Research Article

Composition Operators Mapping Logarithmic Bloch Functions into Hardy Space

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Characterizing the hyperbolic Hardy classes, several g -functions of hyperbolic type are introduced. Using this, necessary and sufficient conditions on the inducing self-maps are established for the boundedness of the composition operators from logarithmic Bloch spaces into Hardy spaces.

1. Introduction

This paper is to characterize the class of holomorphic self-map f of the open complex unit disc D for which the composition operator induced by f maps logarithmic Bloch space boundedly into Hardy space H^p . Main result of this paper is Theorem 2 whose primitive form is as follows.

Theorem 1. *If f is a holomorphic self-map of D and if $0 < p < \infty$, $-1/2 < \alpha < \infty$, then the following are equivalent:*

- (i) $h \circ f \in H^{2p/(1+2\alpha)}$ for all holomorphic h on D satisfying

$$\sup_{z \in D} |h'(z)| (1 - |z|) \left(\log \frac{e}{1 - |z|} \right)^{-\alpha} < \infty; \quad (1)$$

- (ii) $\sup_{0 \leq r < 1} \int_0^{2\pi} (\log(1/(1 - |f(re^{i\theta})|^2)))^p d\theta < \infty$;

- (iii) $\int_0^{2\pi} \left[\int_0^1 (|f'(re^{i\theta})|/(1 - |f(re^{i\theta})|^2))^2 (\log(1/(1 - |f(re^{i\theta})|^2)))^{2\alpha} (1 - r) dr \right]^{p/(1+2\alpha)} d\theta < \infty$.

When $\alpha = 0$ the above equivalence is known. We refer to [1–3], wherein the results are investigated in D and in the ball of $\mathbb{C}^n$, respectively.

Also, the case $p < 1 + 2\alpha$ was considered in [4], so the main case under consideration is $p \geq 1 + 2\alpha$. In the latter case, a different approach, based on duality, is used.

Note that Theorem 1 not only characterizes the composition operators mapping logarithmic Bloch functions into the

Hardy space but also introduces a kind of g -function. The result will be stated precisely and more extensively in Section 3.

The restriction of the range of α is essential. If $\alpha < -1/2$, then it reduces to a trivial result. If $\alpha = -1/2$, then it corresponds to another space instead of Hardy space. These will be treated separately in the last section.

2. Preliminaries

2.1. Hardy Space and Hyperbolic Hardy Class. Let D be the unit disc of the complex plane. For $0 < p < \infty$ and for $|f|$ subharmonic in D , we denote

$$\|f\|_p := \lim_{r \rightarrow 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}. \quad (2)$$

Then the right side limit is monotone increasing. And by definition, the Hardy space $H^p = H^p(D)$ consists of holomorphic f in D for which $\|f\|_p < \infty$, while the Yamashita hyperbolic Hardy class ϱH^p consists of holomorphic self-map f of D for which $\|\varrho(f)\|_p < \infty$. Here $\varrho(z)$ denotes the hyperbolic distance of z and 0 in D , namely,

$$\varrho(z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}. \quad (3)$$

Though ϱH^p is not a linear space, it has, as hyperbolic counterparts, many properties analogous to those of H^p .

For holomorphic self-maps f of D , we let

$$\lambda_f = \log \frac{1}{1 - |f|^2}, f^\# = \frac{|f'|}{1 - |f|^2} \quad (4)$$

following the notation of Yamashita and let

$$M_{\lambda_f}(\theta) = \sup \{ \lambda_f(re^{i\theta}) : 0 \leq r < 1 \}. \quad (5)$$

Then they have the following basic properties:

$$\varrho(f)^p, \lambda_f^p, (f^\#)^p \quad (6)$$

are subharmonic functions for any $p > 0$

$$\varrho f \text{ and } f^\# \text{ are automorphism invariant in the sense that} \quad (7)$$

$$\varrho(f, 0) = \varrho(\varphi_a \circ f, a) \text{ and } (\varphi_a \circ f)^\# = f^\#$$

for any $a \in D$, where

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in D. \quad (8)$$

$$\|\lambda_f\|_p < \infty \iff \|\varrho_f\|_p < \infty \iff f \in \varrho H^p \quad (9)$$

$$\|M_{\lambda_f}\|_{L^p} \leq \|\lambda_f\|_p \quad \text{if } f \in \varrho H^p \quad (10)$$

$$\begin{aligned} \Delta(\lambda_f^p) &= 4p \{ (p-1)|f|^2 + \lambda_f \} (f^\#)^2 \lambda_f^{p-2} \\ &\approx (f^\#)^2 \lambda_f^{p-1} \end{aligned} \quad (11)$$

Here and throughout, L^p means $L^p([0, 2\pi])$, Δ denotes the Laplacian: $\Delta = 4(\partial^2/\partial z \partial \bar{z})$, $\psi \lesssim \phi$ means that ψ is bounded by a positive uniform constant times ϕ , and $\psi \approx \phi$ means that either both sides are zero or the quotient ψ/ϕ lies between two positive uniform constants. We refer to [1, 5–7] for (6)–(11). For a general theory of H^p and ϱH^p , we refer to [8–10] and [7, 11, 12], respectively.

