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Research Article

Stepanov-Like Asymptotical Almost Periodic Functions and an Application

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In this paper, we first study some basic properties of Stepanov-like asymptotical almost periodic functions including the completeness of the space of Stepanov-like asymptotical almost periodic functions. Then, as an application, based on these and the contraction mapping principle, we obtain sufficient conditions for the existence and uniqueness of Stepanov-like asymptotical almost periodic solutions for a class of semilinear delay differential equations.

1. Introduction

Almost periodic functions, which are an important generalization of periodic functions, were introduced into the field of mathematics by Bohr [1, 2]. From the very beginning, the concept of almost periodic function has attracted extensive attention of mathematicians and has led to various extensions and variations of this concept. For example, Stepanov proposed a weaker concept of almost periodic functions in the sense of Bohr. For more details about Stepanov's almost periodic functions, see [3–11].

On the one hand, due to the fact that almost periodic phenomena exist in the real world, more and more scholars are interested in the almost periodicity and its various generalizations. For example, Diagana [12] introduced Stepanov-like pseudo almost periodicity in 2007. The Stepanov-like pseudo almost periodicity is a generalization of the classical pseudo almost periodicity [13]. The concept of Stepanov-like weighted pseudo almost periodicity was introduced by Diagana et al. [14]. This notion is more extensive than Stepanov-like pseudo almost periodicity. Moreover, Diagana also introduced Stepanov-like almost automorphic functions which are a generalization of the classical almost automorphic functions; for more details, see [15]. In 2009, Diagana introduced the notion of Stepanov-like pseudo almost automorphy which generalizes the concept of pseudo almost automorphy [16].

On the other hand, the concept of the asymptotically almost periodicity was introduced into the research field by French mathematician Frechet [17, 18]. Such a notion is a natural generalization of the concept of the almost periodicity in the sense of Bohr. Since then, asymptotically almost periodic functions have become a very important function class and to find asymptotically almost periodic solutions for differential equations has been a hot topic for researchers. For the basic properties of asymptotical almost periodic functions, we refer the reader to [19] and for some recent papers about the existence of asymptotically almost periodic solutions for differential equations arising in theory and application, we refer the reader to [20–25]. However, up to now, few studies have been done on Stepanov-like asymptotical almost periodic functions [26], but these studies are necessary.

Motivated by the above discussions, in this paper, we first study some basic properties of Stepanov-like asymptotical almost periodic functions. Then, based on these properties and by using the contraction mapping principle, we investigate the existence and uniqueness of Stepanov-like asymptotical almost periodic solutions for a class of semilinear delay differential equations.

2. Preliminaries

In this section, we recall some basis definitions and lemmas of Bohr almost periodic functions and Stepanov's

almost periodic functions which are used throughout this paper.

Let $(X, \|\cdot\|_X)$ be a Banach space and $BC(R, X)$ be the collection of bounded continuous functions from R to X with the norm $\|x\| = \sup_{t\in R}\|x(t)\|_X$.

Definition 1 (see [27]). A function $f \in BC(R, X)$ is said to be almost periodic in Bohr sense if for each $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that in every interval of length l of R one can find a number $\tau \in (a, a + l)$ with the property

$$\|f(t+\tau) - f(t)\|_X < \varepsilon, \quad t \in R. \tag{1}$$

We denote the space of all such functions by $AP(R, X)$; the norm of the space is

$$|f|_{AP} = \sup_{t\in R}\|f(t)\|_X. \tag{2}$$

Definition 2 (see [28]). Let Λ be a set of some almost periodic functions in Bohr sense. Then Λ is uniformly almost periodic family if it is uniformly bounded, equicontinuous and for every $\varepsilon > 0$ there exists a number $l > 0$ such that every interval of length l contains a number τ such that

$$\|f(t+\tau) - f(t)\|_X < \varepsilon, \quad \forall t \in R, \ \forall f \in \Lambda. \tag{3}$$

Lemma 3 (see [27]). *Function $f \in AP(R, X)$ is equivalent to the property of relative compactness for the family $F = \{f(t + h); h \in R\}$.*

Definition 4 (see [27]). The space M is defined as follows:

$$M(R, R) = \left\{x : R \longrightarrow R, \sup_{t\in R}\int_t^{t+1}|x(s)|\,ds < +\infty\right\} \tag{4}$$

with the norm defined by

$$|x|_M = \sup_{t\in R}\int_t^{t+1}|x(s)|\,ds. \tag{5}$$

Lemma 5 (see [27]). *The space $M(R, R)$ is a Banach space.*

Definition 6 (see [27]). A function $f \in M(R, R)$ is said to be Stepanov's almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that each interval $(a, a + l) \subset R$ contains a point τ with the property

$$\sup_{t\in R}\int_t^{t+1}|f(s+\tau) - f(s)|\,ds < \varepsilon. \tag{6}$$

We denote the space of all such functions by $S(R, R)$ and the norm of $S(R, R)$ is

$$\|f\|_S = \sup_{t\in R}\int_t^{t+1}|f(s)|\,ds. \tag{7}$$

Lemma 7 (see [27]). *The space $S(R, R)$ is a Banach space.*

Lemma 8 (see [27]). *A function $f \in S(R, R)$ if and only if $\varphi(\cdot,\cdot) \in AP(R, L([0, 1], R))$, where*

$$\varphi(t, x) = f(t + x), \quad t \in R, \ x \in [0, 1] \subset R. \tag{8}$$

Lemma 9 (see [27]). *A function $f \in S(R, R)$ if and only if $\mathcal{F} = \{f(t+h); h \in R\} \subset S(R, R)$ is relatively compact in $S(R, R)$.*

