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*Research Article*

# Toeplitz Operators with Horizontal Symbols Acting on the Poly-Fock Spaces

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We describe the $C^*$-algebra generated by the Toeplitz operators acting on each poly-Fock space of the complex plane $\mathbb{C}$ with the Gaussian measure, where the symbols are bounded functions depending only on $x = \operatorname{Re} z$ and have limit values at $y = -\infty$ and $y = \infty$. The $C^*$ algebra generated with this kind of symbols is isomorphic to the $C^*$-algebra functions on extended reals with values on the matrices of dimension $n \times n$, and the limits at $y = -\infty$ and $y = \infty$ are scalar multiples of the identity matrix.

## 1. Introduction

Recall that the $n$ poly-Fock space is denoted by $F_n^2(\mathbb{C}) \subset L_2(\mathbb{C}, e^{-|z|^2})$ and consists of the $n$-analytic functions which satisfy the equation

$$\left(\frac{\partial}{\partial \bar{z}}\right)^n \varphi = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^n \varphi = 0. \tag{1}$$

The $n$ true poly-Fock space is denote by $F_{(n)}^2(\mathbb{C})$, which consists of the all true-$n$-analytic functions; i.e.,

$$F_{(n)}^2(\mathbb{C}) = F_n^2(\mathbb{C}) \ominus F_{n-1}^2(\mathbb{C}), \tag{2}$$

for $n \geq 1$, and $F_{(0)}^2(\mathbb{C}) = \{0\}$.

It is clear that $F_1^2(\mathbb{C})$ is the classical Fock space on the complex plane $\mathbb{C}$, which is also denoted by $F^2(\mathbb{C})$.

In [1], N. Vasilevski proved that $L_2(\mathbb{C})$ has a decomposition as a direct sum of the $n$ true poly-Fock and $n$ true anti-poly-Fock spaces:

$$L_2\left(\mathbb{C}, e^{-|z|^2}\right) = \bigoplus_{k=1}^{\infty} F_{(n)}^2(\mathbb{C}) \tag{3}$$

Moreover, they proved that the spaces $F_{(n)}^2(\mathbb{C})$ are isomorphic and isometric to $L_2(\mathbb{R}) \otimes H_{n-1}$, where $H_{n-1}$ is the one-dimensional space generated by Hermite function of order $n-1$. Finally, they found the explicit expressions for the reproduction kernels of all these function spaces.

In [2], K. Esmeral and N. Vasilevski introduced the so-called horizontal Toeplitz operators acting on the Fock space and give an explicit description of the $C^*$-algebra generated by them. They showed that any Toeplitz operator with $L_\infty$-symbol, which is invariant under imaginary translations, is unitarily equivalent to the multiplication operator by its "spectral function". They stated that the corresponding spectral functions form a dense subset in the $C^*$-algebra of bounded uniformly continuous functions with respect to the standard metric on $\mathbb{R}$.

The Toeplitz operators acting on spaces of polyanalytic functions have been object of study of several authors in different direction. For example, in [3], Sánchez-Nungaray and Vasilevski studied Toeplitz operators with pseudodifferential symbols acting on poly-Bergman spaces upper half plane. A different approach by Hutník, Maximenko, and Mišková in [4] considers Toeplitz Localization operators on the space of Wavelet transform or the space of short-time Fourier

transform. They studied these operators with symbols that just depend on the first coordinate in the phase space, which are unitary equivalent to multiplication operators of certain specific functions "spectral functions". In particular, the poly-Bergman spaces are spaces of Wavelet transform which is related to Laguerre functions, and the poly-Fock spaces are spaces of short-time Fourier transform which is related to Hermite functions.

In [5], J. Ramírez-Ortega and A. Sánchez-Nungaray described the $C^*$-algebra generated by the Toeplitz operators with bounded vertical symbols and acting over each poly-Bergman space in the upper plane $\mathscr{A}_n^2(\Pi)$. They considered bounded vertical symbols that have limit values at $y = 0, \infty$ and prove that the $C^*$-algebra generated by the Toeplitz operator acting on $\mathscr{A}_n^2(\Pi)$ with this kind of symbols is isomorphic and isometric to the $C^*$-algebra of matrix-valued functions of the compact $[0, \infty]$. Similar result can be found in [6], where M. Loaiza and J. Ramírez-Ortega gave an analogous description to the above for the $C^*$-algebra generated by the Toeplitz operators with bounded homogeneous symbols acting over each poly-Bergman space in the upper plane.

The main result of this paper is the classified $C^*$-algebra generated by the Toeplitz operators with bounded vertical symbols with limits at $-\infty$ and $\infty$ acting over poly-Fock space in the complex plane.

This paper is organized as follows. In Section 2 we introduce preliminary results about the $n$-polyanalytic function spaces and their relationship with the Hermite polynomials. In Section 3 we prove that every Toeplitz operator with bounded horizontal symbol $a(z)$ acting on Fock space is unitary equivalent to a multiplication operator $\gamma^{n,a}(x)I$ acting on $(L_2(\mathbb{R}_+))^n$, where $\gamma^{n,a}(x)$ is a continuous matrix-valued function on $(-\infty, \infty)$. Finally, in Section 4, we describe the pure states of the algebra

$$\begin{aligned}&\mathfrak{D}\\&= \left\{M \in M_n(\mathbb{C}) \otimes C\left(\overline{\mathbb{R}}\right) : M(-\infty), M(\infty) \in \mathbb{C}I\right\}.\end{aligned} \tag{4}$$

We prove that the $C^*$ algebra $\mathscr{T}^{(n)}_{-\infty,\infty}$ generated by Toeplitz operator with bounded vertical symbols that have limit values at $y = -\infty, \infty$ acting on Fock space is isomorphic and isometric to the $C^*$-algebra $\mathfrak{D}$.

## 2. Poly-Fock Space on the Complex Plane

In this work we use the following standard notation: $z = x + iy \in \mathbb{C}$, with the usual complex conjugation $\overline{z}$; thus $|z|^2 = z \cdot z$. The Gaussian measure on $\mathbb{C}$ is given by

$$d\mu(z) = \frac{1}{\pi} e^{-z\cdot\overline{z}} d\nu(z) \tag{5}$$

where $d\nu(z) = dx\,dy$ is the usual Euclidean measure on $\mathbb{R}^2 = \mathbb{C}$.

