Research Article

Non-Nehari Manifold Method for Fractional p -Laplacian Equation with a Sign-Changing Nonlinearity

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We consider the following fractional p -Laplacian equation: $(-\Delta)_p^\alpha u + V(x)|u|^{p-2}u = f(x, u) - \Gamma(x)|u|^{q-2}u$, $x \in \mathbb{R}^N$, where $N \geq 2$, $p_\alpha^* > q > p \geq 2$, $\alpha \in (0, 1)$, $(-\Delta)_p^\alpha$ is the fractional p -Laplacian, and $\Gamma \in L^\infty(\mathbb{R}^N)$ and $\Gamma(x) \geq 0$ for a.e. $x \in \mathbb{R}^N$. f has the subcritical growth but higher than $\Gamma(x)|u|^{q-2}u$; however, the nonlinearity $f(x, u) - \Gamma(x)|u|^{q-2}u$ may change sign. If V is coercive, we investigate the existence of ground state solutions for p -Laplacian equation.

1. Introduction

Consider the following nonlinear Schrödinger equation with fractional p -Laplacian:

$$(-\Delta)_p^\alpha u + V(x)|u|^{p-2}u = f(x, u) - \Gamma(x)|u|^{q-2}u, \quad (1)$$

$$x \in \mathbb{R}^N,$$

where $N \geq 2$, $p_\alpha^* > q > p \geq 2$, $\alpha \in (0, 1)$, and $(-\Delta)_p^\alpha$ is the fractional p -Laplacian. $V(x)$, $\Gamma(x)$, and $f(x, u) : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following assumptions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$, there exists a constant $d_0 > 0$ such that

$$\lim_{|y| \rightarrow \infty} \text{meas} \{x \in \mathbb{R}^N : |x - y| \leq d_0, V(x) \leq M\} = 0, \quad (2)$$

$$\forall M > 0,$$

where $\text{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$;

(Γ) $\Gamma \in L^\infty(\mathbb{R}^N)$, $\Gamma(x) \geq 0$ for a.e. $x \in \mathbb{R}^N$;

(f_1) $f(x, t) : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable, continuous in $t \in \mathbb{R}$ for a.e. $x \in \mathbb{R}^N$ and there are $C > 0$ and $2 \leq p < q < r < p_\alpha^*$ such that

$$|f(x, t)| \leq C(1 + |t|^{r-1}) \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^N, \quad (3)$$

where $p_\alpha^* = Np/(N - p\alpha)$;

(f_2) $f(x, t) = o(|t|^{p-1})$ as $|t| \rightarrow 0$ uniformly in $x \in \mathbb{R}^N$;

(f_3) $F(x, t)/|t|^q \rightarrow \infty$ uniformly in x as $|t| \rightarrow \infty$, where $F(x, t) = \int_0^t f(x, \tau) d\tau$;

(f_4) $t \mapsto f(x, t)/|t|^{q-1}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$.

When $p = 2$, (1) arises in the study of the nonlinear Fractional Schrödinger equation

$$(-\Delta)^\alpha u + V(x)u = f(x, u) - \Gamma(x)|u|^{q-2}u, \quad (4)$$

$$x \in \mathbb{R}^N.$$

Problems with this type have occurred in many different fields, such as continuum mechanics, phase transition phenomena, population dynamics, and game theory, as they are

the typical outcome of stochastically stabilization of Lévy processes; see [1–4].

When $\alpha = 1$ and $\Gamma = 0$, (4) reduces to be the nonlinear Schrödinger equation

$$-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N. \quad (5)$$

Using the Nehari-type monotonicity condition, Szulkin and Weth [5] obtained the existence of ground state solutions for (5). But in this paper, the Nehari manifold is usually not smooth and the Nehari-type monotonicity condition for the nonlinearity is not satisfied; then the Nehari manifold method is invalid. In this paper, we are aimed to obtain ground state solutions for (1) by the so-called non-Nehari manifold method, which is established by Tang [6, 7]. Unlike the Nehari manifold method, the main idea of our approach lies on finding a minimizing sequence for the energy functional outside the Nehari manifold by using the diagonal method.

Now, we are ready to state the main result of this paper.

Theorem 1. *Suppose that (V) , (Γ) and $(f_1) - (f_4)$ hold. Then (1) has a nontrivial ground state solution.*

2. Preliminaries

In the paper, we will denote $o_n(1)$ by the infinitesimal as $n \rightarrow +\infty$. For the sake of simplicity, the norm of the space $L^p(\mathbb{R}^N)$ will be denoted by $\|\cdot\|_p$, and integrals over the whole $\mathbb{R}^N$ will be written $\int$.

