

## Research Article

# A Characterization of the Existence of a Fundamental Bounded Resolution for the Space $C_c(X)$ in Terms of $X$

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We characterize in terms of the topology of a Tychonoff space  $X$  the existence of a bounded resolution for  $C_c(X)$  that swallows the bounded sets, where  $C_c(X)$  is the space of real-valued continuous functions on  $X$  equipped with the compact-open topology.

## 1. Preliminaries

In the sequel, unless otherwise stated,  $X$  is a nonempty completely regular Hausdorff space. We represent by  $C_p(X)$  the ring  $C(X)$  of real-valued continuous functions defined on  $X$  equipped with the *pointwise* topology  $\tau_p$ . As usual, we denote by  $L_p(X)$  the weak\* dual of  $C_p(X)$ . When  $C(X)$  is equipped with the *compact-open* topology  $\tau_c$  we write  $C_c(X)$ . As in [1], we denote by  $C^*(X)$  the linear space of real-valued continuous and bounded functions defined on  $X$ . If  $C^*(X)$  is regarded as a subspace of  $C_c(X)$ , we denote this space by  $C_c^*(X)$ . Since  $C^*(X)$  is dense in  $C_c(X)$ , both  $C_c^*(X)$  and  $C_c(X)$  have the same dual. The Banach space  $C^*(X)$  equipped with the supremum norm has recently been studied in [2]. Let us recall that a family  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  of subsets of a set  $X$  is called a *resolution* for  $X$  if it covers  $X$  and  $A_\alpha \subseteq A_\beta$  whenever  $\alpha \leq \beta$ ; i.e.,  $\alpha(i) \leq \beta(i)$  for every  $i \in \mathbb{N}$  (see [3, Chapter 3]). If  $E$  is a topological vector space, a resolution  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  consisting of *bounded sets* (see [4, Definition 1.4.5]) is referred to as a *bounded resolution*. A bounded resolution  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  on  $E$  that *swallows the bounded sets*, i.e., such that for each bounded set  $Q$  in  $E$  there is  $\gamma \in \mathbb{N}^{\mathbb{N}}$  with  $Q \subseteq A_\gamma$ , is referred to as a *fundamental bounded resolution*. The existence of such a resolution in a locally convex space  $E$  has shown to be equivalent to the existence of a so-called  $\mathfrak{G}$ -base of absolutely convex neighborhoods of the origin in the strong dual  $E'_\beta$  of  $E$ . Besides, fundamental bounded resolutions are essential in order to get a proper extension of the class of (DF)-spaces (see [5] for details).

Fundamental compact resolutions, i.e., resolutions consisting of compact sets which swallow the compact sets, have been widely used, even in Banach space theory [6], since they were introduced in [7]. A well-known result of Christensen [8, Theorem 3.3] asserts that a metrizable space  $X$  is Polish if and only if  $X$  has a fundamental compact resolution. Moreover, it has been shown in [9, Theorem 2] that  $C_c(X)$  has a  $\mathfrak{G}$ -base of neighborhoods of the origin if and only if  $X$  has a fundamental compact resolution. In this note we provide two characterizations, in terms of the domain space  $X$ , of the existence of a fundamental bounded resolution for  $C_c(X)$ , one by means of certain uniformity for  $X$  and the other purely topological. Our main motivation is the two following  $C_p$ -theoretic results.