The Hilbert space $L_2(\mathbb{C}, d\mu) = L_2(\mathbb{C}, e^{|z|^2})$ of $d\mu(z)$ square integrable functions on $\mathbb{C}$ with the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} d\mu(z). \tag{6}$$

The closed subspace $F^2(\mathbb{C})$ of $L_2(\mathbb{C}, e^{|z|^2})$ consisting of all analytic functions is called the Fock or Segal-Bargmann space. Also, the Fock space $F^2(\mathbb{C})$ can be defined as the closure of the set of all smooth functions satisfying the equation $\overline{\partial}\phi = 0$. Similarly, given a natural number $k$, the $k$ poly-Fock space $F_k^2(\mathbb{C})$ is the closure of the set of all smooth functions in $L_2(\mathbb{C}, e^{|z|^2})$ satisfying $\overline{\partial}^n\phi = 0$.

Recall that the Hermite polynomial $H_n(y)$ of degree $n$ is defined by

$$\begin{aligned}H_n(y) &= (-1)^n e^{y^2} \frac{d^n}{dy^n}\left(e^{-y^2}\right)\\&= n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2y)^{n-2m}}{m!\,(n-2m)!}, \quad n = 0, 1, 2, \ldots\end{aligned} \tag{7}$$

and the system of Hermite functions

$$h_n(y) = \left(2^n n! \sqrt{\pi}\right)^{-1/2} e^{-y^2/2} H_n(y), \quad n = 0, 1, 2, \ldots \tag{8}$$

form an orthonormal basis for $L_2(\mathbb{R})$. By abuse of notation we also denote $H_n$ to the one-dimensional space generated by $h_n(y)$ for $n \in \mathbb{Z}_+$. Further, define

$$H_n^{\oplus} = \bigoplus_{k=0}^{n} H_k. \tag{9}$$

The one-dimensional projection $P_{(n)}$ from $L_2(\mathbb{R})$ onto $H_n$ is given by $(P_{(n)}\phi)(y) = \langle \phi, h_n\rangle \cdot h_n(y)$. Thus, $P_n = P_{(0)} \oplus \cdots \oplus P_{(n)}$ is the orthogonal projection from $L_2(\mathbb{R})$ onto $H_n^{\oplus}$, and

$$\begin{aligned}(P_n\phi)(y) &= \sum_{k=0}^{n} \langle \phi, h_k\rangle \cdot h_k(y)\\&= \sum_{k=0}^{n} h_k(y) \int_{\mathbb{R}} \phi(v) h_k(v)\, dv.\end{aligned} \tag{10}$$

On the other hand, we consider the unitary operator $U_1 : L_2(\mathbb{C}, e^{|z|^2}) \longrightarrow L_2(\mathbb{R}^2, dxdy)$ defined by $(U_1\varphi)(z) = \pi^{-1} e^{(1/2)z\overline{z}} \varphi(z)$ that transforms the space $F_k^2(\mathbb{C})$ into $F_k^{(1)}$, the set of all functions in $L_2(\mathbb{R}^2)$, which satisfy the following equation:

$$\left(\frac{\partial}{\partial \overline{z}} + \frac{z}{2}\right)^k f = 0 \tag{11}$$

The image $F_k^{(2)}$ of the space $F_k^{(1)}$ under the unitary transform $U_2 = I \otimes F$ is the closure of the set of all smooth functions in $L_2(\mathbb{R}^2)$ which satisfy the equation

$$\frac{1}{2^k}\left(\frac{\partial}{\partial x} - y + x - \frac{\partial}{\partial y}\right)^k f = 0, \tag{12}$$

where $F$ is the Fourier transform.

Finally, we take the isomorphism $U_3 : L_2(\mathbb{R}^2) \longrightarrow L_2(\mathbb{R}^2)$ defined by

$$(U_3 f)(x, y) = f\left(\frac{1}{\sqrt{2}}(x + y), \frac{1}{\sqrt{2}}(x - y)\right) \tag{13}$$

to transform the space $F_k^{(2)}$ onto the space $F_k^{(3)}$, which is the closure of the set of smooth functions satisfying the equation

$$\frac{1}{2^{k/2}}\left(\frac{\partial}{\partial y}+y\right)^k f=0. \tag{14}$$

In summary, the unitary operator $U = U_3U_2U_1$ provides an isometric isomorphism from the space $L_2(\mathbb{C}, e^{|z|^2})$ into the space $L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}, dy)$, under which the $k$ poly-Fock space $F_k^2$ is mapped into $L_2(\mathbb{R}) \otimes H_k^{\oplus}$. We denote by $B_{(n)}$ and $B_n$ the orthogonal projections from $L_2(\mathbb{C}, e^{|z|^2})$ onto $F_{(n)}^2(\mathbb{C})$ and $F_n^2(\mathbb{C})$, respectively. The true $k$ poly-Fock spaces $F_{(k)}^2(\mathbb{C})$ are defined as follows:

$$\begin{aligned} F_{(1)}^2(\mathbb{C}) &= F_1^2(\mathbb{C}) = F^2(\mathbb{C}), \quad k=1,\\ F_{(k)}^2(\mathbb{C}) &= F_k^2(\mathbb{C}) \ominus F_{k-1}^2(\mathbb{C}), \quad k>1. \end{aligned} \tag{15}$$

Thus, $P_{(k)}$ is true $k$-Bargmann projection from $L_2(\mathbb{C})$ into $F_{(k)}(\mathbb{C})$.

The above construction is due to Vasilevski in [1], using the unitary operator $U$, they obtain the following characterizations:

(1) The true-poly-Fock space $F_{(n)}^2(\mathbb{C})$ is mapped onto $L_2(\mathbb{R}) \otimes H_{n-1}$.

(2) The true-poly-Fock projection $B_{(n)}$ is unitary equivalent to the following one:

$$UB_{(n)}U^{-1} = I \otimes P_{(n-1)}. \tag{16}$$

(3) The poly-Fock space $F_n^2(\mathbb{C})$ is mapped onto $L_2(\mathbb{R}) \otimes H_{n-1}^{\oplus}$.

