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Research Article

The Sub-Supersolution Method and Extremal Solutions of Quasilinear Elliptic Equations in Orlicz-Sobolev Spaces

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We prove the existence of extremal solutions of the following quasilinear elliptic problem $-\sum_{i=1}^{N}(\partial/\partial x_i)a_i(x,u(x),Du(x)) + g(x,u(x),Du(x)) = 0$ under Dirichlet boundary condition in Orlicz-Sobolev spaces $W_0^1 L_M(\Omega)$ and give the enclosure of solutions. The differential part is driven by a Leray-Lions operator in Orlicz-Sobolev spaces, while the nonlinear term $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying a growth condition. Our approach relies on the method of linear functional analysis theory and the sub-supersolution method.

1. Introduction

The aim of this paper is to study some qualitative properties of solutions of the following quasilinear elliptic problem:

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,u(x),Du(x)) + g(x,u(x),Du(x)) = 0 \quad \text{in } \Omega \qquad (1)$$

$$u = 0 \quad \text{on } \partial\Omega,$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$ in Orlicz-Sobolev spaces. The differential part is driven by a Leray-Lions operator, while the nonlinear term $g : \Omega \times \mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying a growth condition.

In [1, Chapter 3], the differential part of (1) is a Leray-Lions operator in Sobolev spaces and the nonlinearity $g(x,s,\xi)$ satisfies the growth condition:

$$|g(x,s,\xi)| \le k_1(x) + c|\xi|^{p-1}, \qquad (2)$$

with the constant $c > 0$ and $k_1(x) \in L_+^{p'}(\Omega)$, for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$, where p' is the conjugate Hölder exponent to p, i.e., $1/p + 1/p' = 1$. In [2], the nonlinearity $g(x,s,\xi)$ satisfies the growth condition:

$$|g(x,s,\xi)| \le k_1(x) + c|\xi|^{r}, \qquad (3)$$

with the constant $c \ge 0$ and $k_1(x) \in L^q(\Omega)$, $k_1 \ge 0$, $q > (p^*)'$, $0 \le r \le p/(p^*)'$, for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$, where p^* is the Sobolev conjugate of p. Faria [2] pointed that the condition (3) is more general than (2) because $p-1 < p/(p^*)'$. However, $p/(p^*)' = p-1-p/N < p-1$ if $0 < p < N$. Hence, the growth condition (3) is not more general than (2).

When trying to weaken the restriction on the Leray-Lions operator and the growth condition (2), one is led to replace $W_0^{1,p}(\Omega)$ with $W_0^1 L_M(\Omega)$ built from an Orlicz space $L_M(\Omega)$ instead of $L^p(\Omega)$, where M is an N-function. The choice $M(t) = |t|^p$, $p > 1$ leads to [1, Theorem 3.17]. A nonstandard example is $M(t) = \int_0^{|u|}\log(1+|t|^q)|t|^{p-2}t\,dt$ (see, e.g., [2, 3]).

Many papers used the surjectivity result for pseudomonotone operators (see, e.g., [1, Theorem 2.99]) defined on reflexive spaces to prove the existence of the solution (see, e.g., [1, 2, 4, 5]). Our method does not need the reflexivity of the spaces. It is well known that the Orlicz space is reflex if and only if both M and its complementary function $\overline{M}$ satisfy Δ_2-condition. However, there exist many spaces without reflexivity. For example, let $M(u) = (1+|u|)\ln(1+|u|) -$

$|u|$; then M satisfies Δ_2-condition, but its complementary function $\overline{M}(v) = \exp(|v|) - |v| - 1$ does not satisfy Δ_2-condition; i.e., $L_M(\Omega)$ is not reflexive.

In this paper, we get rid of the restriction of the reflexivity of the spaces and get a weak solution for (1) in Orlicz-Sobolev spaces by using a linear functional analysis method. We also give the enclosure of solutions and prove the existence of extremal solutions.

This paper is organized as follows. Section 2 contains some preliminaries and some technical lemmas which will be needed. In Section 3, we use the linear functional analysis method to prove the existence of solutions for (1) in separable Orlicz-Sobolev spaces and the sub-supersolutions method to give the enclosure of solutions and the existence of extremal solutions between a subsolution and a supersolution. We also get the compactness and directness of the solutions set.

For some results, we also refer to [6–13].

2. Preliminaries

For quick reference, we recall some basic results of Orlicz spaces. Good references are Adams [14, Chapter 8], Krasnosel'skil [15], Chen [16], and Gossez [17].

2.1. N-Function. Let $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be an N-function; i.e., M is continuous, convex, with $M(u) > 0$ for $u > 0$, $M(u)/u \longrightarrow 0$ as $u \longrightarrow 0$, and $M(u)/u \longrightarrow +\infty$ as $u \longrightarrow +\infty$. Equivalently, M admits the representation $M(u) = \int_0^u \phi(t)dt$, where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing, right continuous function, with $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$, and $\phi(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$.

