Research Article

Stability of the Wave Equation with a Source

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We prove the generalized Hyers-Ulam stability of the wave equation with a source, \( u_t(x,t) - c^2u_{xx}(x,t) = f(x,t) \), for a class of real-valued functions with continuous second partial derivatives in \( x \) and \( t \).

1. Introduction

The stability problem for functional equations or (partial) differential equations started with the question of Ulam [1]: Under what conditions does there exist an additive function near an approximately additive function? In 1941, Hyers [2] answered the question of Ulam in the affirmative for the Banach space cases. Indeed, Hyers’ theorem states that the following statement is true for all \( \varepsilon \geq 0 \): if a function \( f \) satisfies the inequality \( \|f(x + y) - f(x) - f(y)\| \leq \varepsilon \) for all \( x \), then there exists an exact additive function \( F \) such that \( \|f(x) - F(x)\| \leq \varepsilon \) for all \( x \). In that case, the Cauchy additive functional equation, \( f(x + y) = f(x) + f(y) \), is said to have (satisfy) the Hyers-Ulam stability.

Assume that \( V \) is a normed space and \( I \) is an open interval of \( \mathbb{R} \). The \( n \)-th order linear differential equation

\[
a_n(x) y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) + h(x) = 0
\]

is said to have (satisfy) the Hyers-Ulam stability provided the following statement is true for all \( \varepsilon \geq 0 \): if a function \( u : I \to V \) satisfies the differential inequality

\[
\|a_n(x) u^{(n)}(x) + a_{n-1}(x) u^{(n-1)}(x) + \cdots + a_1(x) u'(x) + a_0(x) u(x) + h(x)\| \leq \varepsilon
\]

for all \( x \in I \), then there exists a solution \( u_0 : I \to V \) to the differential equation (1) and a continuous function \( K \) such that \( \|u(x) - u_0(x)\| \leq K(\varepsilon) \) for any \( x \in I \) and \( \lim_{\varepsilon \to 0} K(\varepsilon) = 0 \).

When the above statement is true even if we replace \( \varepsilon \) and \( K(\varepsilon) \) by \( \varphi(x) \) and \( \Phi(x) \), where \( \varphi, \Phi : I \to [0, \infty) \) are functions not depending on \( u \) and \( u_0 \) explicitly, the corresponding differential equation (1) is said to have (satisfy) the generalized Hyers-Ulam stability. (This type of stability is sometimes called the Hyers-Ulam-Rassias stability.)

These terminologies will also be applied for other differential equations and partial differential equations. For more detailed definitions, we refer the reader to [1–9].

To the best of our knowledge, Obloza was the first author who investigated the Hyers-Ulam stability of differential equations (see [10, 11]): assume that \( g, r : (a, b) \to \mathbb{R} \) are continuous functions with \( \int_a^b |g(x)| \, dx < \infty \) and \( \varepsilon \) is an arbitrary positive real number. Obloza’s theorem states that there exists a constant \( \delta > 0 \) such that \( |y(x) - y_0(x)| \leq \delta \) for all \( x \in (a, b) \) whenever a differentiable function \( y : (a, b) \to \mathbb{R} \) satisfies the inequality \( |y'(x) + g(x)y(x) - r(x)| \leq \varepsilon \) for all \( x \in (a, b) \) and a function \( y_0 : (a, b) \to \mathbb{R} \) satisfies \( y_0'(x) + g(x)y_0(x) = r(x) \) for all \( x \in (a, b) \) and \( y(r) = y_0(r) \) for some \( r \in (a, b) \). Since then, a number of mathematicians have dealt with this subject (see [3, 12, 13]).

Prástaro and Rassias are the first authors who investigated the Hyers-Ulam stability of partial differential equations (see [14]). Thereafter, the first author [15], together with Lee, proved the Hyers-Ulam stability of the first-order linear
partial differential equation of the form, \( a u_x(x, y) + b u_y(x, y) + c u(x, y) + d = 0 \), where \( a, b \in \mathbb{R} \) and \( c, d \in \mathbb{C} \) are constants with \( \Re(c) \neq 0 \). As a further step, the first author proved the generalized Hyers-Ulam stability of the wave equation without source (see [16, 17]).

