $b_1 < 1$ and $b_2 < b_2^*$.

Combining our analysis and numerical observation, we give some conjectures.



(a) $d = 2 > l,\ u_0 \equiv m,\ v_0 \equiv 0.1$ (b) $d = 2 > l,\ u_0 = v_0 \equiv m/2$ (c) $d = 2 > l,\ u_0 \equiv 0.1,\ v_0 \equiv m$

(d) $d = l,\ u_0 \equiv m,\ v_0 \equiv 0.1$ (e) $d = l,\ u_0 = v_0 \equiv m/2$ (f) $d = l,\ u_0 \equiv 0.1,\ v_0 \equiv m$

(g) $d = 0.5 < l,\ u_0 \equiv m,\ v_0 \equiv 0.1$ (h) $d = 0.5 < l,\ u_0 = v_0 \equiv m/2$ (i) $d = 0.5 < l,\ u_0 \equiv 0.1,\ v_0 \equiv m$

Figure 1: SDD_1 versus RD: in the situation of Theorem 2, the mass ratio $\int u(t)dx / \int (u+v)(t)dx$ of numerical solutions is plotted at $t = 1000$ with the parameters in (14). In the red and blue region SDD_1 and RD prevail, respectively. In any cases the SDD_1 survives near $b_1 = b_2 = 1$ even if RD is stronger than SDD_1 ($b_2 > b_1$).

Conjecture. For simplicity, we ignore unstable coexistence steady states. Assume ϵ is sufficiently small. Then we conjecture the following:

(1) When SDD_1 competes with RD as in Theorem 2 (see Table 1).

(2) When RD competes with SDD_2 as in Theorem 3 (see Table 2).

(3) When SDD_1 competes with SDD_2 as in Theorem 4 (see Table 3).

Table 1

	$b_1 < 1$	$b_1 > 1$
$b_2 > b_2^*$	RD prevails	
$1 < b_2 < b_2^*$	coexist	SDD_1 survives
$b_2 < 1$	coexist	SDD_1 prevails

Table 2

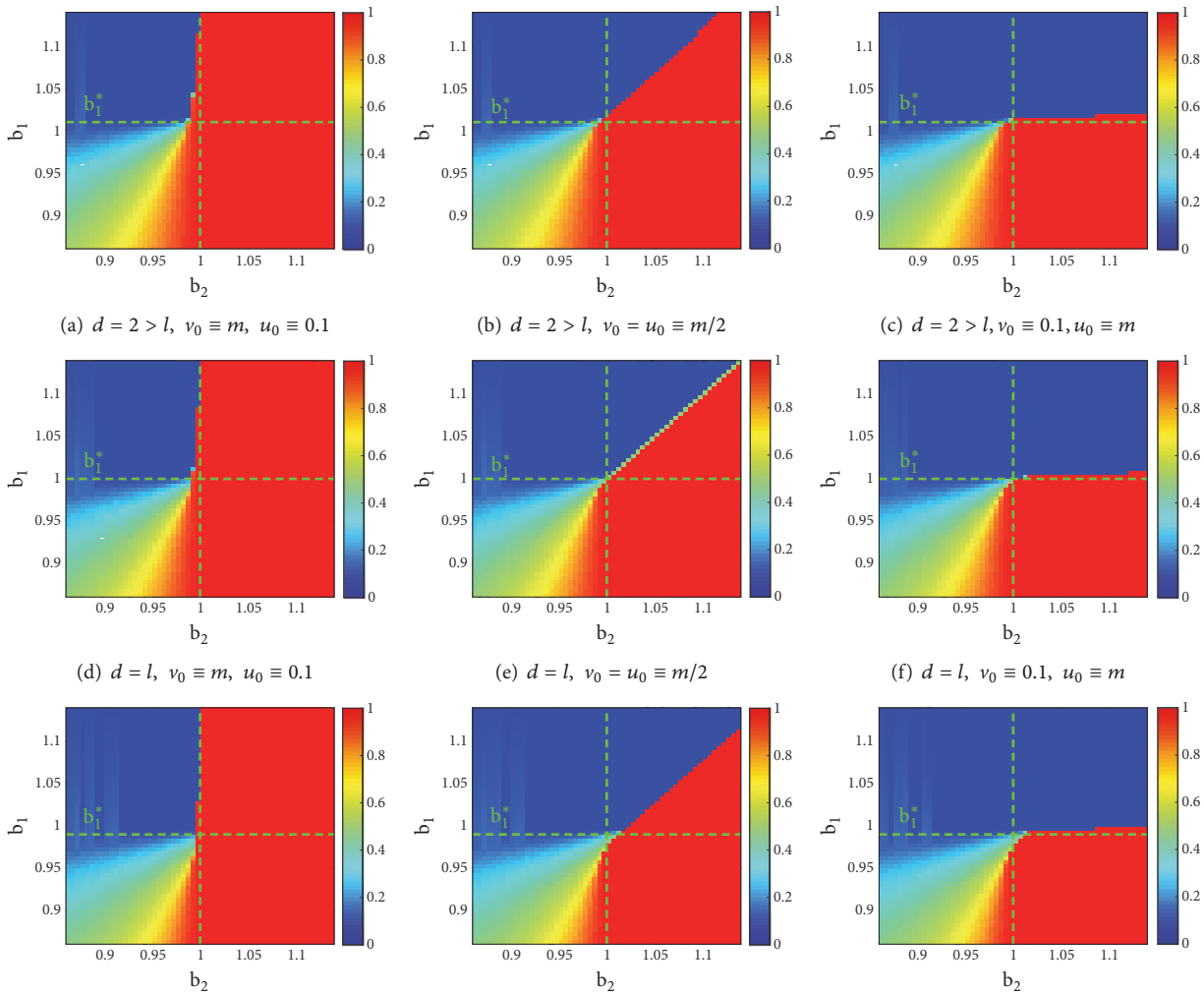
	$b_2 < 1$	$b_2 > 1$
$b_1 > b_1^*$	RD prevails	
$b_1 < b_1^*$	coexist	SDD_2 prevails

3. Proof of Theorem 1

To prove Theorem 1, we need some lemmas. In the rest of the paper, we omit the ϵ-dependence in proofs if there is no confusion.