2.2. Logarithmic Bloch Space. For $-\infty < \alpha < \infty$, $\mathcal{B}_{\log \alpha}$ denotes the weighted Bloch space consisting of holomorphic functions h in D satisfying

$$\begin{aligned} \|h\|_{\mathcal{B}_{\log \alpha}} &:= \sup_{z \in D} |h'(z)| (1 - |z|^2) \left(\log \frac{e}{1 - |z|^2} \right)^{-\alpha} \\ &< \infty. \end{aligned} \quad (12)$$

$\mathcal{B}_{\log \alpha}$ is a Banach space equipped with the norm $|h(0)| + \|h\|_{\mathcal{B}_{\log \alpha}}$. Though the invariance “ $\|h\|_{\mathcal{B}_{\log \alpha}} = \|h \circ \varphi_a\|_{\mathcal{B}_{\log \alpha}}$ for all $a \in D$ and $h \in \mathcal{B}_{\log \alpha}$ ” is satisfied only when $\alpha = 0$, it is not difficult to see that we still have

$$h \in \mathcal{B}_{\log \alpha} \iff h \circ \varphi_a \in \mathcal{B}_{\log \alpha} \quad (13)$$

once $a \in D$ is fixed.

2.3. Bloch Pullback Problem. The composition operator induced by a holomorphic self-map f of D will be denoted as usual by $\mathcal{C}_f$, that is, $\mathcal{C}_f h = h \circ f$. For two function spaces X and Y , we denote $\mathcal{C}(X, Y)$ by the set of self-maps f for which $\mathcal{C}_f(X) \subset Y$. In this notation, (i) $\iff$ (ii) of Theorem 1 can be expressed as

$$\mathcal{C}(\mathcal{B}_{\log \alpha}, H^p) = \varrho H^{(p/2)(1+2\alpha)}, \quad -\frac{1}{2} < \alpha < \infty. \quad (14)$$

The problem of characterizing $\mathcal{C}(\mathcal{B}_{\log \alpha}, H^p)$ is a kind of Bloch pullback problem. Bloch pullback problem was initiated by P. Ahern and W. Rudin. See [13–20] for Bloch-BMOA pullbacks and [1–3, 17, 21] for Bloch-Hardy pullbacks.

2.4. New Hyperbolic $\mathcal{G}$ -Functions. For h holomorphic in D , it is well known that $\mathcal{G}$ -function of Littlewood-Paley defined by

$$\mathcal{G}h(\theta) := \left(\int_0^1 |h'(re^{i\theta})|^2 (1-r) dr \right)^{1/2}, \quad 0 \leq \theta < 2\pi \quad (15)$$

satisfies $\mathcal{G}h \in L^p \iff h \in H^p$ and

$$\|\mathcal{G}h\|_{L^p} \approx \|h - h(0)\|_p. \quad (16)$$

See [10, 22, 23]. Analogously, for holomorphic self-map f of D , the hyperbolic version of $\mathcal{G}$ -function defined by

$$\varrho \mathcal{G}f(\theta) := \int_0^1 [f^\#(re^{i\theta})]^2 (1-r) dr, \quad 0 \leq \theta < 2\pi \quad (17)$$

satisfies $\varrho \mathcal{G}f \in L^p \iff f \in \varrho H^p$ and

$$\|\varrho \mathcal{G}f\|_{L^p} \approx \|f\|_p \quad (18)$$

provided $f(0) = 0$. See [17].