One of typical examples of hyperbolic partial differential equations is the wave equation with a spatial variable \( x \) and a time variable \( t \),

\[
\frac{\partial^2 u}{\partial t^2} (x, t) - c^2 \frac{\partial^2 u}{\partial x^2} (x, t) = f(x, t) \tag{3}
\]

where \( c > 0 \) is a constant, whose solution is a scalar function \( u = u(x, t) \) describing the propagation of a wave at a speed \( c \) in the spatial direction.

In this paper, applying ideas from [16, 18], we investigate the generalized Hyers-Ulam stability of the wave equation (3) with a source, where \( x > a \) and \( t > b \) with \( a, b \in \mathbb{R} \cup \{-\infty\} \). The main advantages of this present paper over the previous papers [16, 17] are that this paper deals with the wave equation with a source and it describes the behavior of approximate solutions of wave equation in the vicinity of origin while the previous one [17] can only deal with domains excluding the vicinity of origin. (Roughly speaking, a solution to a perturbed equation is called an approximate solution.)

\section{Main Results}

We know that if we introduce the characteristic coordinates

\[
\begin{align*}
\xi & = x + ct, \\
\eta & = x - ct,
\end{align*}
\tag{4}
\]

then the wave equation, \( u_t (x, t) = c^2 u_{xx} (x, t) \), is transformed into \( u_{\xi \eta} (\xi, \eta) = 0 \), which seems to be handled easily.

Given real constants \( a \) and \( b \) with \( a, b \in \mathbb{R} \cup \{-\infty\} \), we define

\[
U = \{(x, t) \in \mathbb{R} \times \mathbb{R} : x + ct > a, \ x - ct > b\},
\]

\[
W = \{ (\xi, \eta) \in \mathbb{R} \times \mathbb{R} : \xi > a, \ \eta > b \}. \tag{5}
\]

We note that the map \((x, t) \mapsto (\xi, \eta)\), where \( \xi = x + ct \) and \( \eta = x - ct \), is a one-to-one correspondence from \( U \) onto \( W \) (see Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Figure 1}
\end{figure}

\textbf{Theorem 1.} Assume that \( f : U \to \mathbb{R} \) and \( \varphi : U \to [0, \infty) \) are continuous functions with the properties

\[
\int_U |f(x, t)| \, dx \, dt < \infty,
\]

\[
\int_U \varphi(x, t) \, dx \, dt < \infty.
\]

If a function \( u : U \to \mathbb{R} \) has continuous second partial derivatives and satisfies the inequality

\[
|u_{\xi \eta} (x, t) - c^2 u_{xx} (x, t) - f(x, t)| \leq \varphi(x, t) \tag{7}
\]

for all \((x, t) \in U\), then there exists a function \( v : U \to \mathbb{R} \) with continuous second partial derivatives such that \( v \) is a solution to the wave equation (3) and

\[
|u(x, t) - v(x, t)| \leq \frac{1}{2c} \int_{U_{a,b,x,t}} \varphi(x, y) \, dx \, dy
\]

for all \((x, t) \in U\), where \( U_{a,b,x,t} \) is the interior of the parallelogram having the points

\[
\left( \begin{array}{c}
a + b \\
\frac{a - b}{2}
\end{array} \right),
\left( \begin{array}{c}
x + a - ct \\
\frac{-x + a + ct}{2c}
\end{array} \right),
\left( \begin{array}{c}
x, t \\
\frac{x + b + ct}{2c}, \frac{x - b + ct}{2c}
\end{array} \right)
\]

as its vertices.