(4) The poly-Fock projection $B_n$ is unitary equivalent to the following one:

$$UB_nU^{-1} = I \otimes P_{n-1}. \tag{17}$$

We introduce the isometric embedding $R_{0,(n)} : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}^2)$ by the rule $(R_{0,(n)}f)(x,y) = f(x)h_{n-1}(y)$. Clearly, the adjoint operator $R_{0,(n)}^* : L_2(\mathbb{R}^2) \longrightarrow L_2(\mathbb{R}_+)$ is given by

$$\left(R_{0,(n)}^*\varphi\right)(x) = \int_{\mathbb{R}} \varphi(x,v)\, h_{n-1}(v)\, dv. \tag{18}$$

The previous operators satisfy the following relations:

$$\begin{aligned} R_{0,(n)}^*R_{0,(n)} &= I: \quad L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R})\\ R_{0,(n)}R_{0,(n)}^* &= I \otimes P_{(n-1)}: \quad L_2(\mathbb{R}) \otimes L_2(\mathbb{R})\\ &\qquad \longrightarrow L_2(\mathbb{R}) \otimes H_{n-1}. \end{aligned} \tag{19}$$

On the other hand, we introduce the operator $R_{(n)} = R_{0,(n)}^*U$ from $L_2(\mathbb{C}, e^{|z|^2})$ onto $L_2(\mathbb{R})$, and its restriction to $F_{(n)}^2(\mathbb{C})$ is an isometric isomorphism. Thus, the adjoint operator $R_{(n)}^* = U^*R_{0,(n)}$ is an isometric isomorphism from $L_2(\mathbb{R})$ onto the subspace $F_{(n)}^2(\mathbb{C})$. Hence, these operators satisfy the following relations:

$$\begin{aligned} R_{(n)}^*R_{(n)} &= B_{(n)}: \quad L_2\left(\mathbb{C}, e^{|z|^2}\right) \longrightarrow F_{(n)}^2(\mathbb{C})\\ R_{(n)}R_{(n)}^* &= I: \quad L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}). \end{aligned} \tag{20}$$

Similarly, introduce the isometric embedding $R_{0,n} : (L_2(\mathbb{R}))^n \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$ by the rule

$$\begin{aligned} (R_{0,n}f)(x,y) &= \sum_{k=1}^{n} f_k(x)\, h_{k-1}(y)\\ &= [N_n(y)]\, f(x)^T, \end{aligned} \tag{21}$$

where $f = (f_1, \ldots, f_n)$ and

$$N_n(y) = (h_0(y), \ldots, h_{n-1}(y)) \tag{22}$$

and the superscript $T$ means that we are taking the transpose matrix.

Further, the adjoint operator $R_{0,n}^* : L_2(\mathbb{R}^2) \longrightarrow (L_2(\mathbb{R}))^n$ is given by

$$\begin{aligned} \left(R_{0,n}^*\varphi\right)(x) &= \left(\int_{\mathbb{R}} \varphi(x,y)\, h_0(y)\, dy, \ldots,\right.\\ &\left.\int_{\mathbb{R}} \varphi(x,y)\, h_{n-1}(y)\, dy\right) = \int_{\mathbb{R}} \varphi(x,y)\, N_n(y)\, dy. \end{aligned} \tag{23}$$

Since the image of $R_{0,n}$ is the space $U(F_n^2(\mathbb{C})) = L_2(\mathbb{R}) \otimes H_{n-1}^{\oplus}$, hence these operators satisfy the following relations:

$$\begin{aligned} R_{0,n}^*R_{0,n} &= I: \quad (L_2(\mathbb{R}))^n \longrightarrow (L_2(\mathbb{R}))^n,\\ R_{0,n}R_{0,n}^* &= I \otimes P_{n-1}: \quad L_2\left(\mathbb{R}^2\right) \longrightarrow L_2(\mathbb{R}) \otimes H_{n-1}^{\oplus}. \end{aligned} \tag{24}$$

Now the operator $R_n := R_{0,n}^*U$ from $L_2(\mathbb{C}, e^{|z|^2})$ onto $(L_2(\mathbb{R}))^n$, and its restriction to $F_n^2(\mathbb{C})$ is an isometric isomorphism. Furthermore, the adjoint operator $R_n^* = U^*R_{0,n}$ is an isometric isomorphism from $(L_2(\mathbb{R}))^n$ onto the space $F_n^2(\mathbb{C})$. Hence, these operators satisfy the following relations:

$$\begin{aligned} R_n^*R_n &= B_n: \quad L_2\left(\mathbb{C}, e^{|z|^2}\right) \longrightarrow F_n^2(\mathbb{C}),\\ R_nR_n^* &= I: \quad (L_2(\mathbb{R}))^n \longrightarrow (L_2(\mathbb{R}))^n. \end{aligned} \tag{25}$$

## 3. Toeplitz Operators with Horizontal Symbol

In this section we introduce a certain class of Toeplitz operators acting on the poly-Fock spaces, and we prove that they are unitarily equivalent to multiplication operators by continuous matrix-valued functions on $(-\infty, \infty)$. Let $a(z) = a(x)$ be a function in $L_\infty(\mathbb{R})$ depending only on $x = \operatorname{Re} z$ and we called this function a horizontal symbol.

*Definition 1.* Let $a$ be a function in $L_\infty(\mathbb{R})$. The Toeplitz operator with symbol $a$ acting on true-poly-Fock space (or poly-Fock space) is defined as

$$T_a : \varphi \in F_{(n)}^2(\mathbb{C}) \longmapsto B_{(n)}(a\varphi) \in F_{(n)}^2(\mathbb{C}). \tag{26}$$

or

$$T_a : \varphi \in F_n^2(\mathbb{C}) \longmapsto B_n(a\varphi) \in F_n^2(\mathbb{C}), \tag{27}$$

where $B_{(n)}$ and $B_n$ are the orthogonal projections for true-poly-Fock space and poly-Fock space, respectively.

In [2], K. Esmeral and N. Vasilevski show that every Toeplitz operator $T_a$ with horizontal symbol $a(x) \in L_\infty(\mathbb{R})$ acting on $F^2(\mathscr{C})$ is unitary equivalent to the multiplication operator $\gamma_a(x)I = R_0 T_a R_0^*$ acting on $L_2(\mathbb{R})$, where $R_0$ is defined in Section 2. The function $\gamma_a$ is given by

$$\gamma_a(x) = \sqrt{\pi} \int_{\mathbb{R}} a\left(\frac{y+x}{\sqrt{2}}\right) e^{-y^2} dy. \tag{28}$$

The following theorem is a generalization of above result for Toeplitz operators with horizontal symbols acting on true-poly-Fock space.