We pay attention to the absence of the square root in the defining of $\varrho \mathcal{G}f$ in (17) when we compare it to that of $\mathcal{G}h$ in (15). The difference actually explains lots of known parallelism (see [11]) between H^2 and ϱH^1 . Suggested by (11), we define for $-1/2 < \alpha < \infty$

$$\begin{aligned} \varrho \mathcal{G}_\alpha f(\theta) &:= \left\{ \int_0^1 [f^\#(re^{i\theta})]^2 \lambda_f(re^{i\theta})^{2\alpha} (1-r) dr \right\}^{1/(1+2\alpha)}, \quad (19) \\ &0 \leq \theta < 2\pi. \end{aligned}$$

Note that $\varrho \mathcal{G}_0 f = \varrho \mathcal{G}f$. Of course, main objective of introducing (19) is to establish an equivalence as (18).

3. Equivalence between Norms

3.1. Main Result Revisited. Equipped with the notions introduced in Section 2, Theorem 1 can be stated as the first part of the following.

Theorem 2. Let $0 < p < \infty$, $-1/2 < \alpha < \infty$, and $q = p/(1 + 2\alpha)$. Let f be a holomorphic self-map of D . Then the following (i), (ii), and (iii) are mutually equivalent:

- (i) $f \in \mathcal{C}(\mathcal{B}_{\log \alpha}, H^{2q})$
- (ii) $f \in \varrho H^p$
- (iii) $\varrho \mathcal{G}_\alpha f \in L^p$.

Moreover, if we assume $f(0) = 0$ then

$$\|\varrho \mathcal{G}_\alpha f\|_{L^p} \approx \|\lambda_f\|_p \quad (20)$$

and

$$\|\mathcal{C}_f\|^{2q/p} \approx \|1 + \lambda_f\|_p, \quad (21)$$

where $\|\mathcal{C}_f\|$ denotes

$$\|\mathcal{C}_f\| := \sup \left\{ \|h \circ f\|_{2q} : h \in \mathcal{B}_{\log \alpha}, |h(0)| + \|h\|_{\mathcal{B}_{\log \alpha}} \leq 1 \right\}. \quad (22)$$

It follows by (13) that the restriction $f(0) = 0$ is not an essential one because $\mathcal{C}_f$ is bounded if and only if $\mathcal{C}_g$ is with $g = \varphi_{f(0)} \circ f$. Note that we used the notation $\|\mathcal{C}_f\|$ also for the case $2q < 1$ when H^{2q} are not normed spaces.

3.2. More on $\mathcal{G}$ -Function Equivalence. For (20), we in fact can prove more extensively the following: we define for $-1/2 < \alpha < \infty$ and $0 \leq \epsilon < \infty$ that

$$\varrho \mathcal{G}_{\alpha, \epsilon} f(\theta) := \left\{ \int_0^1 [f^\#(re^{i\theta})]^2 [\epsilon + \lambda_f(re^{i\theta})]^{2\alpha} \cdot (1-r) dr \right\}^{1/(1+2\alpha)}, \quad 0 \leq \theta < 2\pi. \quad (23)$$

Then

Theorem 3. Under the assumption of Theorem 2 with $f(0) = 0$,

$$\|\varrho \mathcal{G}_{\alpha, \epsilon} f\|_{L^p} \approx \|\epsilon + \lambda_f\|_p. \quad (24)$$

Immediately after this with (21) is the following.

Corollary 4. Under the assumption of Theorem 2 with $f(0) = 0$,

$$\|\varrho \mathcal{G}_{\alpha, 1} f\|_{L^p} \approx \|1 + \lambda_f\|_p \approx \|\mathcal{C}_f\|^{2q/p}. \quad (25)$$

To cover all of our results stated up to here, it is sufficient to prove (24) and (21). This will be done in Section 5 after stating preparatory lemmas in Section 4.

4. Preparatory Lemmas

We describe some lemmas, whose proof will be deferred to Section 6, that will be used in proving our main theorem.

Lemma 5. Let f be a holomorphic self-map of D and $-2 \leq \alpha < \infty$. Then, for any positive ϵ the function

$$F_\epsilon := (f^\#)^2 (\epsilon + \lambda_f)^\alpha \quad (26)$$

is subharmonic in D . Furthermore, for $0 \leq \epsilon < \infty$, $M_p(r, F_\epsilon)$ is an increasing function of r and

$$F_\epsilon(r^2 e^{i\theta}) \leq \int_0^{2\pi} P(r, \theta - t) F_\epsilon(re^{it}) \frac{dt}{2\pi}, \quad 0 \leq r < 1, \quad (27)$$

where $P(r, \theta)$ is the Poisson kernel: $P(r, \theta) = (1 - r^2)/(1 - 2r \cos \theta + r^2)$.

The subharmonicity of F_ϵ in Lemma 5 gives the following, where $dA = dx dy$.