Lemma 5 (approximation from above). *Assume the hypotheses in Theorem 1. Then*

$$\theta_{\gamma^{(\epsilon)}} \leq (1+\epsilon)\, m. \tag{17}$$

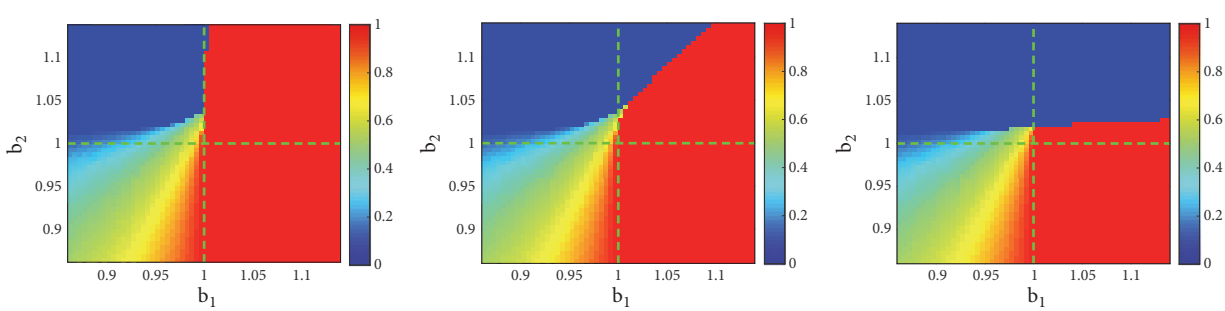


(a) $d = 2 > l,\ v_0 \equiv m,\ u_0 \equiv 0.1$ (b) $d = 2 > l,\ v_0 = u_0 \equiv m/2$ (c) $d = 2 > l, v_0 \equiv 0.1, u_0 \equiv m$

(d) $d = l,\ v_0 \equiv m,\ u_0 \equiv 0.1$ (e) $d = l,\ v_0 = u_0 \equiv m/2$ (f) $d = l,\ v_0 \equiv 0.1,\ u_0 \equiv m$

(g) $d = 0.5 < l,\ v_0 \equiv m,\ u_0 \equiv 0.1$ (h) $d = 0.5 < l,\ v_0 = u_0 \equiv m/2$ (i) $d = 0.5 < l,\ v_0 \equiv 0.1,\ u_0 \equiv m$

Figure 2: RD versus SDD_2: in the situation of Theorem 3, the mass ratio $\int v(t)dx / \int (u+v)(t)dx$ of numerical solutions is plotted at $t = 1000$ with the parameters in (14). In the red and blue region SDD_2 and RD prevail, respectively. Unlike Figure 1, the SDD_2 can go extinct near $b_1 = b_2 = 1$ if $d < l$.



(a) $u_0 \equiv m,\ v_0 \equiv 0.1$ (b) $u_0 = v_0 \equiv m/2$ (c) $u_0 \equiv 0.1,\ v_0 \equiv m$

Figure 3: SDD_1 versus SDD_2: in the situation of Theorem 4, the mass ratio $\int u(t)dx / \int (u+v)(t)dx$ of numerical solutions is plotted at $t = 1000$ with the parameters in (14). In the red and blue region SDD_1 and SDD_2 prevail, respectively. The SDD_1 is advantageous since it survives near $b_1 = b_2 = 1$.

Table 3

	$b_1 < 1$	$b_1 > 1$
$b_2 > 1$	SDD_2 survives	
$b_2 < 1$	coexist	SDD_1 prevails

Proof. Define a map U by $U(u) := \gamma(u/m)u$. Because γ is nondecreasing, U is *strictly* increasing in u so it is invertible. Now define $\overline{U} := l \max_{\overline{\Omega}} m$. Then

$$U(m) = \gamma\left(\frac{m}{m}\right)m = lm \le \overline{U}. \tag{18}$$

Let $\overline{u} := U^{-1}(\overline{U}) > 0$. Then by the monotonicity of U, the above inequality gives $m \le \overline{u}$. Furthermore $\overline{u}$ is a supersolution of (5) because

$$\Delta\overline{U} + \overline{u}[m - \overline{u}] = \overline{u}[m - \overline{u}] \le 0 \tag{19}$$

and $(\partial/\partial\nu)(\overline{U}) = 0$. Similarly, we can show that if we choose $\underline{U} := \delta$ for a sufficiently small $\delta > 0$ then $\underline{u} := U^{-1}(\underline{U})$ is a subsolution of (5) and $\underline{u} \le \overline{u}$. Because the positive solution θ_γ is unique, by the super- and subsolution method,

$$\underline{u} \le \theta_\gamma \le \overline{u}. \tag{20}$$

By the second inequality,

$$\gamma\left(\frac{\theta_\gamma}{m}\right)\theta_\gamma = U\left(\theta_\gamma\right) \le \overline{U} = l \max_{\overline{\Omega}} m. \tag{21}$$

Also from [2, Lemma 1(i)] we know

$$\min_{\overline{\Omega}} m \le \theta_\gamma \le \max_{\overline{\Omega}} m. \tag{22}$$

Now suppose $\theta_\gamma(x_0) > (1+\epsilon)m(x_0)$ at some point $x_0 \in \overline{\Omega}$. Then, at that point, $\theta_\gamma/m > 1+\epsilon$ so by (4),

$$\gamma\left(\frac{\theta_\gamma}{m}\right)\theta_\gamma = h\theta_\gamma \ge h \min_{\overline{\Omega}} m. \tag{23}$$

Combining (21) and (23), we have

$$\frac{\max_{\overline{\Omega}} m}{\min_{\overline{\Omega}} m} \ge \frac{h}{l}, \tag{24}$$

which contradicts to assumption (7). Therefore $\theta_\gamma \le (1+\epsilon)m$. □

Lemma 6 (approximation from below). *Assume the hypotheses in Theorem 1 and fix a number δ such that $0 < \delta < 1$. Then the Lebesgue measure of the set*

$$\Omega^{(\epsilon)} := \left\{x \in \Omega : \left(1 - \epsilon^\delta\right)m > \theta_{\gamma^{(\epsilon)}}\right\} \tag{25}$$

has a bound

$$\left|\Omega^{(\epsilon)}\right| \le C\epsilon^{1-\delta} \tag{26}$$

for some positive constant C independent of ϵ.