We define the Gagliardo seminorm by

$$[u]_{\alpha,p} = \left(\int \frac{|u(x) - u(y)|^p}{|x - y|^{N+\alpha p}} dx dy \right)^{1/p}, \quad (6)$$

where $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a measurable function. Then fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^N)$ is given by

$$\begin{aligned} W^{\alpha,p}(\mathbb{R}^N) \\ = \{u \in L^p(\mathbb{R}^N) : u \text{ is measurable and } [u]_{\alpha,p} < \infty\} \end{aligned} \quad (7)$$

endowed with the norm

$$\|u\|_{\alpha,p} = ([u]_{\alpha,p}^p + \|u\|_p^p)^{1/p}. \quad (8)$$

For the basic properties of fractional Sobolev spaces, we refer the interested reader to [8]. By condition (V), we define the fractional Sobolev space with potential $V(x)$ by

$$E := \left\{ u \in W^{\alpha,p} : \int V(x) |u|^p dx < \infty \right\} \quad (9)$$

endowed with the norm

$$\|u\| := \left([u]_{\alpha,p}^p + \int V(x) |u|^p dx \right)^{1/p}. \quad (10)$$

The energy functional $J : E \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} J(u) &= \frac{1}{p} \int \frac{|u(x) - u(y)|^p}{|x - y|^{N+\alpha p}} dx dy \\ &\quad + \frac{1}{p} \int V(x) |u(x)|^p dx \\ &\quad - \int \left(F(x, u) - \frac{1}{q} \Gamma(x) |u|^q \right) dx. \end{aligned} \quad (11)$$

Under our hypotheses, J is well defined on E . It is well known that $J \in C^1(E, \mathbb{R})$, and its derivative is given by

$$\begin{aligned} \langle J'(u), v \rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+\alpha p}} dx dy \\ &\quad + \int_{\mathbb{R}^N} V(x) |u|^{p-2} uv dx \\ &\quad - \int (f(x, u) v - \Gamma(x) |u|^{q-2} uv) dx, \end{aligned} \quad (12)$$

for $u, v \in E$. It is standard to verify that the weak solutions of (1) correspond to the critical points of J . Now, we review the main embedding result for the space E .

Lemma 2 ([9, Lemma 1]). *Under assumption (V), the embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is compact for any $r \in [p, p_\alpha^*)$.*

In the following lemma, we will show that J has Mountain Pass geometric structure.

Lemma 3. *Suppose that (V) , (Γ) , and $(f_1) - (f_4)$ hold.*

- (i) *There is $\delta_0 > 0$ such that $\rho_0 := \inf_{\|u\|=\delta_0} J(u) > J(0) = 0$.*
- (ii) *For any $u \neq 0$, there exists $t > 0$ such that $J(tu) < 0$.*

Proof. (i) By (f_1) and (f_2) , we have

$$\begin{aligned} |f(x, u)| &\leq \varepsilon |u|^{p-1} + C_\varepsilon |u|^{r-1}, \\ |F(x, u)| &\leq \frac{\varepsilon}{p} |u|^p + \frac{C_\varepsilon}{r} |u|^r. \end{aligned} \quad (13)$$

By $E \hookrightarrow L^s(\mathbb{R}^N)$ for $s \in [p, p_\alpha^*)$ and (13), we have

$$\begin{aligned} J(u) &= \frac{1}{p} \|u\|^p - \int F(x, u) dx + \frac{1}{q} \int \Gamma(x) |u|^q dx \\ &\geq \frac{1}{p} \|u\|^p - \varepsilon C_1 \|u\|^p - C_\varepsilon C_2 \|u\|^r. \end{aligned} \quad (14)$$

By the arbitrariness of ε and $p < r$, we get the conclusion.

(ii) Fix $u \neq 0$; by (f_3) , we have

$$\begin{aligned} \frac{J(tu)}{t^q} &= \frac{1}{p t^{q-p}} \|u\|^p \\ &\quad - \int \left(\frac{F(x, tu)}{(tu)^q} u^q - \frac{1}{q} \Gamma(x) |u|^q \right) dx \end{aligned} \quad (15)$$

$$\rightarrow -\infty,$$

as $t \rightarrow +\infty$. Thus, there exists $t > 0$ such that $J(tu) < 0$. $\square$

Now, we define the Nehari manifold by

$$\mathcal{N} := \{u \in E \setminus \{0\} : \langle J'(u), u \rangle = 0\}. \quad (16)$$

It is easy to prove that $\mathcal{N}$ is not empty. And we have the following lemma.

Lemma 4. Suppose that (V) , (Γ) and $(f_1) - (f_4)$ hold. Let $\theta \in [0, 1) \cup (1, \infty)$ and $u \in \mathcal{N}$; then $J(\theta u) < J(u)$.