**Theorem 1** ([7, Theorem 3.7] or [10, Problem 216]). *The space  $C_p(X)$  has a fundamental compact resolution if and only if  $X$  is countable and discrete.*

**Theorem 2** ([11, Theorem 3.3]). *The space  $C_p(X)$  has a fundamental bounded resolution if and only if  $X$  is countable.*

A space  $X$  is called *K-analytic* if there is an upper semicontinuous compact-valued map  $T$  from the product space  $\mathbb{N}^{\mathbb{N}}$ , where  $\mathbb{N}$  is equipped with the discrete topology, into  $X$  such that  $\bigcup\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\} = X$ . A family  $\mathcal{F}$  of functions from a uniform space  $(X, \mathcal{N})$  into a uniform space  $(Y, \mathcal{M})$  is called *uniformly equicontinuous* [12, Chapter 7, Problem G] if for each  $V \in \mathcal{M}$  there is  $U \in \mathcal{N}$  such that  $(f(x), f(y)) \in V$

whenever  $f \in \mathcal{F}$  and  $(x, y) \in U$ . Let  $\mathcal{N}$  be a uniformity for a (nonempty) set  $X$  and denote by  $\tau_{\mathcal{N}}$  the uniform topology defined by  $\mathcal{N}$ . A base  $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  of the uniformity  $\mathcal{N}$  is called a  $\mathfrak{G}$ -base if  $U_\beta \subseteq U_\alpha$  whenever  $\alpha \leq \beta$ . There is no loss of generality by assuming that each  $U_\alpha$  is a symmetric vicinity.

## 2. Fundamental Bounded Resolutions for $C_c(X)$

Although it can be easily seen that each metrizable locally convex space  $E$  has a fundamental bounded resolution, if the locally convex space  $C_c(X)$  has a fundamental bounded resolution, unlike what happens with  $C_p(X)$ , the space  $X$  needs not be countable and moreover  $C_c(X)$  needs not be metrizable.

**Proposition 3.** *Let  $X$  be a metrizable space. Then  $C_c(X)$  has a fundamental bounded resolution if and only if  $X$  is  $\sigma$ -compact.*

*Proof.* If  $X$  is  $\sigma$ -compact then  $C_c(X)$  is weakly  $K$ -analytic by [13, Proposition 2.2], or it has a compact resolution that swallows the compact sets by [14, Corollary 2.10]. In any case  $C_c(X)$  has a bounded resolution. Since  $X$  is a  $k_{\mathbb{R}}$ -space then  $C_c(X)$  is complete, hence locally complete. So, it follows from Valdivia's [3, Theorem 3.5] that there exists a resolution  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  of  $C_c(X)$  consisting of Banach disks that swallows the bounded sets of  $C_c(X)$ , which shows that  $C_c(X)$  has a fundamental bounded resolution. Conversely, if  $C_c(X)$  has a fundamental bounded resolution then  $C_p(X)$  has a bounded resolution, so [3, Corollary 9.2] shows that  $X$  is  $\sigma$ -compact.  $\square$

*Example 4.* According to the previous proposition  $C_c(\mathbb{R})$  has a fundamental bounded resolution, but  $\mathbb{R}$  is neither countable nor discrete. On the other hand,  $C_c(\mathbb{Q})$  has also a fundamental bounded resolution, but  $\mathbb{Q}$ , although is countable, is not discrete. Observe that  $C_c(\mathbb{R})$  is metrizable, but  $C_c(\mathbb{Q})$  is not. Of course, if  $X$  is hemicompact, or even compact, then  $C_c(X)$  is metrizable, or even a Banach space, and in this case  $C_c(X)$  has obviously a fundamental bounded resolution.

**Theorem 5.** *The space  $C_c(X)$  has a fundamental bounded resolution if and only if  $(X, \mathcal{M})$ , where  $\mathcal{M}$  is the uniformity for  $X$  generated by the pseudometrics*

$$d_A(x, y) = \sup_{f \in A} |f(x) - f(y)| \quad (1)$$

for each bounded set  $A$  of  $C_c(X)$ , has a  $\mathfrak{G}$ -base.