Proof. By [2, Lemma 1(ii)], we know

$$\int_\Omega m \le \int_\Omega \theta_\gamma. \tag{27}$$

By definition of $\Omega^{(\epsilon)}$ and Lemma 5, the right-hand side is estimated by

$$\begin{aligned}\int_\Omega \theta_\gamma &= \int_{\Omega^{(\epsilon)}} \theta_\gamma + \int_{\Omega\setminus\Omega^{(\epsilon)}} \theta_\gamma \\ &< \int_{\Omega^{(\epsilon)}} \left(1 - \epsilon^\delta\right)m + \int_{\Omega\setminus\Omega^{(\epsilon)}} (1+\epsilon)m.\end{aligned} \tag{28}$$

Combining the above inequalities, we have

$$\int_{\Omega^{(\epsilon)}} m < \epsilon^{1-\delta} \int_{\Omega\setminus\Omega^{(\epsilon)}} m. \tag{29}$$

Now the conclusion follows easily. □

Now we are ready to prove the IFD property.

Proof of Theorem 1. (1) First we claim that $\nabla\theta_\gamma$ has a uniform bound independent of ϵ. Because θ_γ is a solution of Neumann problem (5), by the global Lipschitz regularity for Poisson's equation (see, e.g., [13, Theorem 1.3]), we have

$$\begin{aligned}&\left\|\nabla\left(\gamma(s)\theta_\gamma\right)\right\|_{L^\infty(\Omega)} \le C\left\|\theta_\gamma\left(m - \theta_\gamma\right)\right\|_{L^\infty(\Omega)}, \\ &\qquad s := \frac{\theta_\gamma}{m},\end{aligned} \tag{30}$$

where C is a positive constant independent of ϵ. Because θ_γ has a uniform bound independent of ϵ, so does $\nabla(\gamma(s)\theta_\gamma)$. Furthermore, from the computation

$$\nabla\left(\gamma(s)\theta_\gamma\right) = \gamma\nabla\theta_\gamma + \gamma'\theta_\gamma\nabla\left(\frac{\theta_\gamma}{m}\right), \tag{31}$$

we have

$$\left(\underbrace{\gamma + \gamma'\frac{\theta_\gamma}{m}}_{\ge l}\right)\nabla\theta_\gamma = \nabla\left(\gamma\theta_\gamma\right) + \gamma'\left(\frac{\theta_\gamma}{m}\right)^2\nabla m. \tag{32}$$

From this relation, we can observe that $\nabla\theta_\gamma$ also has a uniform bound as claimed; if we divide the both sides by $\gamma + \gamma'(\theta_\gamma/m)$, then we can easily see that the gradient has a bound.

(2) Next we claim that

$$\left\|m - \theta_\gamma\right\|^p_{L^p(\Omega)} = O\left(\epsilon^{p/(p+1)}\right) \quad \text{as } \epsilon \longrightarrow 0. \tag{33}$$

By Lemmas 5 and 6, for sufficiently small ϵ, we have

$$\left|m - \theta_\gamma\right| \le \epsilon^\delta m \quad \text{in } \Omega \setminus \Omega^{(\epsilon)}. \tag{34}$$

Hence

$$\begin{aligned}\int_\Omega \left|m - \theta_\gamma\right|^p &= \int_{\Omega^{(\epsilon)}} \left|m - \theta_\gamma\right|^p + \int_{\Omega\setminus\Omega^{(\epsilon)}} \left|m - \theta_\gamma\right|^p \\ &\le \left(2\max_{\overline{\Omega}} m\right)^p \left|\Omega^{(\epsilon)}\right| + \epsilon^{\delta p}\int_\Omega m^p \\ &\le C\left(\epsilon^{1-\delta} + \epsilon^{\delta p}\right).\end{aligned} \tag{35}$$

By choosing $1 - \delta = \delta p$ or $\delta = 1/(p+1)$, we obtain the claim.

(3) Finally by Gagliardo-Nirenberg interpolation inequality [14, Theorem 5.9],

$$\left\|m-\theta_\gamma\right\|_{L^\infty(\Omega)} \le C\left\|m-\theta_\gamma\right\|_{W^{1,\infty}(\Omega)}^{n/(n+p)}\left\|m-\theta_\gamma\right\|_{L^p(\Omega)}^{p/(n+p)} \tag{36}$$

for any $p \ge 1$. Applying the previous claims, we have

$$\left\|m-\theta_\gamma\right\|_{L^\infty(\Omega)} \le C\epsilon^{p/(p+1)(n+p)}. \tag{37}$$

The exponent of ϵ is maximized when $p = \sqrt{n}$ and the proof is complete. □

4. Proof of Theorem 2

First we introduce the diffusion pressure U for the first species u:

$$U := \gamma(s_1)u, \quad s_1 = \frac{u+b_2v}{m}. \tag{38}$$

Define

$$\mathcal{G}(m,u,v,U) := \gamma\left(\frac{u+b_2v}{m}\right)u - U. \tag{39}$$

Then for given m, v, and U, the function u can be completely determined by the equation $\mathcal{G} \equiv 0$ since γ is a nondecreasing function. Hence, we may write $u = u(m,v,U)$, and (1) is rewritten as

$$\begin{aligned}
U_t &= \frac{\partial\mathcal{G}}{\partial u}\{\Delta U + u[m-u-b_2v]\} \\
&\quad + \frac{\partial\mathcal{G}}{\partial v}\{d\Delta v + v[m-b_1u-v]\} \quad \text{in } \Omega, \\
v_t &= d\Delta v + v[m-b_1u-v] \quad \text{in } \Omega, \\
\frac{\partial}{\partial\nu}U &= \frac{\partial}{\partial\nu}v = 0 \quad \text{on } \partial\Omega, \\
U(x,0) &= U_0(x) \ge 0, \\
v(x,0) &= v_0(x) \ge 0 \\
&\qquad\qquad \text{in } \Omega,
\end{aligned} \tag{40}$$

where $U_0 := \gamma((u_0+b_2v_0)/m)u_0$.