Proof. By (f_4) , we have

$$\begin{aligned} f(x, s) &\leq \frac{f(x, t)}{|t|^{q-1}} |s|^{q-1}, \quad s < t; \\ f(s) &\geq \frac{f(x, t)}{|t|^{q-1}} |s|^{q-1}, \quad s > t. \end{aligned} \quad (17)$$

Then

$$\begin{aligned} F(x, t) - F(x, \theta t) &= \int_{\theta t}^t f(x, s) ds \leq \frac{1 - \theta^q}{q} f(x, t) t, \\ \forall \theta \geq 0, \quad t \in \mathbb{R}. \end{aligned} \quad (18)$$

Let $h(\theta) = p\theta^q - q\theta^p$; then $h'(\theta) = pq(\theta^{q-1} - \theta^{p-1})$. By a simple calculation, we have $h'(1) = 0$ and $h(\theta) > h(1) = p - q$ for all $\theta \in [0, 1) \cup (1, \infty)$. Thus,

$$\frac{\theta^p - 1}{p} - \frac{\theta^q - 1}{q} = \frac{p - q - h(\theta)}{pq} < 0. \quad (19)$$

Let $u \in \mathcal{N}$, it follows from (18) and (19) that

$$\begin{aligned} J(\theta u) &= J(u) + \left(J(\theta u) - J(u) - \frac{\theta^q - 1}{q} \langle J'(u), u \rangle \right) \\ &= J(u) + \left[\frac{\theta^p - 1}{p} - \frac{\theta^q - 1}{q} \right] \|u\|^p \\ &\quad + \int \left[\frac{\theta^q - 1}{q} f(x, u) u - F(x, \theta u) + F(x, u) \right] dx \\ &< J(u). \end{aligned} \quad (20)$$

□

Lemma 5. Suppose that (V) , (Γ) and $(f_1) - (f_4)$ hold; let $m := \inf_{\mathcal{N}} J$; then there exist $\{u_n\} \in E$, $c_* \in (\rho_0, m]$ satisfying

$$\begin{aligned} J(u_n) &\longrightarrow c_*, \\ (1 + \|u_n\|) \|J'(u_n)\| &\longrightarrow 0. \end{aligned} \quad (21)$$

Proof. By (i) of Lemma 3, there exist $\delta_0 > 0$ and $\rho_0 > 0$ such that

$$u \in E, \quad \|u\| = \delta_0 \implies J(u) \geq \rho_0. \quad (22)$$

Choose $v_k \in \mathcal{N}$ such that

$$m \leq J(v_k) < m + \frac{1}{k}, \quad k \in \mathbb{N}. \quad (23)$$

Since $J(tv_k) < 0$ for large $t > 0$, Mountain Pass Lemma implies that there exists $\{u_{k,n}\}_{n \in \mathbb{N}} \subset E$ satisfying

$$\begin{aligned} J(u_{k,n}) &\longrightarrow c_k, \\ \|J'(u_{k,n})\| (1 + \|u_{k,n}\|) &\longrightarrow 0, \end{aligned} \quad (24)$$

$$k \in \mathbb{N},$$

where $c_k \in [\rho_0, \sup_{t \geq 0} J(tv_k)]$. By Lemma 4, we have $J(v_k) = \sup_{t \geq 0} J(tv_k)$. Hence, by (23) and (24), we have

$$\begin{aligned} J(u_{k,n}) &\longrightarrow c_k < m + \frac{1}{k}, \\ \|J'(u_{k,n})\| (1 + \|u_{k,n}\|) &\longrightarrow 0, \end{aligned} \quad (25)$$

$$k \in \mathbb{N}.$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\begin{aligned} J(u_{k,n_k}) &< m + \frac{1}{k}, \\ \|J'(u_{k,n_k})\| (1 + \|u_{k,n_k}\|) &< \frac{1}{k}, \end{aligned} \quad (26)$$

$$k \in \mathbb{N}.$$

Let $u_k = u_{k,n_k}$, $k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\begin{aligned} J(u_n) &\longrightarrow c_* \in [\rho_0, m], \\ \|J'(u_n)\| (1 + \|u_n\|) &\longrightarrow 0. \end{aligned} \quad (27)$$

□

3. Proof of Theorem 1

Proof of Theorem 1. In view of Lemma 5, we find a Cerami sequence $\{u_n\}$ satisfying (21). By (18), we have

$$\frac{1}{q} f(x, u_n) u_n - F(x, u_n) \geq 0, \quad (28)$$

$$\text{for all } x \in \mathbb{R}^N, \quad u_n \in E.$$

Combining (21) and (28), for n big enough, we have

$$\begin{aligned} c_* + 1 &\geq J(u_n) - \frac{1}{q} \langle J'(u_n), u_n \rangle \\ &= \left(\frac{1}{p} - \frac{1}{q} \right) \|u_n\|^p \\ &\quad + \int \left[\frac{1}{q} f(x, u_n) u_n - F(x, u_n) \right] dx \\ &\geq \left(\frac{1}{p} - \frac{1}{q} \right) \|u_n\|^p. \end{aligned} \quad (29)$$