Lemma 10 (see [29]). *Let $f_1, f_2, \ldots, f_n$ be almost periodic functions in Bohr sense from R into Banach space $X_1, X_2, \ldots, X_n$, respectively. Then for each $\varepsilon > 0$, all the functions $f_1, f_2, \ldots, f_n$ have a common set of ε-almost periods.*

3. Stepanov-Like Asymptotic Almost Periodic Functions and Their Basic Properties

Let

$$M_0(R, R) = \left\{h \in M(R, R) \mid \lim_{t\longrightarrow+\infty}\int_t^{t+1}|h(s)|\,ds = 0\right\}, \tag{9}$$

then we give the following definition.

Definition 11 (see [26]). A function $f \in M(R, R)$ is said to be a Stepanov-like asymptotical almost periodic function if it can be expressed as

$$f(t) = g(t) + h(t), \tag{10}$$

where $g \in S(R, R)$, $h \in M_0(R, R)$. The collection of all such functions will be denoted by $AS(R, R)$.

Remark 12. Several equivalent statements of Definition 11 are given by Theorem 1.6.2 in [26].

Remark 13. Obviously, if $f_1, f_2 \in AS(R, R)$ and $\lambda \in R$, then $f_1 + f_2, \lambda f_1 \in AS(R, R)$.

Lemma 14. *The space of $M_0(R, R)$ is a Banach space endowed with the norm*

$$\|f\|_S = \sup_{t\in R}\int_t^{t+1}|f(s)|\,ds. \tag{11}$$

Proof. Let $\{h_n\} \subset M_0(R, R) \subset M(R, R)$ be a Cauchy sequence, then we can find a function $h \in M(R, R)$ such that

$$\lim_{n\longrightarrow\infty}\sup_{t\in R}\int_t^{t+1}|h_n(s) - h(s)|\,ds = 0. \tag{12}$$

Hence

$$\begin{aligned}\lim_{t\longrightarrow+\infty}\int_t^{t+1}|h(s)|\,ds &\le \lim_{t\longrightarrow+\infty}\int_t^{t+1}|h(s) - h_n(s)|\,ds \\ &\quad + \lim_{t\longrightarrow+\infty}\int_t^{t+1}|h_n(s)|\,ds \\ &\le \|h_n - h\|_S, \quad \forall n \in N.\end{aligned} \tag{13}$$

Since $\lim_{n\longrightarrow\infty}\|h_n - h\|_S = 0$, we deduce that

$$\lim_{t\longrightarrow+\infty}\int_t^{t+1}|h(s)|\,ds = 0, \tag{14}$$

that is, $h \in M_0(R, R)$. Thus, $M_0(R, R)$ is a Banach space. The proof is complete. □

Lemma 15. *Let $f = g + h \in AS(R,R), g \in S(R,R), h \in M_0(R,R)$ and $g(t+x) \in AP(R, L([0,1],R)), h(t+x) : R \longrightarrow L([0,1],R), f(t+x) : R \longrightarrow L([0,1],R), x \in [0,1]$, then*

$$\{g(t+x), t \in R\} \subset \overline{\{f(t+x), t \in R\}} \quad (\text{the closure of the range of } f). \tag{15}$$

Proof. Assume that (15) does not hold, then there exist $t_0 \in R$ and $\varepsilon > 0$ such that

$$\begin{aligned} &\|f(t+x) - g(t_0+x)\|_{L([0,1],R)} \\ &\quad = \int_0^1 |f(t+x) - g(t_0+x)|\, dx > 3\varepsilon \quad \forall t \in R. \end{aligned} \tag{16}$$

Since $g(t+x) \in AP(R, L([0,1],R))$, there exists $l > 0$ and for every $n \in Z$, there exists $\tau_n \in [nl - t_0, nl - t_0 + l]$ such that

$$\begin{aligned} &\|g(t_0+x+\tau_n) - g(t_0+x)\|_{L([0,1],R)} \\ &\quad = \int_0^1 |g(t_0+x+\tau_n) - g(t_0+x)|\, dx < \varepsilon. \end{aligned} \tag{17}$$

By using the uniform continuity on R of the almost periodic function $g(t+x)$, there exists $K_0 \in N$ such that $K_0 \geq 2$ and for all $t \in [t_0 + \tau_n - l/K_0, t_0 + \tau_n + l/K_0]$,

$$\begin{aligned} &\|g(t+x) - g(t_0+x+\tau_n)\|_{L([0,1],R)} \\ &\quad = \int_0^1 |g(t+x) - g(t_0+x+\tau_n)|\, dx < \varepsilon. \end{aligned} \tag{18}$$

From (16)-(18), it follows that

$$\begin{aligned} &\int_0^1 |f(t+x) - g(t_0+x)|\, dx \\ &\quad \leq \int_0^1 |f(t+x) - g(t+x)|\, dx \\ &\qquad + \int_0^1 |g(t+x) - g(t_0+x+\tau_n)|\, dx \\ &\qquad + \int_0^1 |g(t_0+x+\tau_n) - g(t_0+x)|\, dx \\ &\qquad \forall t \in \left[t_0 + \tau_n - \frac{l}{K_0}, t_0 + \tau_n + \frac{l}{K_0}\right]. \end{aligned} \tag{19}$$