Let (u^e, v^e) be a steady state solution of (1). The stability of (u^e, v^e) is equivalent to that of the steady state (U^e, v^e) of (40) for $U^e := \gamma((u^e+b_2v^e)/m)u^e$. Let $U = U^e + \Phi$ and $v = v^e + \psi$ with $|\Phi|$ and $|\psi|$ small. Then,

$$\begin{aligned}
u &= u(m, v^e+\psi, U^e+\Phi) \\
&= u^e + \frac{\partial u}{\partial v}\psi + \frac{\partial u}{\partial U}\Phi + \text{higher order terms}
\end{aligned} \tag{41}$$

and we have the following linearized eigenvalue problem:

$$\begin{aligned}
\lambda\Phi &= \frac{\partial\mathcal{G}}{\partial u}\left(\Delta\Phi + (m-2u^e-b_2v^e)\frac{\partial u}{\partial U}\Phi\right. \\
&\quad \left. + \left[(m-2u^e-b_2v^e)\frac{\partial u}{\partial v} - b_2u^e\right]\psi\right) + \frac{\partial\mathcal{G}}{\partial v}\left(d\Delta\psi\right. \\
&\quad \left. + \left(m-b_1u^e-2v^e-b_1v^e\frac{\partial u}{\partial v}\right)\psi - b_1v^e\frac{\partial u}{\partial U}\Phi\right) \\
&\qquad\qquad \text{in } \Omega, \\
\lambda\psi &= d\Delta\psi + \left(m-b_1u^e-2v^e-b_1v^e\frac{\partial u}{\partial v}\right)\psi \\
&\quad - b_1v^e\frac{\partial u}{\partial U}\Phi \quad \text{in } \Omega, \\
\frac{\partial}{\partial\nu}\Phi &= \frac{\partial}{\partial\nu}\psi = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{42}$$

At the semitrivial steady state $(0, \theta_d)$,

$$\begin{aligned}
\frac{\partial u}{\partial U} &= \frac{1}{\gamma(b_2s_d)}, \\
\frac{\partial u}{\partial v} &= 0, \\
\frac{\partial\mathcal{G}}{\partial u} &= \gamma(b_2s_d), \\
\frac{\partial\mathcal{G}}{\partial v} &= 0,
\end{aligned} \tag{43}$$

where

$$s_d := \frac{\theta_d}{m}. \tag{44}$$

So the eigenvalue value problem for (40) at $(0, \theta_d)$ is

$$\begin{aligned}
\lambda\Phi &= \gamma(b_2s_d)\left[\Delta\Phi + \frac{m-b_2\theta_d}{\gamma(b_2s_d)}\Phi\right] \quad \text{in } \Omega, \\
\lambda\psi &= d\Delta\psi + (m-2\theta_d)\psi - \frac{b_1\theta_d}{\gamma(b_2s_d)}\Phi \quad \text{in } \Omega, \\
\frac{\partial}{\partial\nu}\Phi &= \frac{\partial}{\partial\nu}\psi = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{45}$$

Also consider the other semitrivial steady state $(\theta_\gamma, 0)$ and define

$$U_\gamma := \gamma(s_\gamma)\theta_\gamma, \quad s_\gamma := \frac{\theta_\gamma}{m}. \tag{46}$$

Then, similarly, the eigenvalue problem linearized at $(U_\gamma, 0)$ is

$$\begin{aligned}
\lambda\Phi &= \left(\gamma\left(s_\gamma\right) + \gamma'\left(s_\gamma\right)s_\gamma\right) \\
&\quad \cdot \left[\Delta\Phi + \frac{m - 2\theta_\gamma}{\gamma\left(s_\gamma\right) + \gamma'\left(s_\gamma\right)s_\gamma}\Phi - b_2\theta_\gamma\psi\right] \\
&\quad + b_2\gamma'\left(s_\gamma\right)s_\gamma\left[d\Delta\psi + \left(m - b_1\theta_\gamma\right)\psi\right] \quad \text{in } \Omega, \\
\lambda\psi &= d\Delta\psi + \left(m - b_1\theta_\gamma\right)\psi \quad \text{in } \Omega, \\
\frac{\partial}{\partial\nu}\Phi &= \frac{\partial}{\partial\nu}\psi = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{47}$$

We will use the following stability criteria for semitrivial steady states. The proof is similar to that of [6, Lemma 1], but we give it here for completeness.

Lemma 7. *Let θ_γ be unique positive steady states of (5).*

(i) *Let μ_1 be the first eigenvalue of the eigenvalue problem*

$$\begin{aligned}
\mu\phi &= \Delta\phi + \frac{m - b_2\theta_d}{\gamma\left(b_2 s_d\right)}\phi \quad \text{in } \Omega, \\
\frac{\partial}{\partial\nu}\phi &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{48}$$

If $\mu_1 > 0$, then the semitrivial steady state $(0, \theta_d)$ of (1) is linearly unstable. If $\mu_1 < 0$, then $(0, \theta_d)$ is linearly stable.

(ii) *Let ν_1 be the first eigenvalue of the eigenvalue problem*

$$\begin{aligned}
\nu\psi &= d\Delta\psi + \left(m - b_1\theta_\gamma\right)\psi \quad \text{in } \Omega, \\
\frac{\partial}{\partial\nu}\psi &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{49}$$

If $\nu_1 > 0$, then the semitrivial steady state $(\theta_\gamma, 0)$ is linearly unstable. If $\nu_1 < 0$, $(\theta_\gamma, 0)$ is linearly stable.