*Proof.* Let  $E$  be the topological dual of  $C_c(X)$  and denote by  $\mathcal{B}$  the family of all bounded sets of  $C_c(X)$  and by  $\beta(E, C(X))$  the strong topology on  $E$ . By identifying  $X$  with its canonical homeomorphic embedding in  $L_p(X)$ , note that  $X \subseteq L(X) \subseteq E$ . The strong topology  $\beta(E, C(X))$  generates a unique admissible translation-invariant uniformity  $\mathcal{N}$  on  $E$ , so that  $\tau_{\mathcal{N}} = \beta(E, C(X))$ . By considering also  $f \in C(X)$  as a linear functional on  $E$ , observe that for each  $N \in \mathcal{N}$  there is  $A \in \mathcal{B}$  such that

$$\left\{ (u, v) \in E \times E : \sup_{f \in A} |f(u - v)| \leq 1 \right\} \subseteq N. \quad (2)$$

Hence  $M \subseteq X \times X$  belongs to the relative uniformity  $\mathcal{M}$  of  $\mathcal{N}$  to  $X \times X$  if and only if there exists  $A \in \mathcal{B}$  such that

$$\left\{ (x, y) \in X \times X : \sup_{f \in A} |f(x) - f(y)| \leq 1 \right\} \subseteq M. \quad (3)$$

If  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a fundamental bounded resolution for  $C_c(X)$ , by setting

$$U_\alpha = \left\{ (x, y) \in X \times X : \sup_{f \in A_\alpha} |f(x) - f(y)| \leq 1 \right\} \quad (4)$$

we obtain a  $\mathfrak{G}$ -base  $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  of  $\mathcal{M}$ . If  $\alpha \leq \beta$  then  $U_\beta \subseteq U_\alpha$ , and if  $M \in \mathcal{M}$  there is  $A \in \mathcal{B}$  such that  $(x, y) \in M$  whenever  $\sup_{f \in A} |f(x) - f(y)| \leq 1$ , so if  $\gamma \in \mathbb{N}^{\mathbb{N}}$  is such that  $A \subseteq A_\gamma$ , clearly  $U_\gamma \subseteq M$ .

Conversely, if the uniform structure for  $X$  generated by the family of pseudometrics  $\{d_A : A \in \mathcal{B}\}$ , where

$$d_A(x, y) = \sup_{f \in A} |f(x) - f(y)|, \quad (5)$$

has a  $\mathfrak{G}$ -base  $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ , this entails that for each  $A \in \mathcal{B}$  there is  $\delta \in \mathbb{N}^{\mathbb{N}}$  such that  $\sup_{f \in A} |f(x) - f(y)| \leq 1$  for every  $(x, y) \in U_\delta$ . Setting

$$A_\alpha = \left\{ f \in C(X) : \sup_{(x, y) \in U_\alpha} |f(x) - f(y)| \leq 1 \right\}, \quad (6)$$

for each  $\alpha \in \mathbb{N}^{\mathbb{N}}$ , clearly  $A_\alpha \subseteq A_\beta$  if  $\alpha \leq \beta$  and  $A \subseteq A_\delta$ . Consequently  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a fundamental bounded resolution for  $C_c(X)$ .  $\square$

Let  $X$  be a nonempty completely regular Hausdorff topological space and let  $\mathcal{K}(X)$  denote the family of all compact sets of  $X$ .

**Lemma 6.** *A subset  $A$  of  $C(X)$  is bounded for the compact-open topology  $\tau_c$  if and only if there exists a sequence  $\{\mathcal{F}_n(A) : n \in \mathbb{N}\}$  of subsets of  $\mathcal{K}(X)$  such that*

- (1) for each  $n \in \mathbb{N}$ , if  $K \in \mathcal{F}_n(A)$  then  $\sup_{f \in A} \sup_{x \in K} |f(x)| \leq n$
- (2)  $\mathcal{F}_n(A) \subseteq \mathcal{F}_{n+1}(A)$  for each  $n \in \mathbb{N}$
- (3)  $\bigcup_{n=1}^{\infty} \mathcal{F}_n(A) = \mathcal{K}(X)$
- (4) If a set  $Q \in \mathcal{K}(X)$  is such that  $\sup_{x \in Q} |f(x)| \leq n$  for each  $f \in C(X)$  with  $\sup_{x \in K} |f(x)| \leq n$  for all  $K \in \mathcal{F}_n(A)$ , then  $Q \in \mathcal{F}_n(A)$