Lemma 6. Let $0 < p < \infty$ and $0 \leq \epsilon < \infty$. Then for holomorphic self-maps f of D ,

$$\begin{aligned} \|\epsilon + \lambda_f\|_p^p - [\epsilon + \lambda_f(0)]^p &\approx \int_D [f^\#(z)]^2 [\epsilon + \lambda_f(z)]^{p-1} \log \frac{1}{|z|} dA \approx \int_0^1 \int_0^{2\pi} [f^\#(re^{i\theta})]^2 \\ &\cdot [\epsilon + \lambda_f(re^{i\theta})]^{p-1} (1-r) dr d\theta. \end{aligned} \quad (28)$$

We need the following inequalities which are not difficult to guess. See, for example, [24] or [25] for (29) and (30), and see [26] for (31).

Lemma 7. Let $0 < \beta < \infty$ and $-1 < \alpha < \infty$. Then, for $0 \leq r < 1$,

$$\left(\log \frac{e}{1-r} \right)^\beta \leq 1 + \sum_{k=0}^{\infty} (k+1)^{\beta-1} r^{2^k}; \quad (29)$$

$$\sum_{k=0}^{\infty} 2^k (k+1)^\alpha r^{2^k-1} \leq \frac{1}{1-r} \left(\log \frac{e}{1-r} \right)^\alpha; \quad (30)$$

$$\log \log \frac{e}{1-r} \leq \sum_{k=0}^{\infty} (k+1)^{-1} r^{2^k}. \quad (31)$$

5. Proof of Main Results

Let f be a holomorphic self-map of D with $f(0) = 0$. We are sufficient to show (24) and (21). We assume f is not constant because there's nothing left to prove when f is a constant. Let us denote for simplicity by S the boundary of D and σ the arc length measure on S normalized to be $d\sigma(\zeta) = |d\zeta|$.

5.1. Proof of (24). For notational clearance we prove only for $\epsilon = 0$. But replacing $\varrho \mathcal{G}_\alpha f$ and λ_f , respectively, by $\varrho \mathcal{G}_{\alpha, \epsilon} f$

and $\epsilon + \lambda_f$, it is easy to check that the proof below works for general ϵ in the same way.

To show $\|\varrho \mathcal{G}_\alpha f\|_{L^p} \leq \|\lambda_f\|_p$, we divide it into two cases: $q \leq 1$ and $q > 1$.

If $q \leq 1$, then by Hölder's inequality, (10) and Lemma 6, it follows that

$$\begin{aligned} \|\varrho \mathcal{G}_\alpha f\|_{L^p}^p &= \int_S \left\{ \int_0^1 [f^\#(r\zeta)]^2 \lambda_f(r\zeta)^{(2\alpha+1-p)+(p-1)} \right. \\ &\quad \cdot (1-r) dr \Big\}^q d\sigma(\zeta) \\ &\leq \int_S [M_{\lambda_f}(\zeta)]^{p(1-q)} \left\{ \int_0^1 [f^\#(r\zeta)]^2 \lambda_f(r\zeta)^{p-1} \right. \\ &\quad \cdot (1-r) dr \Big\}^q d\sigma(\zeta) \leq \|\lambda_f\|_p^{p(1-q)} \\ &\quad \cdot \left\{ \int_S \int_0^1 [f^\#(r\zeta)]^2 \lambda_f(r\zeta)^{p-1} (1-r) dr d\sigma \right\}^q \\ &\approx \|\lambda_f\|_p^p. \end{aligned} \quad (32)$$

Next, suppose $q > 1$. Set $\beta = 1 + 2\alpha$ and let $\mu := (f^\#)^2 \lambda_f^{2\alpha}$. Note, by (11) and (27),

$$\Delta \lambda_f^\beta \approx (f^\#)^2 \lambda_f^{2\alpha} = \mu \quad (33)$$

and

$$\mu(r^2 e^{i\theta}) \leq \int_0^{2\pi} P(r, \theta - t) \mu(re^{it}) \frac{dt}{2\pi}, \quad 0 \leq r < 1. \quad (34)$$

We make use of the representation

$$\begin{aligned} \|\varrho \mathcal{G}_\alpha f\|_{L^p}^\beta &= \|(\varrho \mathcal{G}_\alpha f)^\beta\|_{L^q} \\ &= \sup_h \int_S h(\zeta) [\varrho \mathcal{G}_\alpha f(\zeta)]^\beta d\sigma(\zeta), \end{aligned} \quad (35)$$

where the supremum is taken with respect to all nonnegative trigonometric polynomials h with $\|h\|_{L^s} \leq 1$, $1/s + 1/q = 1$.