Proof. (i) First suppose $\mu_1 > 0$. For a given λ, consider the eigenvalue problem

$$\begin{aligned}
\sigma\Phi &= \Delta\Phi + \frac{m - b_2\theta_d}{\gamma\left(b_2 s_d\right)}\Phi - \frac{\lambda}{\gamma\left(b_2 s_d\right)}\Phi \quad \text{in } \Omega, \\
\frac{\partial}{\partial\nu}\Phi &= 0 \quad \text{on } \partial\Omega
\end{aligned} \tag{50}$$

and denote its first eigenvalue by σ_1^λ. Then $\sigma_1^0 = \mu_1 > 0$ by (48). Moreover, σ_1^λ will be negative when λ is large enough. Thus, there exists $\lambda_1 > 0$ such that $\sigma_1^{\lambda_1} = 0$ and the corresponding eigenfunction Φ_1 satisfies the first equation in (45) with $\lambda = \lambda_1$. Now due to (11), it is easy to see that the operator

$$\mathscr{L} := d\Delta + \left(m - 2\theta_d\right) - \lambda_1 : C^{2+\alpha}\left(\overline{\Omega}\right) \longrightarrow C^\alpha\left(\overline{\Omega}\right) \tag{51}$$

is invertible. Hence there is a solution of $\mathscr{L}\psi = (b_1\theta_d/\gamma(b_2 s_d))\Phi_1$ and it satisfies the second equation in (45). This yields that $\lambda_1 > 0$ is an eigenvalue of the linearized problem (45), which implies that $(0, \theta_d)$ is linearly unstable.

Next, assume that $\mu_1 < 0$. Suppose that the linearized problem (45) has a nonnegative eigenvalue $\lambda_1 \geq 0$ with the corresponding eigenfunctions (Φ_1, ψ_1). Since θ_d is a positive solution of (11), the operator $\mathscr{L} := d\Delta + (m - 2\theta_d)$ has only strictly negative eigenvalues. But if $\Phi_1 \equiv 0$, by the second equation in (45), the operator $\mathscr{L}$ has a nonnegative eigenvalue λ_1, which is a contradiction. Hence $\Phi_1 \not\equiv 0$. However, by the Rayleigh quotient of (48) and (45), this yields that

$$\begin{aligned}
&\mu_1 \\
&= \sup_{\phi\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega\left[-\left|\nabla\phi\right|^2 + \left(\left(m - b_2\theta_d\right)/\gamma\left(b_2 s_d\right)\right)\phi^2\right]}{\int_\Omega \phi^2} \\
&\geq \frac{\int_\Omega\left[-\left|\nabla\Phi_1\right|^2 + \left(\left(m - b_2\theta_d\right)/\gamma\left(b_2 s_d\right)\right)\Phi_1^2\right]}{\int_\Omega \Phi_1^2} \\
&= \frac{\int_\Omega\left(\lambda_1/\gamma\left(b_2 s_d\right)\right)\Phi_1^2 dx}{\int_\Omega \Phi_1^2 dx} \geq 0.
\end{aligned} \tag{52}$$

This is a contradiction. Therefore, $(0, \theta_d)$ is linearly stable.

(ii) Consider the linearized eigenvalue problem of (5) at U_γ, which is written as

$$\lambda\Phi = \left(\Delta + \left(m - 2\theta_\gamma\right)\frac{\partial u}{\partial U}\right)\Phi. \tag{53}$$

Since θ_γ is a unique positive steady state, it is linearly neutrally stable (in fact it is globally asymptotically stable [2, Theorem 2]). Hence every eigenvalue of the linear operator $\Delta + (m - 2\theta_\gamma)(\partial u/\partial U)$ is nonpositive. Thus, if $\nu_1 > 0$, because of $\partial\mathscr{G}/\partial u = \gamma(s_\gamma) + \gamma'(s_\gamma)s_\gamma > 0$, the operator

$$\mathscr{L} := \Delta + \left(m - 2\theta_\gamma\right)\frac{\partial u}{\partial U} - \nu_1\left(\frac{\partial\mathscr{G}}{\partial u}\right)^{-1} \tag{54}$$

has strictly negative eigenvalues and thus it is invertible. Let $\{\nu_1, \psi_1\}$ be an eigenpair of (49) and Φ_1 be the solution of

$$\begin{aligned}
\mathscr{L}\Phi_1 &= b_2\theta_\gamma\psi_1 \\
&\quad - \left(\frac{\partial\mathscr{G}}{\partial u}\right)^{-1}\frac{\partial\mathscr{G}}{\partial\nu}\left\{d\Delta\psi_1 + \left(m - b_1\theta_\gamma\right)\psi_1\right\}.
\end{aligned} \tag{55}$$

Then, $\{\nu_1, (\phi_1, \Psi_1)\}$ is an eigenpair of the linearized problem (47), which implies that $(U_\gamma, 0)$ is linearly unstable.

Since an eigenvalue of (47) is also an eigenvalue of (49), eigenvalues of (47) are all strictly negative if $\nu_1 < 0$ and hence $(U_\gamma, 0)$ is linearly stable. □

Now we are ready to prove Theorem 2. Due to Lemma 7, it is sufficient to compute the sign of the first eigenvalues μ_1 and ν_1.

Proof of Theorem 2. (i) Recall

$$\mu_1 = \sup_{\phi\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[-|\nabla\phi|^2 + \left((m-b_2\theta_d)/\gamma(b_2 s_d)\right)\phi^2\right]}{\int_\Omega \phi^2}. \quad (56)$$

When $b_2 = 1$, by assumption (7), we can show that $\mu_1 > 0$ (see the proof of [6, Theorem 1]). On the other hand, because $\gamma(s)$ is a nondecreasing function, $(m - b_2\theta_d)/\gamma(b_2 s_d)$ and μ_1 decrease strictly as b_2 increases. By the continuity of μ_1, there exists a constant $b_2^* > 1$ such that $(0,\theta_d)$ is linearly unstable for all $b_2 < b_2^*$ and $(0,\theta_d)$ is linearly stable for all $b_2 > b_2^*$.

Furthermore, since $\min_{\overline{\Omega}} m \le \min_{\overline{\Omega}} \theta_d < \max_{\overline{\Omega}} \theta_d \le \max_{\overline{\Omega}} m$ (one may check this using the maximum principle), when $b_2 \ge \max_{\overline{\Omega}} m/\min_{\overline{\Omega}} m$, $m - b_2\theta_d \le 0$ in Ω. Thus, $\mu_1 \le 0$ for all $b_2 \ge \max_{\overline{\Omega}} m/\min_{\overline{\Omega}} m$. This implies that $b_2^* \le \max_{\overline{\Omega}} m/\min_{\overline{\Omega}} m$.