**Theorem 2.** *For any $a(x) \in L_\infty(\mathbb{R})$, the Toeplitz operator $T_{(n),a}$ acting on $F_{(n)}^2(\mathbb{C})$ is unitary equivalent to the multiplication operator $\gamma_{(n),a} I = R_{(n)} T_{(n),a} R_{(n)}^*$ acting on $L_2(\mathbb{R})$, where the function $\gamma_{(n),a}$ is given by*

$$\gamma_{(n),a}(x) = \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right) (h_{n-1}(y))^2 dy. \tag{29}$$

*Proof.* We know that the operator $R_{(n)}$ is unitary and using (20), we obtain that the Toeplitz operator $T_{(n),a}$ is unitary equivalent to the following operators:

$$\begin{aligned} R_{(n)} T_{(n),a} R_{(n)}^* &= R_{(n)} B_{\Pi,(n)} a B_{\Pi,(n)} R_{(n)}^* \\ &= R_{(n)} R_{(n)}^* R_{(n)} a R_{(n)}^* R_{(n)} R_{(n)}^* \\ &= \left(R_{(n)} R_{(n)}^*\right) R_{(n)} a R_{(n)}^* \left(R_{(n)} R_{(n)}^*\right) \\ &= R_{(n)} a R_{(n)}^* \\ &= R_{0,(n)}^* U_3 U_2 U_1 a(y) U_1^{-1} U_2^{-1} U_3^{-1} R_{0,(n)} \\ &= R_{0,(n)}^* U_3 a(y) U_3^{-1} R_{0,(n)} \\ &= R_{0,(n)}^* a\left(\frac{x+y}{\sqrt{2}}\right) R_{0,(n)}. \end{aligned} \tag{30}$$

Now calculate the explicit expression of the above operator

$$\begin{aligned} &\left(R_{0,(n)}^* a\left(\frac{x+y}{\sqrt{2}}\right) R_{0,(n)} f\right)(x) \\ &\quad = \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right) f(x) (h_{n-1}(y))^2 dy \\ &\quad = \gamma_{(n),a}(x) \cdot f(x), \end{aligned} \tag{31}$$

where $f \in L_2(\mathbb{R})$ and $h_{n-1}$ is the Hermite function of degree $n-1$. □

We called $\gamma_{(n),a}(x)$ the $n$ spectral function for the Toeplitz operator with vertical symbol $a$ in the true-poly-Fock space.

*Remark 3.* Notice that we obtain (28) from (29) taking $n = 1$.

The following result is an extension of above theorem for Toeplitz operators with horizontal symbols acting on poly-Fock space.

**Theorem 4.** *For any $a(x) \in L_\infty(\mathbb{R})$, the Toeplitz operator $T_{n,a}$ acting on $F_n^2(\mathscr{C})$ is unitary equivalent to the matrix multiplication operator $\gamma^{n,a}(x)I = R_n T_{n,a} R_n^*$ acting on $(L_2(\mathbb{R}))^n$, where the matrix-valued function $\gamma^{n,a} = (\gamma_{ij}^{n,a})$ is given by*

$$\gamma^{n,a}(x) := \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right) [N_n(y)]^T N_n(y) dy. \tag{32}$$

*That is,*

$$\gamma_{ij}^{n,a}(x) = \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right) h_{i-1}(y) h_{j-1}(y) dy, \tag{33}$$

*for $i, j = 1, \ldots, n$.*

*Proof.* We have that the operator $R_n$ is unitary and using (25), we obtain that the Toeplitz operator $T_{n,a}$ is unitary equivalent to the following operators:

$$\begin{aligned} R_n T_{n,a} R_n^* &= R_n B_n a B_n R_n^* = R_n R_n^* R_n a R_n^* R_n R_n^* \\ &= \left(R_n R_n^*\right) R_n a R_n^* \left(R_n R_n^*\right) = R_n a R_n^* \\ &= R_{0,n}^* U_3 U_2 U_1 a(y) U_1^{-1} U_2^{-1} U_3^{-1} R_{0,n} \\ &= R_{0,n}^* U_3 a(y) U_3^{-1} R_{0,n} \\ &= R_{0,n}^* a\left(\frac{x+y}{\sqrt{2}}\right) R_{0,n}. \end{aligned} \tag{34}$$

Now calculate the explicit expression of the above operator

$$\begin{aligned} &\left[R_{0,n}^* a\left(\frac{x+y}{\sqrt{2}}\right) R_{0,n} f\right](x) \\ &\quad = R_{0,n}^* \left(a\left(\frac{x+y}{\sqrt{2}}\right) [N_n(y)]^T f(x)\right) \\ &\quad = \int_{\mathbb{R}} \left[a\left(\frac{x+y}{\sqrt{2}}\right) [N_n(y)]^T f(x)\right] N_n(y) dy \\ &\quad = \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right) N_n(y) [N_n(y)]^T f(x) dy \\ &\quad = \gamma^{n,a}(x) f(x), \end{aligned} \tag{35}$$

where $f = (f_1, \ldots, f_n) \in (L_2(\mathbb{R}))^n$ and $N_n(y)$ is given by (22).

Therefore we obtain that each component of $\gamma^{n,a}$ is given by (33), which proves the theorem. □

*Remark 5.* The component function (33) is equal to $\gamma_{ij}^{n,a}(x) = \tilde{a} * (h_i h_j)$, where $*$ denotes the convolution in $\mathbb{R}$, $\tilde{a}(x) = a(x/\sqrt{2}) \in L_\infty(\mathbb{R})$ and $h_i h_j \in L_1(\mathbb{R})$. From [7][p.283, 32.45] it is guaranteed that the function $\gamma_{ij}^{n,a}(x)$ belongs to $C_b(\mathbb{R})$ where $C_b(\mathbb{R})$ is the set of uniformly continuous functions in $\mathbb{R}$.

## 4. Description of the $C^*$-Algebra Generated by Toeplitz Operators with Extended Horizontal Symbols

Denote by $L_\infty^{\{-\infty,+\infty\}}(\mathbb{R})$ the closed subspace of $L_\infty(\mathbb{R})$ which consists of all functions having limit values at the "endpoints" $-\infty$ and $+\infty$; i.e., for each $a \in L_\infty^{\{-\infty,\infty\}}(\mathbb{R})$ the following limits exist

$$\lim_{y\longrightarrow-\infty} a(y) = a_- \quad \text{and} \quad \lim_{y\longrightarrow+\infty} a(y) = a_+. \tag{36}$$

We will identify the functions $a \in L_\infty^{\{0,\infty\}}(\mathbb{R}_+)$ with their extensions $a(z) = a(x)$ to the complex plane $\mathbb{C}$, where $y = \operatorname{Im} z$. We shall say that $a \in L_\infty^{\{0,\infty\}}(\mathbb{R}_+)$ is an extended horizontal symbol.