Since $f = g + h$, by (19), we have

$$\begin{aligned} &\|h(t+x)\|_{L([0,1],R)} = \int_0^1 |h(t+x)|\, dx > \varepsilon \\ &\qquad \forall t \in \left[t_0 + \tau_n - \frac{l}{K_0}, t_0 + \tau_n + \frac{l}{K_0}\right]. \end{aligned} \tag{20}$$

Thus,

$$\begin{aligned} &\int_t^{t+1} |h(s)|\, ds > \varepsilon, \\ &\qquad \forall t \in \left[t_0 + \tau_n - \frac{l}{K_0}, t_0 + \tau_n + \frac{l}{K_0}\right], \end{aligned} \tag{21}$$

which contradicts

$$\lim_{t \longrightarrow +\infty} \int_t^{t+1} |h(s)|\, ds = 0. \tag{22}$$

Consequently, (15) holds. The proof is complete. □

Definition 16. A function $g \in S(R \times X, R)$ if the following three conditions are true:

(i) for every $x \in X$, $g(\cdot, x) \in S(R,R)$,

(ii) the set $\{g(\cdot, x) \mid x \in X\}$ is uniformly bounded in the S-norm and equicontinuous in the S-norm,

(iii) for every $\varepsilon > 0$, there exists a number $l > 0$ such that every interval of length l contains a number τ with the property

$$\sup_{t \in R} \int_t^{t+1} |g(s+\tau, x) - g(s,x)|\, ds < \varepsilon, \quad \forall x \in X. \tag{23}$$

Lemma 17. *For a bounded continuous function $f : R \longrightarrow X$, denote*

$$V_f(\tau) = \sup_{\theta \in R} \|f(\theta+\tau) - f(\theta)\|_X, \quad \tau \in R. \tag{24}$$

Then V_f satisfies the following properties:

(a) $V_f(\tau) \geq 0, V_f(-\tau) = V_f(\tau), \tau \in R$.

(b) $V_f(0) = 0$.

(c) $V_f(t+s) \leq V_f(t) + V_f(s), s, t \in R$

(d) $f \in AP(R,X)$ *if and only if* $V_f \in AP(R,R)$.

Proof. Properties (a) through (c) are easy to show. We only prove the property (d). If $f \in AP(R,X)$, then for each $\varepsilon > 0$ there exists $l > 0$ such that in any interval of length l of R one can find a number $\tau \in (a, a+l)$ with the property

$$V_f(\tau) = \sup_{\theta \in R} \|f(\theta+\tau) - f(\theta)\|_X < \varepsilon. \tag{25}$$

From (c), we have $V_f(t+s) - V_f(t) \leq V_f(s)$ and $V_f(t) = V_f(t+s-s) \leq V_f(t+s) + V_f(-s)$, then $-V_f(s) = -V_f(-s) \leq V_f(t+s) - V_f(t), s, t \in R$. That is,

$$|V_f(t+s) - V_f(t)| \leq V_f(s), \quad s, t \in R. \tag{26}$$

Hence,

$$\sup_{t \in R} |V_f(t+s) - V_f(t)| \leq V_f(s), \quad s, t \in R. \tag{27}$$

It follows from (25) and (27) that

$$\sup_{t \in R} |V_f(t+\tau) - V_f(t)| \leq V_f(\tau) < \varepsilon. \tag{28}$$

Since $f \in AP(R,X)$, it is uniformly continuous. So, we know that, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|h| < \delta$ then $\|f(t+h) - f(t)\|_X < \epsilon$, $\forall t \in R$. Therefore, $V_f(h) < \epsilon$ and by (27) we get $\sup_{t \in R} |V_f(t+h) - V_f(t)| < \epsilon$, which implies that $V_f(t)$ is continuous. Thus, $V_f(t) \in AP(R,R)$.

Conversely, if $V_f \in AP(R,R)$, then for each $\varepsilon > 0$ there exists $l > 0$ such that in any interval of length l of R one can find a number $\tau \in (a, a+l)$ with the property

$$\left|V_f(t+\tau) - V_f(t)\right| < \varepsilon, \quad \forall t \in R. \tag{29}$$

Hence, $V_f(\tau) = \sup_{\theta\in R}\|f(\theta+\tau) - f(\theta)\|_X < \varepsilon$. Since V_f is continuous and $V_f(0) = 0$, then for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|h| < \delta$ we have $V_f(h) < \epsilon$. Therefore, $V_f(h) = \sup_{\theta\in R}\|f(\theta+h) - f(\theta)\|_X < \epsilon$, which implies f is continuous. Thus, $f \in AP(R,X)$. □

Lemma 18. *Let $f \in \Lambda$, where Λ consists of some almost periodic functions in Bohr sense from R to X, $V(\tau) = \sup_{f\in\Lambda} V_f(\tau)$ is finite. Then the family Λ is uniformly almost periodic if and only if $V \in AP(R,R)$.*

Proof. It is easy to find that V satisfies the following properties:

(i) $V(t) \geq 0, V(-t) = V(t)$,

(ii) $V(0) = 0$.