(ii) Recall

$$\nu_1 = \sup_{\psi\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[-d|\nabla\psi|^2 + \left(m - b_1\theta_\gamma\right)\psi^2\right]}{\int_\Omega \psi^2}. \quad (57)$$

By Theorem 1, $\|m-\theta_\gamma\|_{L^\infty(\overline{\Omega})} \longrightarrow 0$ as $\epsilon \longrightarrow 0$. Therefore, for a given $b_1 > 1$, there is a $\delta > 0$ such that

$$m - b_1\theta_\gamma = m - \theta_\gamma - (b_1 - 1)\theta_\gamma \le -\delta \quad \text{for any sufficiently small } \epsilon. \quad (58)$$

Hence for any $\psi \in H^1(\Omega)\setminus\{0\}$,

$$\frac{\int_\Omega \left[-d|\nabla\psi|^2 + \left(m - b_1\theta_\gamma\right)\psi^2\right]}{\int_\Omega \psi^2} \le -\delta, \quad (59)$$

which implies $\nu_1 < 0$.

Similarly, for a given $b_1 < 1$, there is a $\delta > 0$ such that

$$m - b_1\theta_\gamma = m - \theta_\gamma + (1 - b_1)\theta_\gamma \ge \delta \quad \text{for any sufficiently small } \epsilon. \quad (60)$$

Then with $\psi_1 := 1$,

$$\begin{aligned}\nu_1 &= \sup_{\psi\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[-d|\nabla\psi|^2 + \left(m - b_1\theta_\gamma\right)\psi^2\right]}{\int_\Omega \psi^2} \\ &\ge \frac{\int_\Omega \left[-d|\nabla\psi_1|^2 + \left(m - b_1\theta_\gamma\right)\psi_1^2\right]}{\int_\Omega \psi_1^2} \ge \delta > 0.\end{aligned} \quad (61)$$

(iii) Suppose there is a positive coexistence steady state (u,v). Then by (40),

$$\begin{aligned}\Delta U + \frac{m - u - b_2 v}{\gamma\left((u+b_2 v)/m\right)} U &= 0 \quad \text{in } \Omega, \\ \frac{\partial}{\partial\nu} U &= 0 \quad \text{on } \partial\Omega.\end{aligned} \quad (62)$$

Considering the Rayleigh quotient of this equation, we have

$$0 = \sup_{\phi\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[-|\nabla\phi|^2 + \left((m-u-b_2 v)/\gamma\left((u+b_2 v)/m\right)\right)\phi^2\right]}{\int_\Omega \phi^2}. \quad (63)$$

Similarly we also have

$$\int_\Omega (m - b_1 u - v)\, v^2 = \int_\Omega d|\nabla v|^2. \quad (64)$$

Choosing $\phi = v$ in (63),

$$\begin{aligned}0 &\ge \int_\Omega \left[-|\nabla v|^2 + \frac{m-u-b_2 v}{\gamma\left((u+b_2 v)/m\right)} v^2\right] \\ &= \int_\Omega \left[-\frac{1}{d}(m - b_1 u - v)\, v^2 + \frac{1}{\gamma\left((u+b_2 v)/m\right)}(m-u-b_2 v)\, v^2\right] \text{ by (64)} \\ &= \int_\Omega \left[\left(\frac{1}{l} - \frac{1}{d}\right)(m - b_1 u - v)\, v^2 + \left(\frac{1}{\gamma\left((u+b_2 v)/m\right)} - \frac{1}{l}\right)(m-u-b_2 v)\, v^2 + \frac{1}{l}\left\{(b_1 - 1)u + (1 - b_2)v\right\} v^2\right]. \\ &> \int_\Omega \left(\frac{1}{\gamma\left((u+b_2 v)/m\right)} - \frac{1}{l}\right)(m-u-b_2 v)\, v^2 \\ &\qquad \text{by the assumption and (64)}.\end{aligned} \quad (65)$$

Now observe the sign of the integrand. If $s_1 = (u + b_2 v)/m \le 1$, $\gamma(s_1) = l$ so the integrand vanishes. Hence the integral is actually taken over the set where $(u + b_2 v)/m > 1$, and then the last integral is positive. This is a contradiction so there is no coexistence steady state. □

5. Proof of Theorem 3

In this section, we consider the following system:

$$\begin{aligned}u_t &= d\Delta u + u\left[m(x) - u - b_2 v\right] \quad \text{in } \Omega, \\ v_t &= \Delta\left(\gamma(s_2)\, v\right) + v\left[m(x) - b_1 u - v\right] \quad \text{in } \Omega, \\ \frac{\partial}{\partial\nu} u &= \frac{\partial}{\partial\nu}\left(\gamma(s_2)\, v\right) = 0 \quad \text{on } \partial\Omega, \\ u(x,0) &= u_0(x) \ge 0, \\ v(x,0) &= v_0(x) \ge 0 \quad \text{in } \Omega,\end{aligned} \quad (66)$$

where $s_2 := v/m$. We introduce the diffusion pressure V for the second species v:

$$V := \gamma(s_2)\, v,$$
$$V_\gamma := \gamma(s_\gamma)\, \theta_\gamma, \tag{67}$$
$$s_\gamma := \frac{\theta_\gamma}{m}$$

where θ_γ is a unique positive solution of (5). Define

$$\mathcal{G}(m, v, V) := \gamma(s_2)\, v - V \tag{68}$$

then

$$\frac{\partial \mathcal{G}}{\partial v} = \gamma(s_2) + \gamma'(s_2)\, s_2 > 0 \tag{69}$$

and (66) is rewritten as

$$\begin{aligned} u_t &= d\Delta u + u[m - u - b_2 v] \quad \text{in } \Omega, \\ V_t &= \frac{\partial \mathcal{G}}{\partial v}\{\Delta V + v[m - b_1 u - v]\} \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu} u &= \frac{\partial}{\partial \nu} V = 0 \quad \text{on } \partial\Omega, \\ u(x,0) &= u_0(x) \geq 0, \\ V(x,0) &= V_0(x) \geq 0 \quad \text{in } \Omega, \end{aligned} \tag{70}$$