In this section we study the $C^*$-algebra generated by all the Toeplitz operators on $F_n^2(\mathbb{C})$ with extended horizontal symbols.

*Definition 6.* We define some $C^*$-algebras that we will use in this paper.

(i) Denote by $\mathcal{G}_{(n)}^H$ the set of horizontal spectral functions given by

$$\mathcal{G}_{(n)}^H = \left\{\gamma_{(n),a} : a(x) \in L_\infty^{\{-\infty,+\infty\}}(\mathbb{R})\right\}. \tag{37}$$

(ii) Denote by $\mathcal{G}_n^H$ the set of horizontal spectral matrix-valued functions given by

$$\mathcal{G}_n^H = \left\{\gamma^{n,a} : a(x) \in L_\infty^{\{-\infty,+\infty\}}(\mathbb{R})\right\}. \tag{38}$$

(iii) Denote by $\mathcal{T}_{-\infty,\infty}^{(n)}$ the $C^*$-algebra generated by all the Toeplitz operators $T_{(n),a}$ acting on the true-poly-Fock space $F_{(n)}^2(\mathbb{C})$, with $a \in L_\infty^{\{-\infty,\infty\}}(\mathbb{R})$.

(iv) Denote by $\mathcal{T}_{-\infty,\infty}^{n}$ the $C^*$-algebra generated by all the Toeplitz operators $T_{n,a}$ acting on the poly-Fock space $F_n^2(\mathbb{C})$, with $a \in L_\infty^{\{-\infty,\infty\}}(\mathbb{R})$.

**Corollary 7.** *The $C^*$-algebra $\mathcal{T}_{-\infty,\infty}^{(n)}$ is isometrically isomorphic to the $C^*$-algebra $\mathfrak{G}_{(n)}^H$ generated by $\mathcal{G}_{(n)}^H$.*

**Corollary 8.** *The $C^*$-algebra $\mathcal{T}_{-\infty,\infty}^{n}$ is isometrically isomorphic to the $C^*$-algebra $\mathfrak{G}_n^H$ generated by $\mathcal{G}_n^H$.*

The next lemma is important to describe the behavior at the infinity of the spectral matrix-valued-function related to Toeplitz operators with extended horizontal symbols on poly-Fock space.

**Lemma 9.** *We consider a horizontal function $a(x) \in L_\infty^{\{-\infty,+\infty\}}(\mathbb{R})$, and let*

$$a_- = \lim_{y\longrightarrow-\infty} a(x) \quad \text{and} \quad a_+ = \lim_{y\longrightarrow+\infty} a(x). \tag{39}$$

*Then the matrix-valued function $\gamma^{n,a}(x)$ satisfies*

$$a_- I = \lim_{x\longrightarrow-\infty} \gamma^{n,a}(x) \quad \text{and} \quad a_+ I = \lim_{x\longrightarrow+\infty} \gamma^{n,a}(x). \tag{40}$$

*Proof.* First we consider the case when $a_- = 0$ (analogously we have $a_+ = 0$). We proceed to show that the limit value at $-\infty$ of each entry of $\gamma^{n,a}$ is equal to zero. We know that $h_{i-1}h_{j-1} \in L_1(\mathbb{R})$; then

$$C_{ij} = \int_{-\infty}^{\infty} \left|h_{i-1}(y)\, h_{j-1}(y)\right| dy < \infty. \tag{41}$$

Let $\epsilon > 0$ be a fixed number; hence there exists $y_0 \in \mathbb{R}_+$ such that

$$\int_{y_0}^{\infty} \left|h_{i-1}(y)\, h_{j-1}(y)\right| dy < \epsilon. \tag{42}$$

Moreover, there exist $M > 0$ such that $|a(t)| < \epsilon$ for $t < -M$. Under the above assumptions, we estimate the value of each entry of $\gamma^{n,a}(x)$ as follows:

$$\begin{aligned}\left|\gamma_{ij}^{n,a}(x)\right| &\le \int_{-\infty}^{y_0} \left|a\left(\frac{x+y}{\sqrt{2}}\right) h_{i-1}(y)\, h_{j-1}(y)\right| dy \\ &\quad + \int_{y_0}^{\infty} \left|a\left(\frac{x+y}{\sqrt{2}}\right) h_{i-1}(y)\, h_{j-1}(y)\right| dy \\ &\le C_{ij} \max_{-\infty<y<y_0} \left|a\left(\frac{x+y}{\sqrt{2}}\right)\right| + \|a\|_\infty \epsilon.\end{aligned} \tag{43}$$

If $x < -y_0 - \sqrt{2}M$ and $y < y_0$, then $(x+y)/\sqrt{2} < -M$. By the above, it is clear that

$$\left|\gamma_{ij}^{n,a}(x)\right| \le \left(C_{ij} + \|a\|_\infty\right)\epsilon \quad \text{for } x < -y_0 - \sqrt{2}M, \tag{44}$$

which implied the result for the limit value at $-\infty$ when $a_- = 0$. By similar argument the result is valid for the limit value at $\infty$ when $a_+ = 0$.

Now we consider the case when $a_- \neq 0$. If $b(x) = a(x) - a_-$, from the previous case we have that

$$\begin{aligned}\lim_{x\longrightarrow-\infty} \gamma_{ij}^{n,a}(x) &= \lim_{x\longrightarrow-\infty} \gamma_{ij}^{n,b}(x) + \lim_{x\longrightarrow-\infty} \gamma_{ij}^{n,a_-}(x) \\ &= a_- \int_{-\infty}^{\infty} h_{i-1}(y)\, h_{j-1}(y)\, dy = a_- \delta_{ij}.\end{aligned} \tag{45}$$

By similar argument, we obtain $\lim_{x\longrightarrow+\infty} \gamma_{ij}^{n,a}(x) = a_+ \delta_{ij}$. This completes the proof. □

Recall that $C_0(\mathbb{R})$ is the set of continuous functions that vanishes at infinity and $C(\overline{\mathbb{R}})$ is the set of continuous functions belonging to $L_\infty^{\{-\infty,+\infty\}}(\mathbb{R})$.