If we set H the Poisson integral of h on S , then (34), a change of the order of the integration, and (33) give

$$\begin{aligned} \int_S h(\zeta) [\varrho \mathcal{G}_\alpha f(\zeta)]^\beta d\sigma(\zeta) &= \frac{1}{\pi} \int_0^{2\pi} h(e^{i\theta}) d\theta \\ &\quad \cdot \int_0^1 \mu(r^2 e^{i\theta}) (1-r^2) r dr \leq \frac{1}{\pi} \int_0^{2\pi} h(e^{i\theta}) d\theta \\ &\quad \cdot \int_0^1 (1-r^2) r dr \int_0^{2\pi} P(r, \theta - t) \mu(re^{it}) dt \\ &= 2 \int_S d\sigma(\zeta) \int_0^1 H(r\zeta) \mu(r\zeta) (1-r^2) r dr \\ &\approx \int_D H(z) \mu(z) (1-|z|^2) dA(z) \\ &\leq \int_D H(z) \Delta \lambda_f^\beta(z) \log \frac{1}{|z|} dA(z) \end{aligned} \quad (36)$$

while straightforward calculation gives

$$\Delta(\lambda_f^\beta H) = 4H \Delta \lambda_f^\beta + 4(\partial \lambda_f^\beta \bar{\partial} H + \bar{\partial} \lambda_f^\beta \partial H), \quad (37)$$

where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$. Whence by (36) we have

$$\begin{aligned} \int_S h(\zeta) [\varrho \mathcal{G}_\alpha f(\zeta)]^\beta d\sigma(\zeta) &\leq \int_D \Delta(\lambda_f^\beta H)(z) \log \frac{1}{|z|} dA \\ &\quad + 4 \int_D |\partial \lambda_f^\beta \bar{\partial} H + \bar{\partial} \lambda_f^\beta \partial H|(z) \log \frac{1}{|z|} dA \\ &:= (I) + (II). \end{aligned} \quad (38)$$

Now, using Green's theorem with limiting process and Hölder's inequality, we obtain

$$(I) \leq \|\lambda_f^\beta\|_q \|H\|_s \leq \|\lambda_f\|_p^\beta. \quad (39)$$

On the other hand, direct differentiation gives

$$|\partial \lambda_f^\beta| = |\bar{\partial} \lambda_f^\beta| = \beta |f| \lambda_f^{2\alpha} f^\# = \beta |f| \lambda_f^\alpha \sqrt{\mu}, \quad (40)$$

and

$$|\partial H| = |\bar{\partial} H| = \frac{1}{2} |H'|, \quad (41)$$

where H denotes the holomorphic function in D whose real part is H with $H(0) = 0$; whence

$$\begin{aligned} (II) &\leq 4\beta \int_D |H'(z)| |f(z)| \lambda_f^\alpha(z) \sqrt{\mu(z)} \log \frac{1}{|z|} dA \\ &\leq \int_S \int_0^1 |H'(r\zeta)| |f(r\zeta)| \lambda_f^\alpha(r\zeta) \\ &\quad \cdot \sqrt{\mu(r\zeta)} (1-r) dr d\sigma(\zeta). \end{aligned} \quad (42)$$

Since $|f| \leq \sqrt{\lambda_f}$, we have, by Schwarz inequality,

$$(II) \leq \int_S [M_{\lambda_f}(\zeta)]^{\beta/2} \mathcal{G}_H(\zeta) [\varrho \mathcal{G}_\alpha f(\zeta)]^{\beta/2} d\sigma(\zeta). \quad (43)$$

Applying Hölder's inequality with triple $(2q, s, 2q)$ and applying (11) to the last integral, we arrive at

$$(II) \leq \|\lambda_f\|_p^{\beta/2} \|\mathcal{G}_H\|_{L^s} \|\varrho \mathcal{G}_\alpha f\|_{L^p}^{\beta/2}. \quad (44)$$

From (16), we know $\|\mathcal{G}_H\|_{L^s} \leq \|H\|_s$, and it follows from the theorem of M. Riesz ([8, 23]) that $\|H\|_s \leq \|H\|_s \leq 1$. Thus

$$(II) \leq \|\lambda_f\|_p^{\beta/2} \|\varrho \mathcal{G}_\alpha f\|_{L^p}^{\beta/2}. \quad (45)$$