It follows that $\|u_n\|$ is bounded. Passing to a subsequence, we have $u_n \rightharpoonup u_0$ in E . By Lemma 2, we have $u_n \rightarrow u_0$ in $L^r(\mathbb{R}^N)$ for $r \in [p, p^*)$. Then, by (13) and the Hölder inequality, we have

$$\begin{aligned} & \left| \int (f(x, u_n) - f(x, u_0))(u_n - u_0) dx \right| \\ & \leq \varepsilon \int (|u_n|^{p-1} + |u_0|^{p-1}) |u_n - u_0| dx \\ & \quad + C_\varepsilon \int (|u_n|^{r-1} + |u_0|^{r-1}) |u_n - u_0| dx \quad (30) \\ & \leq \varepsilon (\|u_n\|_p^{p-1} + \|u_0\|_p^{p-1}) \|u_n - u_0\|_p \\ & \quad + C_\varepsilon (\|u_n\|_r^{r-1} + \|u_0\|_r^{r-1}) \|u_n - u_0\|_r \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} & \left| \int \Gamma(x) (|u_n|^{q-2} u_n - |u_0|^{q-2} u_0) (u_n - u_0) dx \right| \\ & \leq \|\Gamma\|_\infty \int (|u_n|^{q-1} + |u_0|^{q-1}) |u_n - u_0| dx \quad (31) \\ & \leq (\|u_n\|_q^{q-1} + \|u_0\|_q^{q-1}) \|u_n - u_0\|_q \rightarrow 0. \end{aligned}$$

It follows from (30), (31) and Simon inequality ($(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq (1/2^{p-2})|a - b|^p$) that

$$\begin{aligned} & \langle J'(u_n) - J'(u_0), u_n - u_0 \rangle \\ & = \int \frac{1}{|x - y|^{N+\alpha p}} [|u_n(x) - u_n(y)|^{p-2} \\ & \quad \cdot (u_n(x) - u_n(y)) - |u_0(x) - u_0(y)|^{p-2} \\ & \quad \cdot (u_0(x) - u_0(y))] [u_n(x) - u_n(y) - u_0(x) \\ & \quad + u_0(y)] dx dy + \int V(x) [|u_n|^{p-2} u_n \\ & \quad - |u_0|^{p-2} u_0] (u_n - u_0) dx - \int (f(x, u_n) \\ & \quad - f(x, u_0)) (u_n - u_0) dx + \int \Gamma(x) (|u_n|^{q-2} u_n \\ & \quad - |u_0|^{q-2} u_0) (u_n - u_0) dx \geq \frac{1}{2^{p-2}} \\ & \quad \cdot \int \frac{|(u_n(x) - u_n(y)) - (u_0(x) - u_0(y))|^p}{|x - y|^{N+\alpha p}} dx dy \\ & \quad + \frac{1}{2^{p-2}} \int V(x) |u_n - u_0|^p dx + o_n(1) = \frac{1}{2^{p-2}} \|u_n \\ & \quad - u_0\|^p + o_n(1). \end{aligned} \quad (32)$$

On the other hand, by $\langle J'(u_n), u_n - u_0 \rangle \rightarrow 0$ and $u_n \rightharpoonup u_0$, we have

$$\langle J'(u_n) - J'(u_0), u_n - u_0 \rangle \rightarrow 0. \quad (33)$$

Combining (32) and (33), we have $u_n \rightarrow u_0$ in E . Then, by $J \in C^1(E, \mathbb{R})$, we have $J'(u_0) = 0$. By (28), Lemma 5, and Fatou's lemma, we have

$$\begin{aligned} m & \geq c_* = \lim_{n \rightarrow \infty} \left[J(u_n) - \frac{1}{q} \langle J'(u_n), u_n \rangle \right] \\ & = \left(\frac{1}{p} - \frac{1}{q} \right) \lim_{n \rightarrow \infty} \|u_n\|^p \\ & \quad + \lim_{n \rightarrow \infty} \int \left[\frac{1}{q} f(x, u_n) u_n - F(x, u_n) \right] dx \quad (34) \\ & \geq \left(\frac{1}{p} - \frac{1}{q} \right) \|u_0\|^p \\ & \quad + \int \left[\frac{1}{q} f(x, u_0) u_0 - F(x, u_0) \right] dx \\ & = J(u_0) - \frac{1}{q} \langle J'(u_0), u_0 \rangle = J(u_0). \end{aligned}$$

This shows that $J(u_0) \leq m$ and so $J(u_0) = m = \inf_{\mathcal{A}} J > 0$. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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