\textbf{Proof.} We introduce the characteristic coordinates (4) and we set

\[
\omega(\xi, \eta) = u \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right) = u(x, t),
\]

\[
g(\xi, \eta) = f \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right) = f(x, t),
\]

\[
\psi(\xi, \eta) = \varphi \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c} \right) = \varphi(x, t)
\]

for all \((x, t) \in U\) and corresponding \((\xi, \eta) \in W\) with the relations in (4).

By the chain rule, we get

\[
u_{\xi} (x, t) = \omega_{\xi} (\xi, \eta) \frac{\partial \xi}{\partial x} + \omega_{\eta} (\xi, \eta) \frac{\partial \eta}{\partial x}
\]

\[
= \omega_{\xi} (\xi, \eta) + \omega_{\eta} (\xi, \eta), \tag{11}
\]

\[
u_{t} (x, t) = \omega_{\xi} (\xi, \eta) \frac{\partial \xi}{\partial t} + \omega_{\eta} (\xi, \eta) \frac{\partial \eta}{\partial t}
\]

\[
= c \omega_{\xi} (\xi, \eta) - c \omega_{\eta} (\xi, \eta)
\]
and hence,
\[
\begin{align*}
    u_{tt}(x,t) - c^2 u_{xx}(x,t) &= \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u(x,t) \\
    &= \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( 2cw_\eta(\xi,\eta) \right) = -4c^2 w_\eta(\xi,\eta)
\end{align*}
\]
for all \((x,t) \in U\) and corresponding \((\xi,\eta) \in W\) with the relations in (4).

It then follows from (7) and (10) that
\[
\begin{align*}
    \left| u_{tt}(x,t) - c^2 u_{xx}(x,t) - f(x,t) \right| &= \left| -4c^2 w_\eta(\xi,\eta) - g(\xi,\eta) \right| \leq \psi(\xi,\eta)
\end{align*}
\]
or
\[
\begin{align*}
    w_\eta(\xi,\eta) + \frac{1}{4c^2} g(\xi,\eta) &\leq \frac{1}{4c^2} \psi(\xi,\eta)
\end{align*}
\]
or
\[
\begin{align*}
    -\frac{1}{4c^2} \psi(\xi,\eta) - \frac{1}{4c^2} g(\xi,\eta) &\leq w_\eta(\xi,\eta)
\end{align*}
\]
for any \((\xi,\eta) \in W\).

Considering the conditions in (6) and Figure 2, we can integrate each term of the last inequality from \(a\) to \(\xi\) with respect to the first variable and then integrate each term of the resulting inequality from \(b\) to \(\eta\) with respect to the second variable to obtain
\[
\begin{align*}
    -\frac{1}{4c^2} \int_b^\eta \int_a^\xi \psi(\sigma,\tau) \, d\sigma \, d\tau - \frac{1}{4c^2} \int_b^\eta \int_a^\xi g(\sigma,\tau) \, d\sigma \, d\tau &\leq \int_b^\eta \int_a^\xi w_\eta(\sigma,\tau) \, d\sigma \, d\tau \\
    \leq \frac{1}{4c^2} \int_b^\eta \int_a^\xi \psi(\sigma,\tau) \, d\sigma \, d\tau \\
    \leq \frac{1}{4c^2} \int_b^\eta \int_a^\xi g(\sigma,\tau) \, d\sigma \, d\tau
\end{align*}
\]
for any \((\xi,\eta) \in W\).