Gathering estimates (38), (39), and (45) up, we arrive at

$$\begin{aligned} \int_S h(\zeta) [\varrho \mathcal{G}_\alpha f(\zeta)]^\beta d\sigma(\zeta) &\leq (I) + (II) \\ &\leq \|\lambda_f\|_p^\beta + \|\lambda_f\|_p^{\beta/2} \|\varrho \mathcal{G}_\alpha f\|_{L^p}^{\beta/2} \end{aligned} \quad (46)$$

for all positive trigonometric polynomials h with $\|h\|_{L^s} \leq 1$. That is,

$$\|\varrho \mathcal{E}_\alpha f\|_{L^p}^\beta \leq \|\lambda_f\|_p^\beta + \|\lambda_f\|_p^{\beta/2} \|\varrho \mathcal{E}_\alpha f\|_{L^p}^{\beta/2}. \quad (47)$$

Therefore we obtain via the arithmetic-geometric mean inequality that

$$\|\varrho \mathcal{E}_\alpha f\|_{L^p} \leq \|\lambda_f\|_p. \quad (48)$$

We next show $\|\lambda_f\|_p \leq \|\varrho \mathcal{E}_\alpha f\|_{L^p}$. Set $\beta = 1 + 2\alpha$ again. Take $N > 1$ such that $p < \beta N$. Hölder's inequality with paring $(\beta N/p, \beta N/(\beta N - p))$ gives

$$\begin{aligned} \|\lambda_f\|_p^p &\approx \int_D [f^\#(z)]^2 \lambda_f(z)^{p-1} (1 - |z|^2) dx dy \\ &= \int_D [f^\#(z)]^2 \lambda_f(z)^{p/N + p(1-1/N)-1} \\ &\quad \cdot (1 - |z|^2) dx dy \\ &\leq \int_S \left(\int_0^1 [f^\#]^2 \lambda_f^{2\alpha} (1-r) dr \right)^{p/\beta N} \\ &\quad \cdot \left(\int_0^1 [f^\#]^2 \lambda_f^{2\delta} (1-r) dr \right)^{1-p/\beta N} d\sigma, \end{aligned} \quad (49)$$

where $2\delta = p(1-1/N)(\beta N/(\beta N - p)) - 1$. Applying Hölder's inequality one more time to the last quantity with paring $(N, N/(N-1))$,

$$\begin{aligned} \|\lambda_f\|_p^p &\leq \left(\int_S [\varrho \mathcal{E}_\alpha f]^p d\sigma \right)^{1/N} \left(\int_S [\varrho \mathcal{E}_\delta f]^p d\sigma \right)^{1-1/N} \\ &= \|\varrho \mathcal{E}_\alpha f\|_{L^p}^{p/N} \|\varrho \mathcal{E}_\delta f\|_{L^p}^{p(1-1/N)}. \end{aligned} \quad (50)$$

Therefore by applying the arithmetic-geometric mean inequality and (48), we obtain the desired inequality. This proof is completed.

5.2. Proof of (21). If $h \in \mathcal{B}_{\log \alpha}$, then (16), the definition of $\mathcal{B}_{\log \alpha}$, and (24) gives

$$\begin{aligned} \|h \circ f - h \circ f(0)\|_{2q}^{2q} &\approx \int_S \left(\int_0^1 (1-r) |h'(f) \cdot f'|^2 \right. \\ &\quad \cdot (r\zeta) dr \Big)^q d\sigma(\zeta) \lesssim \|h\|_{\mathcal{B}_{\log \alpha}}^{2q} \\ &\quad \cdot \int_S \left(\int_0^1 (1-r) [f^\#(r\zeta)]^2 \right. \\ &\quad \cdot [1 + \lambda_f(r\zeta)]^{2\alpha} dr \Big)^q d\sigma(\zeta) \\ &= \|h\|_{\mathcal{B}_{\log \alpha}}^{2q} \|\varrho \mathcal{E}_{\alpha,1} f\|_{L^p}^p \approx \|h\|_{\mathcal{B}_{\log \alpha}}^{2q} \|1 + \lambda_f\|_p^p. \end{aligned} \quad (51)$$

This with simple inequality

$$\|h \circ f\|_{2q}^{2q} - |h \circ f(0)|^{2q} \lesssim \|h \circ f - h \circ f(0)\|_{2q}^{2q} \quad (52)$$

gives

$$\|h \circ f\|_{2q}^{2q} \lesssim \left(|h(0)| + \|h\|_{\mathcal{B}_{\log \alpha}} \right)^{2q} \|1 + \lambda_f\|_p^p; \quad (53)$$

whence this verifies $\|\mathcal{E}_f\|^{2q/p} \leq \|1 + \lambda_f\|_p$.