*Proof.* If  $A$  is  $\tau_c$ -bounded and  $n \in \mathbb{N}$ , define

$$\mathcal{F}_n(A) = \left\{ K \in \mathcal{K}(X) : \sup_{f \in A} \sup_{x \in K} |f(x)| \leq n \right\}. \quad (7)$$

Clearly  $\mathcal{F}_n(A) \subseteq \mathcal{F}_{n+1}(A)$  for each  $n \in \mathbb{N}$  and if  $K \in \mathcal{K}(X)$  there is  $m \in \mathbb{N}$  with

$$\sup_{f \in A} \sup_{x \in K} |f(x)| \leq m, \quad (8)$$

so that  $K \in \mathcal{F}_m(A)$ , which shows that  $\bigcup_{n=1}^{\infty} \mathcal{F}_n(A) = \mathcal{K}(X)$ . In addition, if  $K \in \mathcal{F}_n(A)$  the relation  $\sup_{f \in A} \sup_{x \in K} |f(x)| \leq n$  holds by construction. Finally, if we set

$$B_n = \left\{ f \in C(X) : \sup_{x \in K} |f(x)| \leq n \text{ for every } K \in \mathcal{F}_n(A) \right\} \quad (9)$$

then  $A \subseteq B_n$ . Therefore, if  $Q \in \mathcal{K}(X)$  verifies that  $\sup_{f \in B_n} \sup_{x \in Q} |f(x)| \leq n$  then

$$\sup_{f \in A} \sup_{x \in Q} |f(x)| \leq n, \quad (10)$$

so that  $Q \in \mathcal{F}_n(A)$ . Hence  $\{\mathcal{F}_n(A) : n \in \mathbb{N}\}$  satisfies the required conditions.

Conversely, if there exists a sequence  $\{\mathcal{F}_n(A) : n \in \mathbb{N}\}$  of  $\mathcal{K}(X)$  satisfying the four conditions of the statement of the lemma (actually, only the first and the third conditions are required) then clearly  $A$  is  $\tau_c$ -bounded on  $X$ , for if  $P \in \mathcal{K}(X)$  there is  $k \in \mathbb{N}$  with  $P \in \mathcal{F}_k(A)$  such that  $\sup_{f \in A} \sup_{x \in P} |f(z)| \leq k$ .  $\square$

In what follows the fourth condition above, which is independent of  $A$ , will be referred to as the *closure* condition of  $\mathcal{F}_n(A)$ , and we shall say that the family  $\mathcal{F}_n(A)$  is *closed*.

**Definition 7.** A collection  $\{\mathcal{F}_{\alpha,n} : (\alpha,n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$  of closed subsets of  $\mathcal{K}(X)$  will be called a *covering net* of  $\mathcal{K}(X)$  if  $\{\mathcal{F}_{\alpha,n} : n \in \mathbb{N}\}$  is an increasing covering of  $\mathcal{K}(X)$  for each  $\alpha \in \mathbb{N}^{\mathbb{N}}$  such that  $\mathcal{F}_{\beta,n} \subseteq \mathcal{F}_{\alpha,n}$  whenever  $\alpha \leq \beta$  for all  $n \in \mathbb{N}$ .