Moreover, by (c) in Lemma 17 we have

$$\begin{aligned} V(t+s) &= \sup_{f\in\Lambda} V_f(t+s) \leq \sup_{f\in\Lambda}\left(V_f(t) + V_f(s)\right) \\ &\leq \sup_{f\in\Lambda} V_f(t) + \sup_{f\in\Lambda} V_f(s) = V(t) + V(s). \end{aligned} \tag{30}$$

Similar to the proof of (27), we have

$$\sup_{t\in R}|V(t+s) - V(t)| \leq V(s). \tag{31}$$

If $V \in AP(R,R)$, then for each $\varepsilon > 0$, there exists $l > 0$ such that in any interval of length l of R one can find a number $\tau \in (a, a+l)$ with the property

$$\sup_{t\in R}|V(t+\tau) - V(t)| < \varepsilon. \tag{32}$$

Noticing that $V(0) = 0$, hence, $V(\tau) < \varepsilon$. Therefore, $V_f(\tau) < \varepsilon$ for all $f \in \Lambda$. So, the family Λ is a uniformly almost periodic family.

Conversely, suppose that Λ is a uniformly almost periodic family. Then for each $\varepsilon > 0$ there exists $l > 0$ such that in any interval of length l of R one can find a number $\tau \in (a, a+l)$ with the property

$$\sup_{\theta\in R}\left\|f(\theta+\tau) - f(\theta)\right\|_X < \varepsilon, \quad \forall f \in \Lambda. \tag{33}$$

From (33), we obtain that $V_f(\tau) < \varepsilon$, $\forall f \in \Lambda$ and $V(\tau) \leq \varepsilon$. By (31) we obtain that

$$\sup_{t\in R}|V(t+\tau) - V(t)| \leq V(\tau) \leq \varepsilon. \tag{34}$$

Besides, since $f \in \Lambda$ is continuous,

$$V(h) = \sup_{f\in\Lambda}\sup_{\theta\in R}\left\|f(\theta+h) - f(\theta)\right\|_X \longrightarrow 0 \quad \text{as } h \longrightarrow 0. \tag{35}$$

By (31) we obtain that $\sup_{t\in R}|V(t+h) - V(t)| \longrightarrow 0$ as $h \longrightarrow 0$, which implies that V is continuous. Therefore, $V \in AP(R,R)$. The proof is completed. □

Lemma 19. *Let Λ be a uniformly almost periodic family in Bohr sense. Then given a sequence $\{\alpha_n'\}$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha_n'\}$ satisfying the following property: for every $\varepsilon > 0$, there exists a constant N such that*

$$\left\|f(t+\alpha_n) - f(t+\alpha_m)\right\|_X < \varepsilon, \quad n,m > N, \text{ for all } t \in R \text{ and all } f \in \Lambda. \tag{36}$$

Proof. According to Lemma 18, we know that $V(t) = \sup_{f\in\Lambda} V_f(t)$ is almost periodic. By Lemma 3 we obtain that, for every sequence $\{\alpha_n'\}$, there exists a subsequence $\{\alpha_n\}$ satisfying the following property: for each $\varepsilon > 0$, there exists a constant $N > 0$ such that

$$\left|V(t+\alpha_n) - \mathrm{V}(t+\alpha_m)\right| < \varepsilon, \quad n,m > N, \ \forall t \in R. \tag{37}$$

When $t = -\alpha_m$, we obtain $V(\alpha_n - \alpha_m) < \varepsilon$. Thus, $V_f(\alpha_n - \alpha_m) < \varepsilon$ for all $f \in \Lambda$ and $n,m > N$. According to the definition of V_f, we have

$$\left\|f(t+\alpha_n-\alpha_m) - f(t)\right\|_X < \varepsilon, \quad n,m > N, \text{ for all } t \in R \text{ and } \forall f \in \Lambda. \tag{38}$$

Hence, it is easy to see that

$$\left\|f(t+\alpha_n) - f(t+\alpha_m)\right\|_X < \varepsilon, \quad n,m > N, \text{ for all } t \in R \text{ and } \forall f \in \Lambda. \tag{39}$$

The proof is completed. □

Theorem 20. *Let $g \in S(R\times X, R)$. Then for every sequence $\{\alpha_n'\}$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha_n'\}$ such that $\{g(t+\alpha_n, x), n \geq 1\}$ is convergent uniformly with respect to $x \in X$.*

Proof. Since $g \in S(R\times X, R)$, we know that, for every $\varepsilon > 0$, there exists a number $l > 0$ such that every interval of length l contains a number τ satisfying

$$\sup_{t\in R}\int_t^{t+1}\left|g(s+\tau, x) - g(s,x)\right| ds < \varepsilon, \quad \forall x \in X. \tag{40}$$

According to (40), we have

$$\sup_{t\in R}\int_0^1\left|g(t+u+\tau, x) - g(t+u, x)\right| du < \varepsilon, \quad \forall x \in X. \tag{41}$$

Thus,

$$\sup_{t\in R}\left\|g(t+u+\tau, x) - g(t+u, x)\right\|_{L([0,1],R)} < \varepsilon, \quad u \in [0,1], \ \forall x \in X. \tag{42}$$

From Lemma 8, we know that, for every fixed x, $g(t+u,x) \in AP(R, L([0,1],R))$. Therefore, (42) implies that $\{g(t+u,x), x \in X\}$ is a uniformly almost periodic family. From Lemma 19, we have that, for every sequence $\{\alpha_n'\}$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha_n'\}$ such that, for every $\varepsilon > 0$,

$$\begin{aligned}&\sup_{t\in R}\left\|g(t+u+\alpha_n,x)-g(t+u+\alpha_m,x)\right\|_{L([0,1],R)}\\&\quad=\sup_{t\in R}\int_t^{t+1}\left|g(s+\alpha_n,x)-g(s+\alpha_m,x)\right|ds<\varepsilon,\\&\qquad n,m>N,\ \forall x\in X.\end{aligned}\tag{43}$$