The N-function $\overline{M}$ conjugated to M is defined by $\overline{M}(v) = \int_0^v \psi(s)ds$, where $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is given by $\psi(s) = \sup\{t : \phi(t) \le s\}$.

ϕ, ψ are called the right-hand derivatives of M, $\overline{M}$, respectively.

The N-function M is said to satisfy the Δ_2 condition near infinity ($M \in \Delta_2$, for short), if, for some $k > 1$ and $\tilde{u} > 0$, $M(2u) \le kM(u)$, $\forall u \ge \tilde{u}$.

Moreover, one has the following Young inequality: $uv \le M(u) + \overline{M}(v)$, $\forall u, v \ge 0$.

For the N-function M one defines the Sobolev conjugate M_* by $M_*^{-1}(t) = \int_0^t (M^{-1}(\tau)/\tau^{(N+1)/N})d\tau$, $t \ge 0$.

Let P, Q be two N-functions, we say that P grows essentially less rapidly than Q near infinity, denoted as $P \ll Q$, if for every $\varepsilon > 0$, $P(t)/Q(\varepsilon t) \longrightarrow 0$ as $t \longrightarrow +\infty$. This is the case if and only if $\lim_{t\longrightarrow+\infty} Q^{-1}(t)/P^{-1}(t) = 0$.

We will extend these N-functions into even functions on all $\mathbb{R}$.

For a measurable function u on Ω, its modular is defined by $\rho_M(u) = \int_\Omega M(|u(x)|)dx$.

2.2. Orlicz Spaces. Let Ω be an open and bounded subset of $\mathbb{R}^N$ and M be an N-function. The Orlicz class $\mathscr{K}_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that $\rho_M(u) < +\infty$(resp. $\rho_M(u/\lambda) < +\infty$ for some $\lambda > 0$). $L_M(\Omega)$ is a Banach space under the (Luxemburg) norm:

$$\|u\|_{(M)} = \inf\left\{\lambda > 0 : \rho_M\left(\frac{u}{\lambda}\right) \le 1\right\}, \tag{4}$$

and $\mathscr{K}_M(\Omega)$ is a convex subset of $L_M(\Omega)$ but not necessarily a linear space. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if $M \in \Delta_2$; moreover, $L_M(\Omega)$ is separable.

$L_M(\Omega)$ is reflexive if and only if $M \in \Delta_2$ and $\overline{M} \in \Delta_2$.

Convergences in norm and in modular are equivalent if and only if $M \in \Delta_2$.

The dual space of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{(\overline{M})}$.

2.3. Orlicz-Sobolev Spaces. We now turn to the Orlicz-Sobolev space: $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional partial derivatives lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach spaces under the norm

$$\|u\|_{\Omega,M} = \sum_{|\alpha|\le 1} \left\|D^\alpha u\right\|_{(M)}. \tag{5}$$

Denote $\|Du\|_{(M)} = \||Du|\|_{(M)}$ and $\|u\|_{1,M} = \|u\|_{(M)} + \|Du\|_{(M)}$. Clearly, $\|u\|_{1,M}$ is equivalent to $\|u\|_{\Omega,M}$.

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If $M \in \Delta_2$, then $W^1L_M(\Omega) = W^1E_M(\Omega)$. If $M \in \Delta_2$ and $\overline{M} \in \Delta_2$, then $W^1L_M(\Omega) = W^1E_M(\Omega)$ are reflexive; thus the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ are equivalent.

Lemma 1 (See [18, Lemma 2.2]). *For all $u \in W_0^1L_M(\Omega)$, one has*

$$\int_\Omega M\left(\frac{|u(x)|}{\operatorname{diam}\Omega}\right)dx \le \int_\Omega M(|Du(x)|)\,dx, \tag{6}$$

where diam Ω *is the diameter of* Ω.

Lemma 2 (See [19, Lemma 1]). *Let* meas Ω *be bounded and $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $\varphi(0) = 0$, $\varphi(r) \longrightarrow +\infty$ for $r \longrightarrow +\infty$. Then*

$$\frac{\int_\Omega \varphi(|Du(x)|)\,|Du(x)|\,dx}{\int_\Omega |Du(x)|\,dx} \longrightarrow +\infty \qquad \text{if } \int_\Omega |Du(x)|\,dx \longrightarrow +\infty. \tag{7}$$

Lemma 3 (See [20, Lemma 2.1]). *If $u \in W^1L_M(\Omega)$, then u^+, $u^- \in W^1L_M(\Omega)$ and*

$$Du^+ = \begin{cases} Du, & \text{if } u > 0, \\ 0, & \text{if } u \le 0, \end{cases} \qquad Du^- = \begin{cases} 0, & \text{if } u \ge 0, \\ -Du, & \text{if } u < 0. \end{cases} \tag{8}$$

Here $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$.