**Theorem 8.** *The space  $C_c(X)$  has a fundamental bounded resolution if and only if there is a covering net  $\{\mathcal{F}_{\alpha,n} : (\alpha,n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$  such that if  $\{\mathcal{F}_n : n \in \mathbb{N}\}$  is an increasing covering of  $\mathcal{K}(X)$  by closed sets, there exists  $\gamma \in \mathbb{N}^{\mathbb{N}}$  such that  $\mathcal{F}_{\gamma,n} \subseteq \mathcal{F}_n$  for all  $n \in \mathbb{N}$ .*

*Proof.* If there exists a covering net  $\{\mathcal{F}_{\alpha,n} : (\alpha,n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$  for  $X$  which satisfies the property of the statement of the theorem, the sets

$$A_\alpha = \left\{ f \in C(X) : \sup_{x \in K} |f(x)| \leq n \text{ for every } K \in \mathcal{F}_{\alpha,n} \text{ and each } n \in \mathbb{N} \right\} \quad (11)$$

compose a fundamental bounded resolution for  $C(X)$ . Indeed, each set  $A_\alpha$  is  $\tau_c$ -bounded by virtue of the previous lemma, since  $\{\mathcal{F}_{\alpha,n} : n \in \mathbb{N}\}$  is an increasing covering of  $\mathcal{K}(X)$  consisting of closed sets such that

$\sup_{f \in A_\alpha} \sup_{x \in K} |f(x)| \leq n$  for all  $K \in \mathcal{F}_{\alpha,n}$ . Besides  $A_\alpha \subseteq A_\beta$  whenever  $\alpha \leq \beta$  since  $\mathcal{F}_{\beta,n} \subseteq \mathcal{F}_{\alpha,n}$ , and if  $A$  is a  $\tau_c$ -bounded subset of  $C(X)$ , according to the previous lemma there exists an increasing covering  $\{\mathcal{F}_n(A) : n \in \mathbb{N}\}$  of  $\mathcal{K}(X)$  consisting of closed sets. Therefore, by the condition in the statement of the theorem, there exists  $\gamma \in \mathbb{N}^{\mathbb{N}}$  such that  $\mathcal{F}_{\gamma,n} \subseteq \mathcal{F}_n(A)$  for all  $n \in \mathbb{N}$ . Now, if  $f \in A$  then  $\sup_{K \in \mathcal{F}_n(A)} |f(x)| \leq n$  for all  $K \in \mathcal{F}_n(A)$  and  $n \in \mathbb{N}$ , in particular for each  $K \in \mathcal{F}_{\gamma,n}$  and all  $n \in \mathbb{N}$ , which shows that  $f \in A_\gamma$ . Hence  $A \subseteq A_\gamma$ , which proves that  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a fundamental bounded resolution for  $C_c(X)$ .

Conversely, if  $C_c(X)$  has a fundamental bounded resolution  $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  and we set

$$\mathcal{F}_{\alpha,n} = \left\{ K \in \mathcal{K}(X) : \sup_{f \in B_\alpha} \sup_{x \in K} |f(x)| \leq n \right\} \quad (12)$$

then  $\{\mathcal{F}_{\alpha,n} : n \in \mathbb{N}\}$  is an increasing covering of  $\mathcal{K}(X)$  such that  $\mathcal{F}_{\beta,n} \subseteq \mathcal{F}_{\alpha,n}$  whenever  $\alpha \leq \beta$  for all  $n \in \mathbb{N}$ . In addition  $\sup_{f \in B_\alpha} \sup_{x \in K} |f(x)| \leq n$  for  $K \in \mathcal{F}_{\alpha,n}$  and  $n \in \mathbb{N}$  by the definition of  $\mathcal{F}_{\alpha,n}$ . Moreover,  $\mathcal{F}_{\alpha,n}$  satisfies the closure condition, for if

$$B_{\alpha,n} = \left\{ f \in C(X) : \sup_{Q \in \mathcal{F}_{\alpha,n}} \sup_{x \in Q} |f(x)| \leq n \right\} \quad (13)$$

then  $B_\alpha \subseteq B_{\alpha,n}$ , so if  $\sup_{f \in B_\alpha} \sup_{x \in K} |f(x)| \leq n$  then  $\sup_{f \in B_\alpha} \sup_{x \in K} |f(x)| \leq n$ , which means that  $K \in \mathcal{F}_{\alpha,n}$ .