Hence, $\{g(t+\alpha_n,x), n \geq 1\}$ is convergent uniformly with respect to $x \in X$. □

Definition 21. A function $h \in M_0(R \times X, R)$ if the following two conditions are true:

(i) for every $x \in X$, $h(\cdot, x) \in M_0(R,R)$,

(ii) for every $\varepsilon > 0$, there exists a constant $T > 0$ such that

$$\int_t^{t+1}|h(s,x)|\,ds<\varepsilon,\quad \forall t>T,\ \forall x\in X.\tag{44}$$

Definition 22. A function $f \in AS(R \times X, R)$ if it can be expressed as

$$f=g+h,\tag{45}$$

where $g \in S(R \times X, R)$, $h \in M_0(R \times X, R)$.

Theorem 23. *The space of $AS(R,R)$ is a Banach space endowed with the norm*

$$\|f\|_S=\sup_{t\in R}\int_t^{t+1}|f(s)|\,ds.\tag{46}$$

Proof. Let $\{f_n\} \subset AS(R,R)$ be a Cauchy sequence; i.e., for each $\varepsilon > 0$, there exists a natural number $N > 0$ such that

$$\sup_{t\in R}\int_t^{t+1}\left|f_n(s)-f_m(s)\right|ds<\varepsilon,\quad n,m>N.\tag{47}$$

Let $f_n = g_n + h_n$, $n \in N$, where $g_n \in S(R,R)$, $h_n \in M_0(R,R)$, $n \in N$. By using Lemma 15, we have

$$\begin{aligned}&\{g_n(t+x)-g_m(t+x), t\in R\}\\&\quad\subset\overline{\{f_n(t+x)-f_m(t+x), t\in R\}}.\end{aligned}\tag{48}$$

Then, for every $\overline{\varepsilon} > 0$,

$$\begin{aligned}&\sup_{t\in R}\left\|g_n(t+x)-g_m(t+x)\right\|_{L([0,1],R)}\\&\quad<\sup_{t\in R}\left\|f_n(t+x)-f_m(t+x)\right\|_{L([0,1],R)}+\overline{\varepsilon}.\end{aligned}\tag{49}$$

According to the arbitrariness of $\overline{\varepsilon}$, we obtain

$$\begin{aligned}&\sup_{t\in R}\left\|g_n(t+x)-g_m(t+x)\right\|_{L([0,1],R)}\\&\quad\leq\sup_{t\in R}\left\|f_n(t+x)-f_m(t+x)\right\|_{L([0,1],R)}.\end{aligned}\tag{50}$$

Thus,

$$\begin{aligned}&\sup_{t\in R}\int_t^{t+1}\left|g_n(s)-g_m(s)\right|ds\\&\quad=\sup_{t\in R}\int_0^1\left|g_n(x+t)-g_m(x+t)\right|dx\\&\quad\leq\sup_{t\in R}\int_0^1\left|f_n(t+x)-f_m(t+x)\right|dx\\&\quad=\sup_{t\in R}\int_t^{t+1}\left|f_n(s)-f_m(s)\right|ds<\varepsilon,\quad n,m>N,\end{aligned}\tag{51}$$

which means that $\{g_n\} \subset S(R,R)$ is a Cauchy sequence. So $\lim_{n\longrightarrow+\infty} g_n = g \in S(R,R)$. Similarly, we can obtain

$$\begin{aligned}&\sup_{t\in R}\int_t^{t+1}\left|h_n(s)-h_m(s)\right|ds\\&\quad\leq\sup_{t\in R}\int_t^{t+1}\left|f_n(s)-f_m(s)\right|ds\\&\qquad+\sup_{t\in R}\int_t^{t+1}\left|g_n(s)-g_m(s)\right|ds\leq 2\varepsilon,\\&\qquad n,m>N.\end{aligned}\tag{52}$$

According to Lemma 14, we obtain $\lim_{n\longrightarrow+\infty} h_n = h \in M_0(R,R)$ and $\lim_{n\longrightarrow+\infty} f_n = f = g + h \in AS(R,R)$. The proof is completed. □

4. Stepanov-Like Asymptotical Almost Periodic Solutions of Semilinear Delay Differential Equations

In this section, we investigate the existence and uniqueness of Stepanov-like asymptotical almost periodic solutions for the following semilinear differential equation:

$$x'(t)+A(t)x(t)=f(t,x(t),x(t-\tau)),\quad t\in R,\tag{53}$$

where $A \in AP(R,R)$, $f \in AS(R \times R \times R, R)$, and $\tau > 0$ is a constant.

We make some assumption:

(H_1) $A \in AP(R,R)$ and $A^* = \inf_{t\in R} A(t) > 0$.

(H_2) $f = f_1 + f_2 \in AS(R \times R \times R, R)$, where $f_1 \in S(R \times R \times R, R)$, $f_2 \in M_0(R \times R \times R, R)$.