*Remark 10.* Let $a(x) \in L_\infty^{\{-\infty,+\infty\}}(\mathbb{R})$ be an extended horizontal function; then

$$\begin{aligned}&\gamma_{ij}^{n,a} \in C_0(\mathbb{R}) \quad \text{if } i \neq j, \\ &\gamma_{ii}^{n,a} \in C\left(\overline{\mathbb{R}}\right) \\ &\text{and } \gamma_{(n),a} \in C\left(\overline{\mathbb{R}}\right),\end{aligned} \tag{46}$$

where $\gamma_{ij}^{n,a}$ and $\gamma_{(n),a}$ are given by (29) and (32), for $i, j = 1, \ldots, n$, respectively. In particular, it is clear that $\gamma_{nn}^{n,a} = \gamma_{(n),a}$ for each $n \in \mathbb{N}$.

We denote the algebra of all $n \times n$ matrices with complex entries by $M_n(\mathbb{C})$ and we define the $C^*$-algebra $\mathfrak{C} = M_n(\mathbb{C}) \otimes C(\overline{\mathbb{R}})$ that consists of the algebra of all $n \times n$ matrices with entries in $C(\overline{\mathbb{R}})$. We introduce the algebra $\mathfrak{D}$, which is a $C^*$-subalgebra of $\mathfrak{C}$ defined by

$$\mathfrak{D} = \{M \in \mathfrak{C} : M(-\infty), M(\infty) \in \mathbb{C}I\}. \tag{47}$$

It is clear that $\mathfrak{G}_n^H$ is a $C^*$-subalgebra of $\mathfrak{D}$. We want to prove that $\mathfrak{G}_n^H = \mathfrak{D}$, where $\mathfrak{D}$ is an algebra of type I. Thus, using a Stone-Weierstrass theorem [8], we just need to show that $\mathfrak{G}_n^H$ separates all the pure states of $\mathfrak{D}$. Now, we proceed to describe the pure states of $\mathfrak{D}$.

Notice that $\mathfrak{D}$ is a $C^*$-bundle and the fibers are given by

$$\mathfrak{D}(x_0) = \begin{cases} M_n(\mathbb{C}), & x \in (-\infty, \infty) \\ \mathbb{C}, & x = -\infty \text{ or } x = \infty \end{cases} \tag{48}$$

Moreover, the set of its pure states of $\mathfrak{D}$ is determined by the pure states on the fibers; i.e., each pure state of $\mathfrak{D}$ has the form

$$f(M) = f_{x_0}(M(x_0)), \quad M \in \mathfrak{D}, \tag{49}$$

where $x_0 \in [-\infty, \infty]$, and $f_{x_0}$ is a pure state of $\mathfrak{D}(x_0)$; see [9] for more details.

In [10], T. K. Lee characterizes the set of all states of the matrix algebra $M_n(\mathbb{C})$. The author shows that each pure state of $M_n(\mathbb{C})$ is given by a functional $f_v$ defined as

$$f_w(Q) = \langle Qv, v\rangle \quad \text{for } Q \in M_n(\mathbb{C}) \tag{50}$$

where $v \in \mathbb{S}^n = \{z \in \mathbb{C}^n : \|z\| = 1\}$ and $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{C}^n$. Moreover, if $v, w \in \mathbb{S}^n$ such that $f_v = f_w$ then $v = tw$, where $t \in \mathbb{C}$ and $|t| = 1$.

In consequence, we have that the set of pure states of $\mathfrak{D}$ consists of all functionals of the form

$$f_{x_0,v}(M) = \langle M(x_0)v, v\rangle, \quad M \in \mathfrak{D}, \tag{51}$$

where $x_0 \in [-\infty, \infty]$ and $v \in \mathbb{S}^n$.

In the case when $x_0 = -\infty$ or $x_0 = \infty$, we just have one pure state which can be realized by $f_{x_0,v}$ for each $v \in \mathbb{S}^n$; i.e.,

$$f_{x_0,v} = f_{x_0,w} \quad \text{for } x_0 = -\infty \text{ or } \infty, \tag{52}$$

for all $v, w \in \mathbb{S}^n$.

Now, we will show that $\mathfrak{G}_n^H$ separates all the pure states of $\mathfrak{D}$. For this task we are going to use horizontal symbols of the form $c_\beta(z) = \chi_{(-\infty,\beta]}(y)$, where $\chi_{(-\infty,\beta]}$ is the characteristic function of the set $(-\infty, \beta] \subset \mathbb{R}$. Thus for this function the spectral matrix-valued-function has the form

$$\begin{aligned} \gamma^{n,c_\beta}(x) &= \int_{\mathbb{R}} c\left(\frac{x+y}{\sqrt{2}}\right) N_n(y)\left[N_n(y)\right]^T dy \\ &= \int_{-\infty}^{\sqrt{2}\beta - x} N_n(y)\left[N_n(y)\right]^T dy. \end{aligned} \tag{53}$$

To simplify the notation we made the following convention: $\gamma^{n,\beta} = \gamma^{n,c_\beta}$, where $c_\beta(z) = \chi_{(-\infty,\beta]}(y)$ and $\beta \in \overline{\mathbb{R}}$.

Let $v \in \mathbb{S}^n$; we define the function $x_0 \in \overline{\mathbb{R}} \longmapsto f_{x_0,v}(\gamma^{n,\beta}(x))$, whose explicit form is given by

$$\begin{aligned} f_{x_0,v}\left(\gamma^{n,\beta}(x)\right) &= \left\langle \gamma^{n,\beta}(x_0)v, v\right\rangle, \\ &= \left\langle \int_{-\infty}^{\sqrt{2}\beta - x_0} N_n(y)\left[N_n(y)\right]^T dy\, v, v\right\rangle \\ &= \int_{-\infty}^{\sqrt{2}\beta - x_0} \left\langle N_n(y)\left[N_n(y)\right]^T v, v\right\rangle dy \\ &= \int_{-\infty}^{\sqrt{2}\beta - x_0} \langle v, N_n(y)\rangle \langle N_n(y), v\rangle dy \\ &= \int_{-\infty}^{\sqrt{2}\beta - x_0} \left|\langle v, N_n(y)\rangle\right|^2 dy. \end{aligned} \tag{54}$$

In particular, from Lemma 9 we have that $f_{-\infty,v}(\gamma^{n,\beta}(x)) = 1$ and $f_{\infty,v}(\gamma^{n,\beta}(x)) = 0$. Moreover, it is clear that $0 < f_{x_0,v}(\gamma^{n,\beta}(x)) < 1$ for all $x_0 \in \mathbb{R}$. Hence we separated the unique pure state at $x_0 = -\infty$ (or $x_0 = \infty$) from every other pure state of the $C^*$-algebra $\mathfrak{D}$.