If we define the function \(z : W \to \mathbb{R}\) by
\[
\begin{align*}
    z(\xi,\eta) &= -\frac{1}{4c^2} \int_b^\eta \int_a^\xi g(\sigma,\tau) \, d\sigma \, d\tau + w(\xi, b) \\
    &+ w(a, \eta) - w(a, b),
\end{align*}
\]
then we have
\[
\begin{align*}
    \left| w(\xi,\eta) - z(\xi,\eta) \right| &\leq \frac{1}{4c^2} \int_b^\eta \int_a^\xi \psi(\sigma,\tau) \, d\sigma \, d\tau
\end{align*}
\]
for all \((\xi,\eta) \in W\). Moreover, we get
\[
\begin{align*}
    z(\xi,\eta) &= -\frac{1}{4c^2} g(\xi,\eta).
\end{align*}
\]
We now set \(v(x,t) = z(\xi,\eta) = z(x + ct, x - ct)\) and, analogously to (11), we compute the partial derivatives:
\[
\begin{align*}
    v_x(x,t) &= z_\xi(\xi,\eta) + z_\eta(\xi,\eta), \\
    v_{xx}(x,t) &= z_{\xi\xi}(\xi,\eta) + 2z_{\xi\eta}(\xi,\eta) + z_{\eta\eta}(\xi,\eta), \\
    v_t(x,t) &= cz_\xi(\xi,\eta) - cz_\eta(\xi,\eta), \\
    v_{tt}(x,t) &= c^2 z_{\xi\xi}(\xi,\eta) - 2c^2 z_{\xi\eta}(\xi,\eta) + c^2 z_{\eta\eta}(\xi,\eta).
\end{align*}
\]
In view of (10), (12), (19), and (20), we get
\[
\begin{align*}
    v_{tt}(x,t) - c^2 v_{xx}(x,t) &= -4c^2 z_{\eta\eta}(\xi,\eta) = g(\xi,\eta) \\
    &= f(x,t)
\end{align*}
\]
for all \((x,t) \in U\), that is, \(v\) is a solution to wave equation (3).

We compute the Jacobian determinant
\[
\begin{align*}
    J(x,t) &= \det \left( \frac{\partial (\xi, \eta)}{\partial (\sigma, \tau)} \right) = \det \left( \frac{1}{1 - c} \right) = -2c.
\end{align*}
\]
By (10) and (18), we obtain
\[
\begin{align*}
    |u(x,t) - v(x,t)| &\leq \frac{1}{4c^2} \int_b^\eta \int_a^\xi \psi(\sigma,\tau) \, d\sigma \, d\tau \\
    &= \frac{1}{2c} \int \int_{U_{a,b,x,t}} \phi(p,q) \, dp \, dq
\end{align*}
\]
for all \((x,t) \in U\) (see Figure 2).

Remark 2. In general, it is somewhat tedious to estimate the upper bound of inequality (8). However, in view of (10) and (18), we can compute the upper bound less tediously:
\[
\begin{align*}
    |u(x,t) - v(x,t)| &\leq \int_b^\eta \int_a^\xi \frac{1}{4c^2} \psi(\sigma,\tau) \, d\sigma \, d\tau \\
    &= \int_b^\eta \int_a^\xi \frac{1}{2c} \phi(\sigma + \tau, \sigma - \tau) \, d\sigma \, d\tau
\end{align*}
\]
for all \((x,t) \in U\).
When \( a = b = -\infty \) in Theorem 1, \( U = W = \mathbb{R} \times \mathbb{R} \). In that case, by Theorem 1 and Remark 2, we have the following corollary.

**Corollary 3.** Assume that \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( \varphi : \mathbb{R} \times \mathbb{R} \to [0, \infty) \) are continuous functions satisfying the conditions

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, t)| \, dx \, dt < \infty, \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, t) \, dx \, dt < \infty. 
\] (25)

If a function \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) has continuous second partial derivatives and satisfies the inequality

\[
|u_{tt}(x, t) - c^2 u_{xx}(x, t) - f(x, t)| \leq \varphi(x, t) 
\] (26)

for all \( x, t \in \mathbb{R} \), then there exists a function \( v : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) with continuous second partial derivatives such that \( v \) is a solution to the wave equation (3) and

\[
|u(x, t) - v(x, t)| \leq \frac{1}{4c^2} \int_{-\infty}^{x-ct} \int_{-\infty}^{x+ct} \varphi\left(\frac{\sigma + \tau}{2}, \frac{\sigma - \tau}{2c}\right) \, d\sigma \, d\tau 
\] (27)

for all \( x, t \in \mathbb{R} \).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this article.

**Authors’ Contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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