(H_3) There exist constants $L_1, L_2 > 0$ such that, for all $t \in R$ and for all $u_1, u_2, v_1, v_2 \in R$,

$$\begin{aligned}&\left|f_1(t,u_1,v_1)-f_1(t,u_2,v_2)\right|\\&\quad\leq L_1\left(\left|u_1-u_2\right|+\left|v_1-v_2\right|\right),\\&\left|f_2(t,u_1,v_1)-f_2(t,u_2,v_2)\right|\\&\quad\leq L_2\left(\left|u_1-u_2\right|+\left|v_1-v_2\right|\right).\end{aligned}\tag{54}$$

Lemma 24. *Let $x \in AS(R,R)$ and $(H_2),(H_3)$ hold; then $f(\cdot,x(\cdot),x(\cdot-\tau)) \in AS(R,R)$.*

Proof. Since $x \in AS(R,R)$, we have $x = x_1 + x_2$, where $x_1 \in S(R,R)$, $x_2 \in M_0(R,R)$. Then function $f(t,x(t),x(t-\tau))$ can be written in the form:

$$\begin{aligned} f(t,x(t),x(t-\tau)) = {} & f_1(t,x_1(t),x_1(t-\tau)) \\ & + f(t,x(t),x(t-\tau)) \\ & - f(t,x_1(t),x_1(t-\tau)) \\ & + f(t,x_1(t),x_1(t-\tau)) \\ & - f_1(t,x_1(t),x_1(t-\tau)). \end{aligned} \tag{55}$$

Step 1. We prove $f_1(\cdot,x_1(\cdot),x_1(\cdot-\tau)) \in S(R,R)$. Let $\varepsilon > 0$. By Lemma 9, for every sequence $\{h_k; k \geq 1\} \subset R$, there exists a subsequence $\{h_{1k}; k \geq 1\} \subset \{h_k; k \geq 1\}$ such that the sequence $\{x_1(t+h_{1k}); k \geq 1\}$ is convergent. From Theorem 20, it follows that $\{f_1(t+h_{1k},u,v); k \geq 1\}$ is also convergent uniformly with respect to $u,v \in R$. Therefore, for any $\varepsilon > 0$, there exist positive integers N_1 and N_2 such that

$$\sup_{t\in R}\int_t^{t+1}\left|x_1(s+h_{1k}) - x_1(s+h_{1p})\right| ds < \frac{\varepsilon}{4L_1}, \quad k,p \geq N_1 \tag{56}$$

and

$$\sup_{t\in R}\int_t^{t+1}\left|f_1(s+h_{1k},u,v) - f_1(s+h_{1p},u,v)\right| ds < \frac{\varepsilon}{2}, \quad k,p \geq N_2,\ \forall u,v \in R. \tag{57}$$

Hence

$$\begin{aligned} & \sup_{t\in R}\int_t^{t+1}\Big|f_1(s+h_{1k},x_1(s+h_{1k}),x_1(s+h_{1k}-\tau)) \\ & \quad - f_1(s+h_{1p},x_1(s+h_{1p}),x_1(s+h_{1p}-\tau))\Big| ds \\ & \leq \sup_{t\in R}\int_t^{t+1}\Big|f_1(s \\ & \quad + h_{1k},x_1(s+h_{1k}),x_1(s+h_{1k}-\tau)) - f_1(s \\ & \quad + h_{1k},x_1(s+h_{1p}),x_1(s+h_{1p}-\tau))\Big| ds \\ & \quad + \sup_{t\in R}\int_t^{t+1}\Big|f_1(s \\ & \quad + h_{1k},x_1(s+h_{1p}),x_1(s+h_{1p}-\tau)) - f_1(s \\ & \quad + h_{1p},x_1(s+h_{1p}),x_1(s+h_{1p}-\tau))\Big| ds \\ & \leq L_1 \sup_{t\in R}\int_t^{t+1}\Big(\left|x_1(s+h_{1k}) - x_1(s+h_{1p})\right| \\ & \quad + \left|x_1(s+h_{1k}-\tau) - x_1(s+h_{1p}-\tau)\right|\Big) ds \\ & \quad + \frac{\varepsilon}{2} \leq L_1\frac{\varepsilon}{2L_1} + \frac{\varepsilon}{2} = \varepsilon, \\ & \qquad \text{for } k,p \geq N = \max\{N_1,N_2\}. \end{aligned} \tag{58}$$

Therefore, $f_1(\cdot,x_1(\cdot),x_1(\cdot-\tau)) \in S(R,R)$.

Step 2. We prove that

$$f(t,x(t),x(t-\tau)) - f(t,x_1(t),x_1(t-\tau)) \in M_0(R,R) \tag{59}$$

and

$$f(t,x_1(t),x_1(t-\tau)) - f_1(t,x_1(t),x_1(t-\tau)) \in M_0(R,R). \tag{60}$$

From (H_3), we have

$$\begin{aligned} & \lim_{t\longrightarrow\infty}\int_t^{t+1}\Big|f(s,x(s),x(s-\tau)) \\ & \quad - f(s,x_1(s),x_1(s-\tau))\Big| ds \leq \lim_{t\longrightarrow\infty}\int_t^{t+1}(L_1 \\ & \quad + L_2)\left(\left|x_2(s)\right| + \left|x_2(s-\tau)\right|\right) ds = 0. \end{aligned} \tag{61}$$