Now we define the function $h_v(y) = |\langle v, N_n(y)\rangle|^2$ for $y \in \mathbb{R}$. Notice that this function can be expressed as $h_v(y) = q_v(y)e^{-y^2}$, where

$$q_v(y) = \left|v_0 H_0(y) + \cdots + v_{n-1}(-1)^{n-1} H_{n-1}(y)\right|^2 \tag{55}$$

is a polynomial of degree at most $2n-2$ nonnegative valued.

The following lemma provides us a tool to show that the $C^*$-algebra $\mathfrak{G}_n^H$ separates the pure states of $\mathfrak{D}$ of the form $f_{x_0,v}, f_{x_1,w}$, where $x_0 \neq x_1$ with $x_0, x_1 \in \mathbb{R}$ and $v, w \in \mathbb{S}^n$.

**Lemma 11.** *We assume that $v, w \in \mathbb{S}^n$, $x_0, x_1 \in \mathbb{R}$, and $\gamma^{n,\beta} \in \mathfrak{G}_n^H$, where $\beta \in \mathbb{R}$. If we have that $f_{x_0,v}(\gamma^{n,\beta}(x)) = f_{x_1,w}(\gamma^{n,\beta}(x))$ for all $\beta \in \mathbb{R}$, then $x_0 = x_1$ and $|\langle v, N_n(y)\rangle|^2 = |\langle w, N_n(y)\rangle|^2$ for all $y \in \mathbb{R}$.*

*Proof.* By hypothesis we have that $f_{x_0,v}(\gamma^{n,\beta}(x)) = f_{x_1,w}(\gamma^{n,\beta}(x))$, which is equivalent to the following:

$$\int_{-\infty}^{\sqrt{2}\beta - x_0} q_v(y)e^{-y^2} dy = \int_{-\infty}^{\sqrt{2}\beta - x_0} q_w(y)e^{-y^2} dy, \tag{56}$$

where $q_v$ is given by (55). Taking the derivative with respect to $\beta$ of the above equation yields

$$\begin{aligned} &q_v\left(\sqrt{2}\beta - x_0\right)e^{-(\sqrt{2}\beta - x_0)^2} \\ &\quad = q_w\left(\sqrt{2}\beta - x_1\right)e^{-(\sqrt{2}\beta - x_1)^2} \end{aligned} \tag{57}$$

for all $\beta \in \mathbb{R}$. We can rewrite the above equation as

$$\begin{aligned} q_v\left(\sqrt{2}\beta - x_0\right) &= q_w\left(\sqrt{2}\beta - x_1\right)e^{-(\sqrt{2}\beta - x_1)^2 + (\sqrt{2}\beta - x_0)^2} \\ &= q_w\left(\sqrt{2}\beta - x_1\right)e^{2\sqrt{2}\beta(x_1 - x_0) + x_0^2 - x_1^2} \end{aligned} \tag{58}$$

We know that $q_v, q_w$ are polynomials, which implies that the above equation is valid if and only if the exponential part is constant with respect to $\beta$. Hence we obtain that $x_0 = x_1$; using this fact it is clear that $q_v(\sqrt{2}\beta - x_0) = q_w(\sqrt{2}\beta - x_0)$ for all $\beta \in \mathbb{R}$. Therefore we have that $x_0 = x_1$ and $q_v = q_w$ which is equivalent to $|\langle v, N_n(y)\rangle|^2 = |\langle w, N_n(y)\rangle|^2$ for all $y \in \mathbb{R}$. □

*Remark 12.* A consequence of the above lemma is that if $x_0 \neq x_1$ and $v, w \in \mathbb{S}^n$, then there exist $\beta_0 \in \mathbb{R}$ such that $f_{x_0,v}(\gamma^{n,\beta_0}(x)) \neq f_{x_1,w}(\gamma^{n,\beta_0}(x))$; i.e., the spectral matrix-valued-function $\gamma^{n,\beta_0}$ separated the pure states $f_{x_0,v}$ and $f_{x_1,w}$.

Let $y_1, \cdots, y_n$ be real numbers different from each other and recall that $N_n(y) = e^{-y^2}(H_0(y), H_1(y), \ldots, H_{n-1}(y))$. We define the matrix $N$ where the row $k$ is equal to $N_n(y_k)$. Thus

$$N = D\begin{pmatrix} H_0(y_1) & H_1(y_1) & \cdots & H_{n-1}(y_1) \\ H_0(y_2) & H_1(y_2) & \cdots & H_{n-1}(y_2) \\ \vdots & \vdots & & \vdots \\ H_0(y_n) & H_1(y_n) & \cdots & H_{n-1}(y_n) \end{pmatrix}, \tag{59}$$

where $D$ is the diagonal matrix given by $\operatorname{diag}\{e^{-y_1^2/2}, \cdots, e^{-y_n^2/2}\}$. Notice that $2^{k/2}/\sqrt{\pi k!}$ is the leading coefficient of $H_k(y)$; thus

$$H_k(y) = \frac{2^{k/2}}{\sqrt{\pi k!}}\left(y^k + \text{lower degree terms}\right) \tag{60}$$

Now, we calculate the determinant of $N$ using the properties of multilineality, alternativity, and Vandermonde's formula; we have the following relations:

$$\begin{aligned} \det N &= \frac{2^{(n-1)n/4}\cdot \det D}{\pi^{n/2}\prod_{k=0}^{n-1}\sqrt{k!}} \\ &\cdot \begin{vmatrix} 1 & y_1 + a_0 & y_1^2 + b_1 y_1 + b_0 & \cdots & y_1^{n-1} + \cdots \\ 1 & y_2 + a_0 & y_2^2 + b_1 y_2 + b_0 & \cdots & y_2^{n-1} + \cdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & y_n + a_0 & y_n^2 + b_1 y_n + b_0 & \cdots & y_n^{n-1} + \cdots \end{vmatrix} \\ &= \frac{2^{(n-1)n/4}\cdot e^{-(y_1^2+\cdots+y_n^2)}}{\pi^{n/2}\prod_{k=0}^{n-1}\sqrt{k!}} \begin{vmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \cdots & y_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & y_n & y_n^2 & \cdots & y_n^{n-1} \end{vmatrix} \\ &= \frac{2^{(n-1)n/4}\cdot e^{-(y_1^2+\cdots+y_n^2)}}{\pi^{n/2}\prod_{k=0}^{n-1}\sqrt{k!}} \prod_{1\le i<j\le n}\left(y_j - y_i\right) \neq 0. \end{aligned} \tag{61}$$

To complete the proof of the fact that the $C^*$-algebra $\mathfrak{G}_n^H$ separates all the pure states of $\mathfrak{D}$, only missing step is to separate the pure states of the forms $f_{x_0,w}$ and $f_{x_0,v}$, where $v, w \in \mathbb{S}^n$ and $x_0 \in \mathbb{R}$.