3. Main Results

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with Lipschitz boundary, M, P be two N-functions, and $\overline{M}, \overline{P}$ be the complementary functions of M, P, respectively. Assume that M satisfies the Δ_2 condition near infinity and $P \ll M$. By Theorem 2.2 and Proposition 2.1 in [21] the embeddings $W_0^1 L_M(\Omega) \hookrightarrow L_P(\Omega)$ and $W_0^1 L_M(\Omega) \hookrightarrow L_M(\Omega)$ are compact.

Let A be the following quasilinear elliptic differential operator in divergence form:

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), Du(x)), \tag{9}$$

where the coefficients $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $i = 1, \ldots, N$, are assumed to satisfy the following:

(H1) Each function $a_i(x, s, \xi)$ is a Carathéodory function. Also there exists a positive constant β and a nonnegative function $k_0 \in E_{\overline{M}}(\Omega)$ such that

$$\begin{aligned} &\left|a_i(x, s, \xi)\right| \\ &\quad \leq \beta\left[k_0(x) + \overline{P}^{-1}(M(|s|)) + \overline{M}^{-1}(M(|\xi|))\right] \end{aligned} \tag{10}$$

for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

(H2) $\sum_{i=1}^{N}(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(H3) $\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geq \nu M(|\xi|) - k(x)$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$, with some constant $\nu > 0$ and a function $k \in L^1(\Omega)$.

The differential operator A can be seen as a mapping from $W_0^1 L_M(\Omega)$ into its dual space $(W_0^1 L_M(\Omega))^*$ given by

$$\langle Au, v\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, u(x), Du(x)) \frac{\partial v(x)}{\partial x_i} dx \quad \text{for all } u, v \in W_0^1 L_M(\Omega). \tag{11}$$

Example 4. (1) The p-Laplacian operator $\Delta_p = \operatorname{div}(|Du|^{p-2} Du)$ is form A with the coefficients a_i, $i = 1, \ldots, N$, given by $a_i(x, s, \xi) = |\xi|^{p-2}\xi_i$ (see, e.g., [1, Example 2.110]).

(2) Let $p(t)$ be a given positive and continuous function which increases from 0 to $+\infty$ and $a_i(x, s, \xi) = (p(|\xi|)/|\xi|)\xi_i$. Then a_i, $i = 1, \ldots, N$, satisfy the conditions (H1)-(H3).

Consider the following nonlinear elliptic equation:

$$\begin{aligned} Au + g(x, u, Du) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{12}$$

Here, $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is assumed to be a Carathéodory function.

Let G denote the Nemytskij operator related to g by

$$G(u)(x) = g(x, u(x), Du(x)), \quad x \in \Omega. \tag{13}$$

For $u, v \in L^0(\Omega)$, we use the standard notations: $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$, $u^+ := u \vee 0$, $u^- := -u \wedge 0$, $u \leq v \Longleftrightarrow u(x) \leq v(x)$ for a.e. $x \in \Omega$. A weak solution of (12) is called a solution for short.

By Lemma 3, $W^1 L_M(\Omega)$ is closed under $\vee$ and $\wedge$. In fact, since $u \vee v = v + (u - v)^+$ and $u \wedge v = v - (u - v)^-$, $u \vee v$, $u \wedge v \in W^1 L_M(\Omega)$, for any $u, v \in W^1 L_M(\Omega)$.

The following lemma can be found in [5, Remark 3.1] as the setting of Musielak-Orlicz spaces. However, we give another proof.

Lemma 5. *(a) $W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$) is closed under "$\vee$" and "$\wedge$", i.e., if $u, v \in W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$), then $u \vee v, u \wedge v \in W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$).*

(b) The mappings $\vee$ and $\wedge$: $W^1 L_M(\Omega) \times W^1 L_M(\Omega) \longrightarrow W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega) \times W_0^1 L_M(\Omega) \longrightarrow W_0^1 L_M(\Omega)$) are continuous, i.e., for any sequences $\{u_n\}, \{v_n\}$ in $W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$), if $u_n \longrightarrow u, v_n \longrightarrow v$ in $W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$), then $u_n \vee v_n \longrightarrow u \vee v$, $u_n \wedge v_n \longrightarrow u \wedge v$ in $W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$), as $n \longrightarrow \infty$.

Proof. (a) By Lemma 3, $W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$) is closed under $\vee$ and $\wedge$. In fact, since $u \vee v = v + (u - v)^+$ and $u \wedge v = v - (u - v)^-$, $u \vee v, u \wedge v \in W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$), for any $u, v \in W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$).