Now, if  $\{\mathcal{F}_n : n \in \mathbb{N}\}$  is any increasing covering of  $\mathcal{K}(X)$  consisting of closed sets, define

$$P = \left\{ f \in C(X) : \sup_{x \in K} |f(x)| \leq n \text{ for every } K \in \mathcal{F}_{\alpha,n} \text{ and each } n \in \mathbb{N} \right\} \quad (14)$$

and observe that if  $K \in \mathcal{F}_n$  then  $\sup_{f \in P} \sup_{x \in K} |f(x)| \leq n$ , which according to the preceding lemma ensures that  $P$  is a  $\tau_c$ -bounded set. Since  $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a fundamental bounded resolution for  $C_c(X)$  there exists  $\delta \in \mathbb{N}^{\mathbb{N}}$  such that  $P \subseteq B_\delta$ . Now if  $Q \in \mathcal{F}_{\delta,n}$  then  $\sup_{f \in B_\delta} \sup_{x \in Q} |f(x)| \leq n$  so that, in particular,  $\sup_{f \in P} \sup_{x \in Q} |f(x)| \leq n$ . We claim that  $Q \in \mathcal{F}_n$ . Indeed, since  $\sup_{x \in Q} |f(x)| \leq n$  for each  $f \in P$ , we have that  $\sup_{x \in Q} |f(x)| \leq n$  holds for each  $f \in C(X)$  such that  $\sup_{x \in K} |f(x)| \leq n$  for every  $K \in \mathcal{F}_n$  by virtue of the definition of  $P$ . Therefore, the closure property of  $\mathcal{F}_n$  yields  $K \in \mathcal{F}_n$ . This shows that  $\mathcal{F}_{\delta,n} \subseteq \mathcal{F}_n$  for every  $n \in \mathbb{N}$ , which, bearing in mind the properties of the family  $\{\mathcal{F}_{\alpha,n} : (\alpha,n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$  established before, guarantees that  $\{\mathcal{F}_{\alpha,n} : (\alpha,n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}\}$  is a covering net for  $X$  consisting of closed sets that satisfies the required property.  $\square$

In what follows we shall refer to a Tychonoff space  $X$  satisfying the conditions of the statement of Theorem 8 as a *cn-space*. It is shown in [5, Proposition 3.2] that if  $X$  is a *cn-space* or, which is equivalent, if  $C_c(X)$  has a fundamental

bounded resolution, then  $C_c(X)$  has a *countable  $cs^*$ -network* at the origin [15, 16]. The next theorem shows that in order for  $X$  to be a *cn*-space it suffices that  $C_c^*(X)$  have a fundamental bounded resolution.

**Theorem 9.** *Let  $X$  be completely regular. The space  $C_c^*(X)$  has a fundamental bounded resolution if and only if  $X$  is a *cn*-space.*

*Proof.* If  $X$  is a *cn*-space, Theorem 8 ensures that  $C_c(X)$  has a fundamental bounded resolution, which implies that  $C_c^*(X)$ , as a subspace of  $C_c(X)$ , also has a fundamental bounded resolution. Conversely, if  $C_c^*(X)$  has a closed fundamental bounded resolution  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ , for each  $\alpha \in \mathbb{N}^{\mathbb{N}}$  let  $B_\alpha$  denote the closure of  $A_\alpha$  in  $C_c(X)$ . Let us show that  $\mathcal{B} = \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  is a fundamental bounded resolution for  $C_c(X)$ . Indeed, let us denote by  $\mathcal{K}(X)$  the family of all compact sets of  $X$  and pick an arbitrary bounded subset  $B$  of  $C_c(X)$ . For each  $K \in \mathcal{K}(X)$  and  $f \in B$ , set  $M(f, K) = \sup\{|f(x)| : x \in K\}$  and define  $f_K(x) = f(x)$  if  $|f(x)| \leq M(f, K)$  and  $f_K(x) = \text{sign}(f(x)) \cdot M(f, K)$  if  $|f(x)| > M(f, K)$ . Clearly  $f_K \in C^*(X)$ , besides  $f_K(x) = f(x)$  for  $x \in K$  and  $|f_K(x)| \leq |f(x)|$  for all  $x \in X$ . Define  $P_B = \{f_K : f \in B, K \in \mathcal{K}(X)\}$  and note that  $f$  is an adherent point of  $P_B$  in  $C_c(X)$ . Therefore,  $B$  is contained in the closure of  $P_B$  in  $C_c(X)$ . If  $Q$  is any compact subset of  $X$ , the fact that  $B$  is  $\tau_c$ -bounded guarantees that