We next show $\|1 + \lambda_f\|_p \leq \|\mathcal{E}_f\|^{2q/p}$. For each nondyadic $t \in [0, 1]$, let

$$h_t(z) = 1 + \sum_{k=0}^{\infty} \gamma_k(t) (k+1)^\alpha z^{2^k}, \quad z \in D, \quad (54)$$

where $\gamma_k = \gamma_k(t)$ is the Rademacher function (see [8, 23]) defined by

$$\gamma_k(t) = \text{sign} \sin(2^{k+1}\pi t). \quad (55)$$

Then h_t is holomorphic in D and by (30) there is a positive constant C_α such that

$$(1 - |z|) |h'_t(z)| \leq C_\alpha \left(\log \frac{e}{1 - |z|} \right)^\alpha, \quad (56)$$

so that $h_t \in \mathcal{B}_{\log \alpha}$ with $\|h_t\|_{\mathcal{B}_{\log \alpha}} \leq C_\alpha$. Thus, the definition of $\|\mathcal{E}_f\|$ gives

$$\int_S |h_t \circ f(r\zeta)|^{2q} d\sigma(\zeta) \leq (1 + C_\alpha)^{2q} \|\mathcal{E}_f\|^{2q}. \quad (57)$$

On the other hand, it follows by (29) that

$$\begin{aligned} \|1 + \lambda_f\|_p^p &= \lim_{r \rightarrow 1} \int_S \left(\log \frac{e}{1 - |f(r\zeta)|^2} \right)^p d\sigma(\zeta) \\ &\lesssim \lim_{r \rightarrow 1} \int_S \left(1 + \sum_{k=0}^{\infty} (k+1)^{2\alpha} |f(r\zeta)^2|^{2^k} \right)^q d\sigma(\zeta), \end{aligned} \quad (58)$$

while Khinchin's inequality (see [23] Theorem 8.4) gives

$$\begin{aligned} &\left(1 + \sum_{k=0}^{\infty} (k+1)^{2\alpha} |f(r\zeta)^2|^{2^k} \right)^q \\ &\leq \int_0^1 |h_t \circ f(r\zeta)|^{2q} dt. \end{aligned} \quad (59)$$

Now (57), (58), and (59) together verify $\|1 + \lambda_f\|_p \leq \|\mathcal{E}_f\|^{2q/p}$. This proof is completed.

6. Proof of Lemmas

For measurable functions f in D , let us denote as usual

$$M_p(r, f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad 0 \leq r < 1. \quad (60)$$

6.1. *Proof of Lemma 5.* Let $0 < \epsilon < \infty$ and $e^\epsilon = c$. From

$$\begin{aligned} F_\epsilon &:= (f^\#)^2 (\epsilon + \lambda_f)^\alpha \\ &= |f'|^2 (1 - |f|^2)^{-2} \left(\log \frac{c}{1 - |f|^2} \right)^\alpha \end{aligned} \quad (61)$$

we get

$$\frac{\partial}{\partial z} F_\epsilon = \phi F_\epsilon, \quad \frac{\partial}{\partial \bar{z}} F_\epsilon = \overline{\frac{\partial}{\partial z} F_\epsilon}, \quad (62)$$

where

$$\phi = \left\{ \frac{f''}{f'} + \left[2 + \alpha \left(\log \frac{c}{1 - |f|^2} \right)^{-1} \right] \frac{f' \bar{f}}{1 - |f|^2} \right\}. \quad (63)$$

Thus,

$$\frac{\partial^2}{\partial z \partial \bar{z}} F_\epsilon = \left\{ |\phi|^2 + \frac{\partial \phi}{\partial \bar{z}} \right\} F_\epsilon. \quad (64)$$

Noting that

$$\begin{aligned} \frac{\partial \phi}{\partial \bar{z}} &= (f^\#)^2 \left\{ 2 + \alpha \left(\log \frac{c}{1 - |f|^2} \right)^{-1} \right. \\ &\quad \left. - \alpha |f|^2 \left(\log \frac{c}{1 - |f|^2} \right)^{-2} \right\} \end{aligned} \quad (65)$$

and $\alpha \geq -2$, we have

$$\frac{\partial^2}{\partial z \partial \bar{z}} F_\epsilon \geq 0. \quad (66)$$

Thus, F_ϵ is subharmonic if $0 < \epsilon < \infty$.