(b) Let $u_n \longrightarrow u, v_n \longrightarrow v$ in $W^1 L_M(\Omega)$ (resp., $W_0^1 L_M(\Omega)$), as $n \longrightarrow \infty$. Suppose that there exists $\varepsilon_1 > 0$ such that

$$\|u_n \vee v_n - u \vee v\|_{(M)} > \varepsilon_1 > 0 \tag{14}$$

for any $n \in \mathbb{N}$, then $\rho_M((u_n \vee v_n - u \vee v)/\varepsilon_1) > 1$.

Therefore, we have $\int_\Omega M((8/\varepsilon_1)|u_n(x) - u(x)|)dx \longrightarrow 0$, and $\int_\Omega M((8/\varepsilon_1)|v_n(x) - v(x)|)dx \longrightarrow 0$, as $n \longrightarrow \infty$. By passing to a subsequence if necessary, $u_n \longrightarrow u$, $v_n \longrightarrow v$, a.e. in Ω, as $n \longrightarrow \infty$, and there exist $f_1, f_2 \in L^1(\Omega)$ such that $M((8/\varepsilon_1)|u_n(x) - u(x)|) \leq f_1(x)$, and $M((8/\varepsilon_1)|v_n(x) - v(x)|) \leq f_2(x)$, which yields that $M((4/\varepsilon_1)|u_n(x)|) \leq (1/2)f_1 + (1/2)M((8/\varepsilon_1)|u(x)|)$, $M((4/\varepsilon_1)|v_n(x)|) \leq (1/2)f_2 + (1/2)M((8/\varepsilon_1)|v(x)|)$, for a.e. $x \in \Omega$.

Hence, $u_n \vee v_n \longrightarrow u \vee v$ a.e. in Ω, as $n \longrightarrow \infty$, and

$$\begin{aligned} &M\left(\frac{1}{\varepsilon_1}\left|(u_n \vee v_n)(x) - (u \vee v)(x)\right|\right) \\ &\quad \leq \frac{1}{4}\left[M\left(\frac{4}{\varepsilon_1}|u_n(x)|\right) + M\left(\frac{4}{\varepsilon_1}|v_n(x)|\right)\right. \\ &\quad \left. + M\left(\frac{4}{\varepsilon_1}|u(x)|\right) + M\left(\frac{4}{\varepsilon_1}|v(x)|\right)\right] \leq \frac{1}{4}\left[\frac{1}{2}f_1\right. \\ &\quad + \frac{1}{2}M\left(\frac{8}{\varepsilon_1}|u(x)|\right) + \frac{1}{2}f_2 + \frac{1}{2}M\left(\frac{8}{\varepsilon_1}|v(x)|\right) \\ &\quad \left. + M\left(\frac{4}{\varepsilon_1}|u(x)|\right) + M\left(\frac{4}{\varepsilon_1}|v(x)|\right)\right], \end{aligned} \tag{15}$$

for a.e. $x \in \Omega$.

By Lebesgue's theorem, we get $\int_\Omega M((1/\varepsilon_1)|(u_n \vee v_n)(x) - (u \vee v)(x)|)dx \longrightarrow 0$, as $n \longrightarrow \infty$; this is a contradiction. Consequently, $\|u_n \vee v_n - u \vee v\|_{(M)} \longrightarrow 0$, as $n \longrightarrow \infty$. Similarly, we can deduce that $\|D(u_n \vee v_n) - D(u \vee v)\|_{(M)} \longrightarrow 0$, $\|u_n \wedge v_n - u \wedge v\|_{(M)} \longrightarrow 0$ and $\|D(u_n \wedge v_n) - D(u \wedge v)\|_{(M)} \longrightarrow 0$, as $n \longrightarrow \infty$; that is, the mappings $\vee$ and $\wedge$ are continuous. □

A function u is called a (weak) solution of (12) if $u \in W_0^1 L_M(\Omega)$, $G(u) \in L_{\overline{P}}(\Omega)$ and u satisfies the following:

$$\langle Au, v\rangle + \int_\Omega G(u)(x)\, v(x)\, dx = 0, \quad \text{for all } v \in W_0^1 L_M(\Omega). \tag{16}$$

A function u is called a subsolution (resp. supersolution) of (12) if $u \in W_0^1 L_M(\Omega)$, $G(u) \in L_{\overline{P}}(\Omega)$, and (16) holds with "=" replaced with "≤" (resp. "≥") for every nonnegative functions v in $W_0^1 L_M(\Omega)$.