**Lemma 13.** *Given $v, w \in \mathbb{S}^n$ and $x_0 \in \mathbb{R}$ being fixed, consider the spectral matrix-valued-functions $\gamma^{n,\alpha}(x), \gamma^{n,\beta}(x) \in \mathfrak{G}_n^H$ for all $\alpha, \beta \in \mathbb{R}$, which are given by (32). If $f_{x_0,v}(\gamma^{n,\alpha}(x)\gamma^{n,\beta}(x)) = f_{x_0,w}(\gamma^{n,\alpha}(x)\gamma^{n,\beta}(x))$ for all $\alpha, \beta \in \mathbb{R}$, then $v = tw$, where $t \in \mathbb{C}$ and $|t| = 1$.*

*Proof.* From Lemma 11, we have $|\langle v, N_n(y)\rangle|^2 = |\langle w, N_n(y)\rangle|^2$ for all $y$, which implies that there exist a function $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\langle v, N_n(y)\rangle = e^{i\theta(y)}\langle w, N_n(y)\rangle \quad \text{for all } y \in \mathbb{R}. \tag{62}$$

For each $u \in \mathbb{S}^n$, we define the function $H_u : \mathbb{R}^2 \longrightarrow \mathbb{C}$ given by

$$\begin{aligned} H_u(\alpha, \beta) &= f_{x_0,v}\left(\gamma^{n,\alpha}(x)\gamma^{n,\beta}(x)\right) \\ &= \int_{-\infty}^{\sqrt{2}\alpha - x_0} \overline{u}^T N_n(y)\left[N_n(y)\right]^T dy \\ &\quad \cdot \int_{-\infty}^{\sqrt{2}\beta - x_0} N_n(t)\left[N_n(t)\right]^T u\, dt. \end{aligned} \tag{63}$$

Without loss of generality we can assume that $x_0 = 0$, just to simplify the calculations. Thus we calculate the second derivative of $H_u$ with respect to $\alpha$ and $\beta$; we obtain

$$\begin{aligned} &\frac{\partial^2 H_u}{\partial\alpha\partial\beta}\left(\frac{\alpha}{\sqrt{2}}, \frac{\beta}{\sqrt{2}}\right) \\ &\quad = \overline{u}^T N_n(\alpha)\left[N_n(\alpha)\right]^T N_n(\beta)\left[N_n(\beta)\right]^T u \end{aligned} \tag{64}$$

$$= \langle N_n(\alpha), u\rangle\langle N_n(\alpha), N_n(\beta)\rangle\langle u, N_n(\beta)\rangle \tag{65}$$

By hypothesis, we have that $H_v = H_w$ which implies that $\partial^2 H_v/\partial\alpha\partial\beta = \partial^2 H_w/\partial\alpha\partial\beta$; using this fact and (62) and (64) we obtain

$$\begin{aligned} &\langle N_n(\alpha), w\rangle\langle N_n(\alpha), N_n(\beta)\rangle\langle w, N_n(\beta)\rangle \\ &\quad = e^{i(\theta(\beta)-\theta(\alpha))}\langle N_n(\alpha), w\rangle\langle N_n(\alpha), N_n(\beta)\rangle \\ &\quad \cdot \langle w, N_n(\beta)\rangle \end{aligned} \tag{66}$$

for all $\alpha, \beta \in \mathbb{R}$.

It is clear that there exist $\beta_0 \in \mathbb{R}$ fixed such that $\langle w, N_n(\beta_0)\rangle \neq 0$. Thus, turn out the above equation as follows:

$$\left(1 - e^{i(\theta(\beta_0)-\theta(\alpha))}\right)\langle N_n(\alpha), w\rangle\langle N_n(\alpha), N_n(\beta_0)\rangle = 0 \tag{67}$$

for all $\alpha \in \mathbb{R}$.

Notice that $\langle N_n(\alpha), w\rangle\langle N_n(\alpha), N_n(\beta_0)\rangle$ is nonzero polynomial with respect to $\alpha$; thus the above equation implies that the function $\theta$ is constant. From (62), we obtain that $\langle v - e^{i\theta_0}w, N_n(y)\rangle = 0$ for all $y \in \mathbb{R}$. Using (61) it is clear that $v - e^{i\theta_0} = 0$, which implies the result. □

*Remark 14.* The above result completes the proof that the algebra $\mathfrak{G}_n^H$ separates to all pure states of $\mathfrak{D}$.

**The noncommutative Stone-Weierstrass conjecture:** let $\mathscr{B}$ be a $C^*$-subalgebra of a $C^*$-algebra $\mathscr{A}$, and suppose that $\mathscr{B}$ separates all the pure states of $\mathscr{A}$ (and 0 if $\mathscr{A}$ is nonunital). Then $\mathscr{A} = \mathscr{B}$.

This conjecture for a $C^*$-algebra type I was proved by I. Kaplansky in [8]. In consequence, we have proved that the algebra $\mathfrak{G}_n^H$ is equal to $\mathfrak{D}$. From Corollary 8 we have that the algebra of Toeplitz operators $\mathscr{T}^n_{-\infty,\infty}$ is isometric and isomorphic to algebra $\mathfrak{D}$. In summary, we have the following result.

**Theorem 15.** *The $C^*$-algebra $\mathscr{T}^n_{-\infty,\infty}$ is isomorphic and isometric to the $C^*$-algebra $\mathfrak{D}$. The isomorphism is given by*

$$\mathscr{T}^n_{-\infty,\infty} : T_{n,a} \longmapsto \left(SymT_{n,a}\right)(x) = \gamma^{n,a}(x), \qquad (68)$$

*where $\gamma^{n,a}(x)$ is given in (32).*

**Corollary 16.** *The $C^*$-algebra $\mathscr{T}^{(n)}_{-\infty,\infty}$ is isomorphic and isometric to the commutative $C^*$-algebra $C[-\infty,\infty]$. The isomorphism is given by*

$$\mathscr{T}^{(n)}_{-\infty,\infty} : T_{(n),a} \longmapsto \left(SymT_{(n),a}\right)(x) = \gamma_{(n),a}(x), \qquad (69)$$

*where $\gamma_{(n),a}(x)$ is given in (29).*

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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