For $0 < \epsilon < \infty$, subharmonicity of F_ϵ implies that $M_p(r, F_\epsilon)$ is an increasing function of r :

$$M_p(r_1, F_\epsilon) \leq M_p(r_2, F_\epsilon) \quad \text{if } r_1 < r_2 < 1. \quad (67)$$

Also, by subharmonicity,

$$F_\epsilon(r^2 e^{i\theta}) \leq \int_0^{2\pi} P(r, \theta - t) F_\epsilon(re^{it}) \frac{dt}{2\pi}, \quad 0 \leq r < 1. \quad (68)$$

Letting $\epsilon \rightarrow 0$, monotone convergence theorem guarantees (27) and (28) also for $\epsilon = 0$. This proof is completed.

6.2. *Proof of Lemma 6.* First let $0 < \epsilon < \infty$. Apply Green's Theorem of the form

$$\frac{d}{dr} \int_0^{2\pi} \psi d\theta = \frac{1}{r} \iint_{rD} \Delta \psi dx dy \quad (69)$$

valid for $\psi = (\epsilon + \lambda_f)^p \in C^2(r\bar{D})$. Recalling (11), it follows by a limiting process after integrating with respect to dr that

$$\begin{aligned} \|\epsilon + \lambda_f\|_p^p - [\epsilon + \lambda_f(0)]^p &\approx \int_0^{2\pi} \frac{d\theta}{2\pi} \\ &\cdot \int_0^1 [f^\#(\rho e^{i\theta})]^2 [\epsilon + \lambda_f(\rho e^{i\theta})]^{p-1} \log \frac{1}{\rho} \rho d\rho. \end{aligned} \quad (70)$$

Thus, the first equivalence follows.

The second equivalence follows from Lemma 5: setting $F_\epsilon := (f^\#)^2 (\epsilon + \lambda_f)^{p-1}$, the inequality

$$r \log \frac{1}{r} \leq 1 - r \leq \log \frac{1}{r} \quad (71)$$

and the increasing property of $M_1(r, F_\epsilon)$ give

$$\begin{aligned} \int_D F_\epsilon(z) \log \frac{1}{|z|} dA &\leq \int_0^1 \int_0^{2\pi} F_\epsilon(re^{i\theta}) (1 - r) dr d\theta \\ &= \int_0^1 \int_0^{2\pi} F_\epsilon(r^2 e^{i\theta}) (1 - r^2) 2r dr d\theta \\ &\leq 4 \int_0^1 \int_0^{2\pi} F_\epsilon(re^{i\theta}) \log \frac{1}{r} r dr d\theta \\ &= 4 \int_D F_\epsilon(z) \log \frac{1}{|z|} dA. \end{aligned} \quad (72)$$

Next, letting $\epsilon \rightarrow 0$ we obtain (28) for $\epsilon = 0$ also. This proof is completed.

7. Remarks and Acknowledgement

In view of our main result, Theorem 2, there arises a natural question: what is $\mathcal{C}(\mathcal{B}_{\log \alpha}, H^p)$ when $\alpha \leq -1/2$?

Let $\alpha < -1/2$. If we denote the identity map of D by I , that is, $I(z) = z, z \in D$, then

$$I \in \mathcal{C}(\mathcal{B}_{\log \alpha}, H^p) \quad \text{for any } p \quad (73)$$

simply because

$$\begin{aligned} \int_0^1 \left(\frac{|I'(r\zeta)|}{1 - |I(r\zeta)|^2} \right)^2 \left(\log \frac{1}{1 - |I(r\zeta)|^2} \right)^{2\alpha} (1 - r) dr \\ \in L^\infty(S) \subset L^p(S) \end{aligned} \quad (74)$$

which gives $\mathcal{G}(h \circ I) \in L^p(S)$, equivalently $h \circ I \in H^p$, for all $h \in \mathcal{B}_{\log \alpha}$.

Let f be holomorphic self-map of D satisfying $f(0) = 0$. Then f is subordinate to I . By (73) and Littlewood Subordination Theorem ([8]), we have $f \in \mathcal{C}(\mathcal{B}_{\log \alpha}, H^p)$ for any p . But by (13) we can remove the condition $f(0) = 0$. Therefore we have

$$\mathcal{C}(\mathcal{B}_{\log \alpha}, H^p) = \{\text{holomorphic self-map of } D\} \quad (75)$$

for any p .

The case $\alpha = -1/2$ was considered in [26] (partially) in connection with (31) and the concept of the hyperbolic Nevanlinna class (see [27]).

Data Availability

No data were used to support this study.

Disclosure

The present author noticed the publication of [4, 25, 26] after finishing this paper.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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