By Young inequality and $M \in \Delta_2$, there exist $K_1 > 1$ and $\tilde{u}_1 > 0$, such that $\overline{M}(\phi(u)) + M(u) = u\phi(u) \le M(2u) \le K_1 M(u) + M(2\tilde{u}_1)$ for all $u > 0$. Hence,

$$\overline{M}(\phi(u)) \le (K_1 - 1) M(u) + M(2\tilde{u}_1). \tag{17}$$

Theorem 6. *Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of problem (12), respectively, such that $\underline{u} \le \overline{u}$. Assume (H1)-(H3) and the following local growth condition for the nonlinearity g:*

$$|g(x,s,\xi)| \le k_1(x) + c\overline{P}^{-1}(M(|\xi|)) \tag{18}$$

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$, and all $s \in [\underline{u}(x), \overline{u}(x)]$, with $k_1 \in E_{\overline{P}}(\Omega)$, $k_1 \ge 0$, $c \ge 0$. Then there exists at least one solution $u \in W_0^1 L_M(\Omega)$ of problem (12) with $u \in [\underline{u}, \overline{u}] := \{v \in W_0^1 L_M(\Omega) : \underline{u} \le v \le \overline{u}\}$.

Proof. Denote $W_0^1 L_M(\Omega) = V$. For $x \in \Omega$, $u \in V$, we put

$$Tu(x) = \begin{cases} \overline{u}(x), & \text{if } u(x) > \overline{u}(x) \\ u(x), & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x) \\ \underline{u}(x), & \text{if } u(x) < \underline{u}(x) \end{cases} \quad \text{for } u \in V. \tag{19}$$

Then $Tu = u \vee \underline{u} + u \wedge \overline{u} - u$. By Lemma 5, $T : V \longrightarrow V$ is continuous. It is easy to see that T is bounded.

We define the cutoff function $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x,s) = \begin{cases} \phi(s - \overline{u}(x)), & \text{if } s > \overline{u}(x) \\ 0, & \text{if } \underline{u}(x) \le s \le \overline{u}(x) \\ -\phi(\underline{u}(x) - s), & \text{if } s < \underline{u}(x), \end{cases} \tag{20}$$

for $x \in \Omega$, $s \in \mathbb{R}$. Then f satisfies the following condition:

$$|f(x,s)| \le \phi(|\underline{u}| + |\overline{u}| + |s|), \tag{21}$$

for $x \in \Omega$ and all $s \in \mathbb{R}$.

Since M is convex and $M \in \Delta_2$, there exist $K_2 > 1$ and $\tilde{u}_2 > 0$ such that $M(|u|) \le (K_2/2)[M(u - \overline{u}(x)) + M(|\overline{u}(x)|)] + M(2\tilde{u}_2)$ whenever $u > \overline{u}(x)$, and $M(|u|) \le (K_2/2)[M(\underline{u}(x) - u) + M(|\underline{u}(x)|)] + M(2\tilde{u}_2)$ whenever $u < \underline{u}(x)$ for $x \in \Omega$, $u \in \mathbb{R}$. For all $u \in V$, we have

$$\begin{aligned}
\int_\Omega f(x,u(x))\, u(x)\, dx
&= \int_{\{u>\overline{u}\}} \phi(u(x) - \overline{u}(x))(u(x) - \overline{u}(x))\, dx \\
&\quad + \int_{\{u>\overline{u}\}} \phi(u(x) - \overline{u}(x))\, \overline{u}(x)\, dx \\
&\quad + \int_{\{u<\underline{u}\}} \phi(\underline{u}(x) - u(x))(\underline{u}(x) - u(x))\, dx \\
&\quad - \int_{\{u<\underline{u}\}} \phi(\underline{u}(x) - u(x))\, \underline{u}(x)\, dx \\
&\ge \int_{\{u>\overline{u}\}} M(u(x) - \overline{u}(x))\, dx \\
&\quad - \int_{\{u>\overline{u}\}} \overline{M}(\varepsilon_1 \phi(u(x) - \overline{u}(x)))\, dx \\
&\quad - \int_{\{u>\overline{u}\}} M\left(\frac{1}{\varepsilon_1}\overline{u}(x)\right) dx \\
&\quad + \int_{\{u<\underline{u}\}} M(\underline{u}(x) - u(x))\, dx \\
&\quad - \int_{\{u<\underline{u}\}} \overline{M}(\varepsilon_1 \phi(\underline{u}(x) - u(x)))\, dx \\
&\quad - \int_{\{u<\underline{u}\}} M\left(\frac{1}{\varepsilon_1}\underline{u}(x)\right) dx \\
&\ge [1 - \varepsilon_1(K_1 - 1)] \int_{\{u>\overline{u}\}} M(u(x) - \overline{u}(x))\, dx \\
&\quad + [1 - \varepsilon_1(K_1 - 1)] \int_{\{u<\underline{u}\}} M(\underline{u}(x) - u(x))\, dx \\
&\quad - C_1 \\
&\ge [1 - \varepsilon_1(K_1 - 1)] \frac{2}{K_2} \int_\Omega M(|u(x)|)\, dx - C_2 \\
&= \frac{1}{K_2} \int_\Omega M(|u(x)|)\, dx - C_2,
\end{aligned} \tag{22}$$

where $\varepsilon_1 = 1/2(K_1 - 1)$ and the constants $C_1, C_2 > 0$.