$$\begin{aligned} \sup_{g \in P_B} \sup_{x \in Q} |g(x)| &= \sup_{f \in B, K \in \mathcal{K}(X)} \sup_{x \in Q} |f_K(x)| \\ &\leq \sup_{f \in B} \sup_{x \in Q} |f(x)| < \infty, \end{aligned} \quad (15)$$

which shows that  $P_B$  is a  $\tau_c$ -bounded subset of  $C^*(X)$ . Hence, there is  $\gamma \in \mathbb{N}^{\mathbb{N}}$  such that  $P_B \subseteq A_\gamma$ , so that  $B \subseteq B_\gamma$ . Consequently  $\mathcal{B}$  is a fundamental bounded resolution for  $C_c(X)$ , as stated. Another application of Theorem 8 shows that  $X$  is a *cn*-space.  $\square$

The existence of a bounded resolution on a locally convex space  $E$  does not imply the existence of a fundamental bounded resolution for  $E$ , as the following example shows.

*Example 10.* If  $X$  is an infinite Talagrand compact set, the weak\* dual  $L_p(C_p(X))$  of  $C_p(C_p(X))$  has a bounded resolution but it has no fundamental bounded resolution.

*Proof.* Since  $C_p(X)$  is  $K$ -analytic,  $L_p(C_p(X))$  is also  $K$ -analytic by [17, Proposition 0.5.13]. So  $L_p(C_p(X))$  has even a compact resolution by [3, Theorem 3.2]. Suppose by contradiction that  $L_p(C_p(X))$  has a fundamental bounded resolution  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ . Identifying  $C_p(X)$  with its homeomorphic copy in  $L_p(C_p(X))$ , for each  $\alpha \in \mathbb{N}^{\mathbb{N}}$  set  $B_\alpha = A_\alpha \cap C_p(X)$  and consider the family  $\mathcal{B} = \{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ . We claim that  $B_\alpha$  is a functionally bounded subset of  $C_p(X)$ . Indeed, if  $F \in C(C_p(X))$  according to [17, Proposition 0.5.11] there exists a (unique) continuous functional  $u_F$  of  $L_p(C_p(X))$  such that  $u_F|_{C_p(X)} = F$ . Since  $A_\alpha$  is bounded, there is  $C > 0$  such that  $|u_F(a)| < C$  for all  $a \in A_\alpha$ . In particular,  $|F(b)| = |u_F(b)| < C$  for every  $b \in B_\alpha$ . So  $\mathcal{B}$  is a functionally

bounded resolution in  $C_p(X)$ . If  $B$  is a functionally bounded subset of  $C_p(X)$ , then  $B$ , considered as a subset of  $L_p(C_p(X))$ , is bounded. Therefore, there is  $\alpha \in \mathbb{N}^{\mathbb{N}}$  with  $B \subseteq B_\alpha$ . Hence  $\mathcal{B}$  swallows the functionally bounded subsets of  $C_p(X)$ . Since  $C_p(X)$  is Lindelöf and hence a  $\mu$ -space, the family  $\{\overline{B_\alpha}^{\tau_p} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  consists of compact subsets and swallows the compact sets of  $C_p(X)$ . So  $C_p(X)$  has a fundamental compact resolution. But according to Theorem 1, the space  $X$  should be countable and discrete. Therefore,  $X$  being compact is finite, a contradiction.  $\square$

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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