where $V_0 := \gamma(v_0/m)v_0$. The corresponding eigenvalue value problems linearized at the semitrivial steady states $(\theta_d, 0)$ and $(0, V_\gamma)$ are

$$\begin{aligned} \lambda\phi &= d\Delta\phi + (m - 2\theta_d)\phi - \frac{b_2}{l}\theta_d \Psi \quad \text{in } \Omega, \\ \lambda\Psi &= l\Delta\Psi + (m - b_1\theta_d)\Psi \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu}\phi &= \frac{\partial}{\partial \nu}\Psi = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{71}$$

and

$$\begin{aligned} \lambda\phi &= d\Delta\phi + (m - b_2\theta_\gamma)\phi \quad \text{in } \Omega, \\ \lambda\Psi &= (\gamma(s_\gamma) + \gamma'(s_\gamma)\, s_\gamma) \\ &\quad \cdot \left\{\Delta\Psi + \frac{m - 2\theta_\gamma}{\gamma(s_\gamma) + \gamma'(s_\gamma)\, s_\gamma}\Psi - b_1\theta_\gamma\phi\right\} \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu}\phi &= \frac{\partial}{\partial \nu}\Psi = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{72}$$

respectively.

Omitting the proof, we use the following stability criteria for semitrivial steady states; the proof is similar to the one of Lemma 7.

Lemma 8. *Let θ_γ be unique positive steady states of (5).*

(i) *Let μ_1 be the first eigenvalue of*

$$\begin{aligned} \mu\Psi &= l\Delta\Psi + (m - b_1\theta_d)\Psi \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu}\Psi &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{73}$$

If $\mu_1 > 0$, then the semitrivial steady state $(\theta_d, 0)$ of (66) is linearly unstable. If $\mu_1 < 0$, then $(\theta_d, 0)$ is linearly stable.

(ii) *Let ν_1 be the first eigenvalue of*

$$\begin{aligned} \nu\phi &= d\Delta\phi + (m - b_2\theta_\gamma)\phi \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu}\phi &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{74}$$

If $\nu_1 > 0$, then $(0, \theta_\gamma)$ is linearly unstable. If $\nu_1 < 0$, $(0, \theta_\gamma)$ is linearly stable.

Because system (66) has no cross-diffusion (a cross-diffusion is a Fokker-Planck diffusion depending on both u and v), we can prove that it is a strongly monotone dynamical system.

Lemma 9 (monotonicity). *Let (u, v) and $(\tilde{u}, \tilde{v})$ be solutions of (66) with initial data (u_0, v_0) and $(\tilde{u}_0, \tilde{v}_0)$, respectively. If $u_0(x) \leq \tilde{u}_0(x)$ and $v_0(x) \geq \tilde{v}_0(x)$, then $u(x,t) < \tilde{u}(x,t)$ and $v(x,t) > \tilde{v}(x,t)$.*

We omit the proof; it is a slight modification of the proof in [8, pp.266–267].

Now we give a proof Theorem 3. Due to Lemma 8, it suffices to check the sign of μ_1 and ν_1 for the linear stability.

Proof of Theorem 3 . (i) Since

$$\mu_1 = \sup_{\Psi \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[-l|\nabla\Psi|^2 + (m - b_1\theta_d)\Psi^2\right]}{\int_\Omega \Psi^2}, \tag{75}$$

μ_1 decreases strictly as b_1 increases. If $b_1 \geq \max_{\overline{\Omega}} m / \min_{\overline{\Omega}} m$, then $m - b_1\theta_d \leq 0$ and $\mu_1 \leq 0$.

If $b_1 < \int_\Omega m / \int_\Omega \theta_d$, choosing $\Psi \equiv 1$ in the above Rayleigh quotient, we obtain

$$\mu_1 \geq \frac{1}{|\Omega|}\int_\Omega m - b_1\theta_d > 0. \tag{76}$$

Therefore, there exists a constant b_1^* such that $\int_\Omega m / \int_\Omega \theta_d \leq b_1^* \leq \max_{\overline{\Omega}} m / \min_{\overline{\Omega}} m$, $\mu_1 < 0$ when $b_1 > b_1^*$, and $\mu_1 > 0$ when $b_1 < b_1^*$.

(ii) The proof is the same as the one for Theorem 2 (ii).

(iii) By (i), (ii), and the assumption, the semitrivial steady states $(\theta_d, 0)$ and $(0, \theta_\gamma)$ are either both stable or both unstable. Hence there is a coexistence steady state by virtue of the monotonicity of system [15].

(iv) By an argument almost similar to the proof of Theorem 2 (iii), we can show there is no coexistence steady state. Also $(0, \theta_\gamma)$ is linearly stable by (ii). Therefore by the theory of monotone dynamical system [16, Proposition 9.1 and Theorem 9.2], the linearly stable semitrivial steady state $(0, \theta_\gamma)$ is globally asymptotically stable as in [15, 17]. □

6. Proof of Theorem 4

We consider the following system:

$$\begin{aligned} u_t &= \Delta(\gamma(s_1)u) + u[m(x) - u - b_2 v] \quad \text{in } \Omega, \\ v_t &= \Delta(\gamma(s_2)v) + v[m(x) - b_1 u - v] \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu}(\gamma(s_1)u) &= \frac{\partial}{\partial \nu}(\gamma(s_2)v) = 0 \quad \text{on } \partial\Omega, \\ u(x,0) &= u_0(x) \geq 0, \\ v(x,0) &= v_0(x) \geq 0 \quad \text{in } \Omega, \end{aligned} \tag{77}$$

where $s_1 := (u + b_2 v)/m$ and $s_2 := v/m$. Set

$$\begin{aligned} U &:= \gamma(s_1)u, \\ V &:= \gamma(s_2)v, \\ U_\gamma &:= \gamma(s_\gamma)\theta_\gamma, \\ s_\gamma &:= \frac{\theta_\gamma}{m}. \end{aligned} \tag{78}$$