Define $\Gamma_T : V \longrightarrow V^*$,

$$\begin{aligned}
\langle \Gamma_T u, w\rangle &:= \int_\Omega \sum_{i=1}^N a_i(x, Tu(x), Du(x)) \frac{\partial w(x)}{\partial x_i} dx \\
&\quad + \lambda \int_\Omega f(x, u(x))\, w(x)\, dx \\
&\quad + \int_\Omega G(Tu)(x)\, w(x)\, dx,
\end{aligned} \tag{23}$$

$\forall w \in V$, where $\lambda > 0$ is a parameter to be specified later. Then Γ_T is well defined.

Since $M \in \Delta_2$, there exists a sequence $\{w_j\}_{j=1}^{\infty} \subset \mathscr{D}(\Omega)$ such that $\{w_j\}_{j=1}^{\infty}$ dense in V. Let $V_m = \operatorname{span}\{w_1, \ldots, w_m\}$ and consider $\Gamma_T|_{V_m} \cdot \int_\Omega |Du| dx$ and $\|Du\|_{(M)}$ are two norms of V_m equivalent to the usual norm of finite dimensional vector spaces.

Similar to the proof of Proposition 3.1 in [22], we can deduce that the mapping $u \longrightarrow \Gamma_T|_{V_m} u : V_m \longrightarrow V_m^*$ is continuous.

By (H3), (18), and (22),

$$\begin{aligned}\langle \Gamma_T u, u \rangle &\geq \nu \int_\Omega M(|Du(x)|)\,dx - \int_\Omega k(x)\,dx + \frac{\lambda}{K_2} \\ &\cdot \int_\Omega M(|u(x)|)\,dx - \lambda C_2 \\ &- \left[c\varepsilon_2 \int_\Omega M(|DTu(x)|)\,dx \right. \\ &\left. + c \int_\Omega P\left(\frac{1}{\varepsilon_2}|u(x)|\right) dx \right] - C_0 \|u\|_{1,M} \geq \frac{\nu}{2} \\ &\cdot \int_\Omega M(|Du(x)|)\,dx + \left(\frac{\lambda}{K_2} - 1 - cK_3\right) \\ &\cdot \int_\Omega M(|u(x)|)\,dx - C_3 - C_0 \|u\|_{1,M},\end{aligned} \tag{24}$$

for every $u \in V$, where $\varepsilon_2 = \nu/2c$, $K_3 > 0$ such that $P((1/\varepsilon_2)|u(x)|) \leq K_3 M(|u(x)|)$ and the constants $C_3, C_0 > 0$. Let $\lambda > K_2(1 + cK_3)$. Then we can deduce that

$$\langle \Gamma_T u, u \rangle \geq \frac{\nu}{2} \int_\Omega M(|Du(x)|)\,dx - C_3 - C_0 \|u\|_{1,M}. \tag{25}$$

By Lemma 1, we get

$$\begin{aligned}\frac{\int_\Omega M(|Du(x)|)\,dx}{\|u\|_{1,M}} &\geq \frac{1}{1 + \operatorname{diam}\Omega} \\ &\cdot \frac{\int_\Omega M(|Du(x)|)\,dx}{\|Du\|_{(M)}} \geq \frac{C}{2(1 + \operatorname{diam}\Omega)} \\ &\cdot \frac{\int_\Omega \phi(|Du(x)/2|)(|Du(x)|/2)\,dx}{\int_\Omega (|Du(x)|/2)\,dx},\end{aligned} \tag{26}$$

where the constant $C > 0$. By Lemma 2, we immediately have

$$\frac{\int_\Omega M(|Du(x)|)\,dx}{\|u\|_{1,M}} \longrightarrow +\infty \quad \text{as } \|u\|_{1,M} \longrightarrow +\infty. \tag{27}$$

Combining (25) and (27), we obtain

$$\frac{\langle \Gamma_T u, u \rangle}{\|u\|_{1,M}} \longrightarrow +\infty \quad \text{as } \|u\|_{1,M} \longrightarrow +\infty. \tag{28}$$