According to the definition of f and Definition 21, we have

$$\begin{aligned} & f(t,x_1(t),x_1(t-\tau)) - f_1(t,x_1(t),x_1(t-\tau)) \\ & \quad = f_2(t,x_1(t),x_1(t-\tau)) \end{aligned} \tag{62}$$

and

$$\lim_{t\longrightarrow\infty}\int_t^{t+1}\left|f_2(s,x_1(s),x_1(s-\tau))\right| ds = 0. \tag{63}$$

Hence, $f(\cdot,x(\cdot),x(\cdot-\tau)) \in AS(R,R)$. The proof is complete. □

Theorem 25. *Assume (H_1)-(H_3) hold. If $\Delta < 1$, where $\Delta = 4(L/A^*)$, $L = \max\{L_1,L_2\}$, then system (53) has a unique Stepanov-like asymptotical almost periodic solution.*

Proof. For any $\varphi \in AS(R,R)$, consider the linear differential equation

$$x'(t) + A(t)x(t) = f(t,\varphi(t),\varphi(t-\tau)). \tag{64}$$

Since (H_1) holds, by the exponential dichotomy of linear differential equation, (64) has a unique bounded solution

$$x^{\varphi}(t) = \int_{-\infty}^{t} e^{-\int_s^t A(u)du} f(s,\varphi(s),\varphi(s-\tau))\, ds, \quad t \in R. \tag{65}$$

Define an operator $F : AS(R,R) \longrightarrow M(R,R)$ by setting $F\varphi = x^{\varphi}$ for every $\varphi \in AS(R,R)$.

Step 1 (F is self-mapping). From Lemma 24, we have $\widetilde{f}(\cdot) = f(\cdot, \varphi(t), \varphi(\cdot - \tau)) \in AS(R,R)$. Let $\widetilde{f} = \widetilde{f_1} + \widetilde{f_2}$, where $\widetilde{f_1} \in S(R,R)$, $\widetilde{f_2} \in M_0(R,R)$, and define

$$F_1(t) = \int_{-\infty}^{t} e^{-\int_s^t A(u)du} \widetilde{f_1}(s)\, ds,$$
$$F_2(t) = \int_{-\infty}^{t} e^{-\int_s^t A(u)du} \widetilde{f_2}(s)\, ds. \tag{66}$$

By Lemmas 8 and 10, there exists $l > 0$ such that every interval of length l contains a number θ such that

$$|A(t+\theta) - A(t)| < \varepsilon,$$
$$\sup_{t\in R} \int_t^{t+1} \left|\widetilde{f_1}(s+\theta) - \widetilde{f_1}(s)\right| ds < \varepsilon. \tag{67}$$

Then

$$\begin{aligned}
\|F_1(t+\theta) - F_1(t)\|_S &= \sup_{t\in R} \int_t^{t+1} \left| \int_{-\infty}^{s+\theta} e^{-\int_w^{s+\theta} A(u)du} \widetilde{f_1}(w)\, dw \right. \\
&\quad \left. - \int_{-\infty}^{s} e^{-\int_w^s A(u)du} \widetilde{f_1}(w)\, dw \right| ds \\
&= \sup_{t\in R} \int_t^{t+1} \left| \int_{-\infty}^{s} e^{-\int_w^s A(u+\theta)du} \widetilde{f_1}(w+\theta)\, dw \right. \\
&\quad \left. - \int_{-\infty}^{s} e^{-\int_w^s A(u)du} \widetilde{f_1}(w)\, dw \right| ds \\
&\le \sup_{t\in R} \int_t^{t+1} \left| \int_{-\infty}^{s} e^{-\int_w^s A(u+\theta)du} \widetilde{f_1}(w+\theta)\, dw \right. \\
&\quad \left. - \int_{-\infty}^{s} e^{-\int_w^s A(u)du} \widetilde{f_1}(w+\theta)\, dw \right| ds \\
&\quad + \sup_{t\in R} \int_t^{t+1} \left| \int_{-\infty}^{s} e^{-\int_w^s A(u)du} \widetilde{f_1}(w+\theta)\, dw \right. \\
&\quad \left. - \int_{-\infty}^{s} e^{-\int_w^s A(u)du} \widetilde{f_1}(w)\, dw \right| ds \\
&= \sup_{t\in R} \int_t^{t+1} \left| \int_{-\infty}^{s} \left(e^{-\int_w^s A(u+\theta)du} - e^{-\int_w^s A(u)du} \right) \right. \\
&\quad \left. \cdot \widetilde{f_1}(w+\theta)\, dw \right| ds \\
&\quad + \sup_{t\in R} \int_t^{t+1} \int_{-\infty}^{s} e^{-\int_w^s A(u)du} \left| \widetilde{f_1}(w+\theta) - \widetilde{f_1}(w) \right| dw\, ds \\
&\le \sup_{t\in R} \int_t^{t+1} \left| \int_{-\infty}^{s} e^{-A^*(s-w)} \left(\int_w^s |A(u+\theta) - A(u)|\, du \right) \right. \\
&\quad \left. \cdot \widetilde{f_1}(w+\theta)\, dw \right| ds \\
&\quad + \sup_{t\in R} \int_t^{t+1} \int_{-\infty}^{s} e^{-\int_w^s A(u)du} \left| \widetilde{f_1}(w+\theta) - \widetilde{f_1}(w) \right| dw\, ds \\
&\le \varepsilon \sup_{t\in R} \int_t^{t+1} \int_0^{+\infty} e^{-A^*\sigma} \sigma \left| \widetilde{f_1}(s-\sigma+\theta) \right| d\sigma\, ds \\
&\quad + \int_0^{+\infty} e^{-A^*\sigma} \sup_{t\in R} \int_t^{t+1} \left| \widetilde{f_1}(s-\sigma+\theta) - \widetilde{f_1}(s-\sigma) \right| ds\, d\sigma \\
&\le \left\| \widetilde{f_1} \right\|_S \varepsilon \frac{1}{(A^*)^2} + \varepsilon \frac{1}{A^*},
\end{aligned} \tag{68}$$

where $\sigma = s - w$. Hence, $F_1 \in S(R,R)$.