Then the corresponding eigenvalue problems linearized at the semitrivial steady states $(U_\gamma, 0)$ and $(0, U_\gamma)$ are

$$\begin{aligned} \lambda\Phi &= \left(\gamma(s_\gamma) + \gamma'(s_\gamma)s_\gamma\right) \\ &\quad \cdot \left\{\Delta\Phi + \frac{m - 2\theta_\gamma}{\gamma(s_\gamma) + \gamma'(s_\gamma)s_\gamma}\Phi - \frac{b_2}{l}\theta_\gamma\Psi\right\} \\ &\quad + b_2\gamma'(s_\gamma)s_\gamma\left\{\Delta\Psi + \frac{m - b_1\theta_\gamma}{l}\Psi\right\} \quad \text{in } \Omega, \\ \lambda\Psi &= l\Delta\Psi + \left(m - b_1\theta_\gamma\right)\Psi \quad \text{in } \Omega, \\ \frac{\partial}{\partial\nu}\Phi &= \frac{\partial}{\partial\nu}\Psi = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{79}$$

and

$$\begin{aligned} \lambda\Phi &= \gamma(b_2 s_\gamma)\left\{\Delta\Phi + \frac{m - b_2\theta_\gamma}{\gamma(b_2 s_\gamma)}\Phi\right\} \quad \text{in } \Omega, \\ \lambda\Psi &= \left(\gamma(s_\gamma) + \gamma'(s_\gamma)s_\gamma\right) \\ &\quad \cdot \left\{\Delta\Psi + \frac{m - 2\theta_\gamma}{\gamma(s_\gamma) + \gamma'(s_\gamma)s_\gamma}\Psi - \frac{b_1\theta_\gamma}{\gamma(b_2 s_\gamma)}\Phi\right\} \quad \text{in } \Omega, \\ \frac{\partial}{\partial\nu}\Phi &= \frac{\partial}{\partial\nu}\Psi = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{80}$$

respectively.

In a similar way to the proofs of Lemmas 7 and 8, we obtain the following stability criterion. We omit the proof.

Lemma 10. *Let θ_γ be unique positive steady states of (5).*

(i) *Let μ_1 be the first eigenvalue of*

$$\begin{aligned} \mu\Psi &= l\Delta\Psi + \left(m - b_1\theta_\gamma\right)\Psi \quad \text{in } \Omega, \\ \frac{\partial}{\partial\nu}\Psi &= 0. \quad \text{on } \partial\Omega. \end{aligned} \tag{81}$$

If $\mu_1 > 0$, then the semitrivial steady state $(\theta_\gamma, 0)$ of (77) is linearly unstable. If $\mu_1 < 0$, $(\theta_\gamma, 0)$ is linearly stable.

(ii) *Let ν_1 be the first eigenvalue of*

$$\begin{aligned} \nu\phi &= \Delta\phi + \frac{m - b_2\theta_\gamma}{\gamma(b_2 s_\gamma)}\phi \quad \text{in } \Omega, \\ \frac{\partial}{\partial\nu}\phi &= 0. \quad \text{on } \partial\Omega. \end{aligned} \tag{82}$$

If $\nu_1 > 0$, then $(0, \theta_\gamma)$ is linearly unstable. If $\nu_1 < 0$, $(0, \theta_\gamma)$ is linearly stable.

Proof of Theorem 4 . (i), (ii) The proof is similar to the proofs of Theorems 2 and 3.

(iii) If there is a coexistence steady state (u, v),

$$0 = \sup_{\psi \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[-|\nabla\psi|^2 + \left((m - u - b_2 v)/\gamma((u + b_2 v)/m)\right)\psi^2\right]}{\int_\Omega \psi^2}. \tag{83}$$

Choosing $\psi = \gamma(v/m)v$,

$$\begin{aligned} 0 &\geq \int_\Omega \left[-\left|\nabla\gamma\left(\frac{v}{m}\right)v\right|^2 + \frac{m - u - b_2 v}{\gamma((u + b_2 v)/m)}\gamma^2\left(\frac{v}{m}\right)v^2\right] = \int_\Omega \gamma\left(\frac{v}{m}\right) \\ &\quad \cdot v^2\left[\frac{\gamma(v/m)}{\gamma((u + b_2 v)/m)}(m - u - b_2 v) - (m - b_1 u - v)\right]. \end{aligned} \tag{84}$$

If we show

$$A := \frac{\gamma(v/m)}{\gamma((u+b_2v)/m)}(m-u-b_2v)-(m-b_1u-v) \quad (85)$$

is strictly positive, it causes a contradiction so the proof is complete.

Suppose $m \geq u + b_2v$. Then $\gamma(v/m)/\gamma((u+b_2v)/m) = \gamma(v/m)/l \geq 1$ so

$$\begin{aligned} A &\geq (m-u-b_2v)-(m-b_1u-v) \\ &= (b_1-1)u+(1-b_2)v > 0. \end{aligned} \quad (86)$$

Similarly, if $m < u + b_2v$ and $m \geq v$, then

$$\begin{aligned} &\frac{\gamma(v/m)}{\gamma((u+b_2v)/m)}(m-u-b_2v) \\ &\quad = \frac{l}{\gamma((u+b_2v)/m)}(m-u-b_2v) \\ &\quad \geq (m-u-b_2v). \end{aligned} \quad (87)$$

Thus $A \geq (b_1-1)u+(1-b_2)v > 0$.

Finally suppose $m < u + b_2v$ and $m < v$. Note that

$$\begin{aligned} -(m-b_1u-v) &= -m+u+b_2v+(b_1-1)u \\ &\quad +(1-b_2)v \\ &> (b_1-1)u+(1-b_2)v \geq (1-b_2)v \\ &> (1-b_2)m. \end{aligned} \quad (88)$$

So if $(1+\epsilon)m \geq u + b_2v$ and $\epsilon < (l/h)(1-b_2)$, we have

$$A \geq -\epsilon\frac{h}{l}m+(1-b_2)m > 0. \quad (89)$$

When $(1+\epsilon)m < u + b_2v$, by the structure of γ in (4), $\gamma(v/m)/\gamma((u+b_2v)/m) = \gamma(v/m)/h \leq 1$. Hence

$$A \geq (b_1-1)u+(1-c_1)v > 0. \quad (90)$$

This completes the proof of Theorem 4. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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