By Remark 2.1 in [22], for every m, there is a Galerkin solution $u_m \in V_m$ such that

$$\langle \Gamma_T u_m, v \rangle = 0, \quad \forall v \in V_m. \tag{29}$$

By the density of $\{w_m\}$, we get

$$\langle \Gamma_T u_m, v \rangle = 0, \quad \forall v \in V. \tag{30}$$

As the same proof in [22], we can deduce that the sequence $\{u_m\}$ is bounded in V and there exists $u_0 \in V$ and a subsequence $\{u_k\}$ of $\{u_m\}$, such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } V \text{ for } \sigma\left(\prod L_M, \prod E_{\overline{M}}\right), \tag{31}$$

$$u_k \longrightarrow u_0 \quad \text{strongly in } L_M(\Omega), \tag{32}$$

$$u_k \longrightarrow u_0 \quad \text{strongly in } L_P(\Omega), \tag{33}$$

$$u_k \longrightarrow u_0 \quad \text{a.e. in } \Omega, \tag{34}$$

$$\Gamma_T u_k \rightharpoonup 0 \quad \text{weakly in } V^* \text{ for } \sigma\left(\prod L_{\overline{M}}, \prod E_M\right), \tag{35}$$

as $k \longrightarrow +\infty$.

From (21), $\{f(x, u_k(x))\}$ is bounded in $L_{\overline{M}}(\Omega)$. By Lemma 4.4 of [17],

$$\begin{aligned}&f(x, u_k(x)) \rightharpoonup f(x, u_0(x)) \\ &\qquad \text{weakly in } L_{\overline{M}}(\Omega) \text{ for } \sigma\left(L_{\overline{M}}(\Omega), E_M(\Omega)\right),\end{aligned} \tag{36}$$

as $k \longrightarrow +\infty$.

On the other hand, thanks to (32) and (33), we have

$$\begin{aligned}&\int_\Omega f(x, u_k(x))(u_k(x) - u_0(x))\,dx \longrightarrow 0, \\ &\int_\Omega g(x, Tu_k(x), DTu_k(x))(u_k(x) - u_0(x))\,dx \longrightarrow 0,\end{aligned} \tag{37}$$

as $k \longrightarrow +\infty$. Thus we obtain that

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N a_i(x, Tu_k(x), Du_k(x)) \\ &\quad \cdot \left(\frac{\partial u_k(x)}{\partial x_i} - \frac{\partial u_0(x)}{\partial x_i}\right) dx \longrightarrow 0, \\ &\qquad\qquad \text{as } k \longrightarrow +\infty.\end{aligned} \tag{38}$$

Similar to the proof of Proposition 3.1 in [22], we can construct a subsequence still denoted by $\{u_k\}$ such that

$$Du_k \longrightarrow Du_0 \quad \text{a.e. in } \Omega, \text{ as } k \longrightarrow +\infty. \tag{39}$$

Hence,

$$\begin{aligned}&\sum_{i=1}^N a_i(x, Tu_k, Du_k) \longrightarrow \sum_{i=1}^N a_i(x, Tu_0, Du_0) \\ &\qquad\qquad \text{a.e. in } \Omega,\end{aligned} \tag{40}$$

as $k \longrightarrow +\infty$.

Following the lines of Theorem 1 in [2], we can deduce that $\underline{u} \leq u_k \leq \overline{u}$ for every $k \in \mathbb{N}$. By (34), $\underline{u} \leq u_0 \leq \overline{u}$.

Denote $\Omega_0 = \{x \in \Omega : \underline{u}(x) \leq u_0(x) \leq \overline{u}(x)\}$ and $\Omega_k = \{x \in \Omega : \underline{u}(x) \leq u_k(x) \leq \overline{u}(x)\}$. Then meas $\Omega \setminus \Omega_k =$ meas $\Omega \setminus$

$\Omega_0 = 0$, for every $k \in \mathbb{N}$. It follows from (39) that, passing to a subsequence if necessary,

$$DTu_k \longrightarrow DTu_0 \quad \text{a.e. in } \Omega, \text{ as } k \longrightarrow +\infty. \tag{41}$$

Therefore,

$$g\left(x, Tu_k, DTu_k\right) \longrightarrow g\left(x, Tu_0, DTu_0\right) \quad \text{a.e. in } \Omega, \text{ as } k \longrightarrow +\infty. \tag{42}$$

Since $\{u_k\}$ and $\{Tu_k\}$ are bounded in V, $\{\sum_{i=1}^{N} a_i(x, Tu_k, Du_k)\}$ is bounded in $L_{\overline{M}}(\Omega)$ and $\{g(x, Tu_k, DTu_k)\}$ is bounded in $L_{\overline{P}}(\Omega)$. By Lemma 4.4 of [17], $\Gamma_T u_k \rightharpoonup \Gamma_T u_0$ weakly in V^* for $\sigma(\prod L_{\overline{M}}, \prod E_M)$. Thanks to (35), one has $(\Gamma_T u_0, v) = 0$, for any $v \in V$. Therefore, we obtain that u_0 is a solution of (12). □