Since $\widetilde{f_2} \in M_0(R,R)$, we have

$$\begin{aligned}
&\lim_{t\longrightarrow+\infty} \int_t^{t+1} \left| F_2(s) \right| ds \\
&\quad = \lim_{t\longrightarrow+\infty} \int_t^{t+1} \left| \int_{-\infty}^{s} e^{-\int_w^s A(u)du} \widetilde{f_2}(w)\, dw \right| ds \\
&\quad \le \lim_{t\longrightarrow+\infty} \int_t^{t+1} \int_{-\infty}^{s} e^{-A^*(s-w)} \left| \widetilde{f_2}(w) \right| dw\, ds \\
&\quad = \lim_{t\longrightarrow+\infty} \int_t^{t+1} \int_0^{+\infty} e^{-A^*\xi} \left| \widetilde{f_2}(s-\xi) \right| d\xi\, ds \\
&\quad = \lim_{t\longrightarrow+\infty} \int_0^{+\infty} e^{-A^*\xi} \int_t^{t+1} \left| \widetilde{f_2}(s-\xi) \right| ds\, d\xi,
\end{aligned} \tag{69}$$

where $\xi = s - w$. According to Lebesgue's dominated convergence theorem, we obtain $F_2 \in M_0(R,R)$. Therefore, F is self-mapping.

Step 2 (F is a contraction mapping). $\forall \varphi, \psi \in AS(R,R)$, we have

$$\begin{aligned}
&\|F\varphi - F\psi\|_S \\
&\quad = \sup_{t\in R} \int_t^{t+1} \left| \int_{-\infty}^{s} e^{-\int_w^s A(u)du} f(w, \varphi(w), \varphi(w-\tau))\, dw \right. \\
&\quad \left. - \int_{-\infty}^{s} e^{-\int_w^s A(u)du} f(w, \psi(w), \psi(w-\tau))\, dw \right| ds \\
&\quad \le \sup_{t\in R} \int_t^{t+1} \int_{-\infty}^{s} e^{-A^*(s-w)} 2L \left(|\varphi(w) - \psi(w)| \right. \\
&\quad \left. + |\varphi(w-\tau) - \psi(w-\tau)| \right) dw\, ds \\
&\quad = 2L \sup_{t\in R} \int_t^{t+1} \int_0^{+\infty} e^{-A^*\sigma} \left(|\varphi(s-\sigma) - \psi(s-\sigma)| \right. \\
&\quad \left. + |\varphi(s-\sigma-\tau) - \psi(s-\sigma-\tau)| \right) d\sigma\, ds \\
&\quad \le 2L \int_0^{+\infty} e^{-A^*\sigma} 2 \|\varphi - \psi\|_S\, du = \frac{4L}{A^*} \|\varphi - \psi\|_S .
\end{aligned} \tag{70}$$

Hence, T has a unique fixed point in $AS(R,R)$. Therefore, system (53) has a unique Stepanov-like asymptotical almost periodic solution. The proof is complete. □

An example: consider the following equation:

$$\begin{aligned} x'(t) + (\sin t + 5)\, x(t) &= \frac{1}{2}(\sin x(t) + \sin x(t-\tau)) \\ &\quad + \frac{1}{2}e^{-|t|}(\cos x(t) + \cos x(t-\tau)), \end{aligned} \tag{71}$$

where τ is a positive constant. In this case, $A(t) = \sin t + 5$,

$$\begin{aligned} f(t, x(t), x(t-\tau)) &= \frac{1}{2}(\sin x(t) + \sin x(t-\tau)) \\ &\quad + \frac{1}{2}e^{-|t|}(\cos x(t) + \cos x(t-\tau)), \\ f_1(t, x(t), x(t-\tau)) &= \frac{1}{2}(\sin x(t) + \sin x(t-\tau)), \\ f_2(t, x(t), x(t-\tau)) &= \frac{1}{2}e^{-|t|}(\cos x(t) + \cos x(t-\tau)). \end{aligned} \tag{72}$$

Obviously, $A^* = 4$,

$$\begin{aligned} &\left|f_1(t, x(t), x(t-\tau)) - f_1(t, y(t), y(t-\tau))\right| \\ &\quad \le \frac{1}{2}\left(\left|x(t) - y(t)\right| + \left|x(t-\tau) - y(t-\tau)\right|\right), \\ &\left|f_2(t, x(t), x(t-\tau)) - f_2(t, y(t), y(t-\tau))\right| \\ &\quad \le \frac{1}{2}\left(\left|x(t) - y(t)\right| + \left|x(t-\tau) - y(t-\tau)\right|\right). \end{aligned} \tag{73}$$

Hence, $4L/A^* = 1/2 < 1$. Thus, all the conditions of Theorem 25 are satisfied. By Theorem 25, system (71) has a unique Stepanov-like asymptotical almost periodic solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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