Under the assumptions of Theorem 6, we define

$$\mathcal{S} = \left\{u \in W_0^1 L_M(\Omega) : u \text{ is a solution of (12) and } \underline{u} \le u \le \overline{u}\right\}. \tag{43}$$

Theorem 7. *Under the assumptions of Theorem 6, the set $\mathcal{S}$ is compact in $W_0^1 L_M(\Omega)$.*

Proof. Let $\{u_n\}$ be a sequence in $\mathcal{S}$. It follows from the coerciveness of Γ_T that $\{u_n\}$ is bounded in $W_0^1 L_M(\Omega)$. As the same proof of Theorem 6, there exists that u_0 is a solution of (12) and $\underline{u} \le u_0 \le \overline{u}$, i.e., $u_0 \in \mathcal{S}$. □

To show that the set $\mathcal{S}$ is directed with respect to the usual pointwise order, the following additional assumption on the coefficients $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is required.

(H4) Let a nonnegative function $k \in L_{\overline{M}}(\Omega)$ and a continuous function $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ exist such that

$$\left|a_i(x, s, \xi) - a_i\left(x, s', \xi\right)\right| \le \left[k(x) + \overline{P}^{-1}(M(|s|)) + \overline{P}^{-1}\left(M\left(\left|s'\right|\right)\right) + \overline{M}^{-1}(M(|\xi|))\right]\omega\left(\left|s - s'\right|\right) \tag{44}$$

holds for a.e. $x \in \Omega$, for all $s, s' \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where ω satisfies $\int_{0^+}(dr/\omega(r)) = +\infty$, that is, for every $\varepsilon > 0$, $\int_0^{\varepsilon}(dr/\omega(r)) = +\infty$.

Similar to the proof of [1, Theorem 3.20], we can deduce the following result.

Theorem 8. *Assume hypotheses (H1)-(H4), and let u_1 and u_2 be subsolutions of (12) such that the Nemytskii operator*

$$G : \left[u_1 \wedge u_2, u_1 \vee u_2\right] \longrightarrow L_{\overline{M}}(\Omega) \tag{45}$$

is well defined. Then $u_1 \vee u_2$ is a subsolution of (12). Analogously, if u_1 and u_2 are supersolutions of (12) with the same assumption on the Nemytskii operator G, then $u_1 \wedge u_2$ is a supersolution.

Theorem 9. *Let the assumptions of Theorem 6 and (H4) hold. Then the following assertions about $\mathcal{S}$ are true.*

(a) *$\mathcal{S}$ is a direct set in both directions; that is, if $u_1, u_2 \in \mathcal{S}$ then there exist $u, v \in \mathcal{S}$ such that $u \ge u_1 \vee u_2$ and $v \le u_1 \wedge u_2$.*

(b) *$\mathcal{S}$ possesses extremal elements; i.e., there are $u_*, u^* \in \mathcal{S}$ such that $u_* \le u \le u^*$, for all $u \in \mathcal{S}$.*

Proof. (a) Let $u_1, u_2 \in \mathcal{S}$. Then u_1 and u_2 are both subsolutions and supersolutions of (12). It follows, from Theorem 8, $u_1 \vee u_2$ is a subsolution and $u_1 \wedge u_2$ is a supersolution of (12). The claim in (a) is now a straightforward consequence of Theorem 6.

(b) Since $W_0^1 L_M(\Omega)$ is separable, there exists a countable, dense subset $\{w_n : n \in \mathbb{N}\}$ of $\mathcal{S}$. Let $u_1 = w_1$. By (a), we can select $u_{n+1} \in \mathcal{S}$ such that $u_n \vee w_n \le u_{n+1} \le \overline{u}$. Thus, we get a bounded increasing sequence $\{u_n\} \subset \mathcal{S}$. Consequently, $\lim_{n\longrightarrow\infty} u_n(x) = \sup_{n\in\mathbb{N}} u_n(x) := u^*(x)$, for a.e. $x \in \Omega$, and there exists a subsequence $\{u_k\} \subset \{u_n\}$ such that $u_k \rightharpoonup u^*$ weakly in $W_0^1 L_M(\Omega)$ for $\sigma(\prod L_M, \prod E_{\overline{M}})$ as $k \longrightarrow +\infty$. Similar to the proof of Theorem 6, we can deduce that $u^* \in \mathcal{S}$. From the density of $\{w_n : n \in \mathbb{N}\}$, we can get that u^* is the greatest element of $\mathcal{S}$. The existence of the smallest element of $\mathcal{S}$ can be deduced in the same way. □

Remark 10. A special case in Theorem 6 is that $P = M$. In this case, choice $M(t) = |t|^p$ leads to Theorem 3.17 in [1].

Remark 11. The above results can be extended to the more general situation of Musielak-Orlicz-Sobolev spaces following our method developed in this paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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