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Research Article

The Exact Iterative Solution of Fractional Differential Equation with Nonlocal Boundary Value Conditions

Jinxiu Mao,[1] **Zengqin Zhao**,[1] **and Chenguang Wang**[2]

[1] *School of Mathematics, Qufu Normal University, Qufu, Shandong 273165, China*
[2] *Department of Mathematics, Jining University, Qufu, Shandong 273155, China*

Correspondence should be addressed to Jinxiu Mao; maojinxiu1982@163.com

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We deal with a singular nonlocal fractional differential equation with Riemann-Stieltjes integral conditions. The exact iterative solution is established under the iterative technique. The iterative sequences have been proved to converge uniformly to the exact solution, and estimation of the approximation error and the convergence rate have been derived. An example is also given to demonstrate the results.

Dedicated to our advisors

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines; see [1–5]. Much attention has been paid to study fractional differential equations both with initial and boundary conditions; see, for example, [6, 7]. In [8, 9], they focused on sign-changing solution for some fractional differential equations. In [10], they get the existence of solutions for impulsive fractional differential equations. In [11–13], they get the existence and multiplicity of nontrivial solutions for a class of fractional differential equations. The mainly techniques authors need are fixed point theory, variational method, and global bifurcation techniques.

Also, ordinary differential equations and partial differential equations involving nonlocal boundary conditions have been studied extensively in recent years, see [14–22], including integral boundary conditions and multipoint boundary conditions.

In [23], authors obtained results on the uniqueness of positive solution for problem

$$\begin{aligned} &\mathrm{D}^p x(t) + p(t) f(t, x(t)) + q(t) = 0, \quad t \in (0,1), \\ &x(0) = x'(0) = 0, \\ &x(1) = 0, \end{aligned} \tag{1}$$

where $2 < p \leq 3$ is a real number. Under the assumption that

$$\left| f(t,u) - f(t,v) \right| \leq k\lambda_1 |u - v|, \tag{2}$$

where $k \in [0,1)$, and λ is the first eigenvalue of the corresponding linear operator.

Motivated by the above works, we study the following nonlocal boundary value problems:

$$\begin{aligned} &\mathrm{D}^q x(t) + f(t, x(t)) = 0, \quad t \in (0,1), \\ &x(1) = x'(1) = 0, \\ &x(0) = \int_0^1 x(t)\, d\Lambda(t), \end{aligned} \tag{3}$$

where $D^q x$ denotes the left-handed Riemann-Liouville derivative of order q and $2 < q \leq 3$ is a real number. $\lambda[x] = \int_0^1 x(t) d\Lambda(t)$ denotes a Stieltjes integral with a suitable function Λ of bounded variation. Different from [23] and other works, we only use the iterative methods to obtain the existence and uniqueness of positive solution. Moreover, the estimation of the approximation error and the convergence rate have also been derived.

For clarity in presentation, we also list below some assumptions to be used later in the paper.

(H_1) : $f:(0,1)\times(0,+\infty)\longrightarrow[0,+\infty)$ is continuous, and for $(t,u)\in(0,1)\times(0,+\infty)$, f is increasing with respect to u and there exists a constant $k\in(0,1)$ such that, for $\forall\sigma\in(0,1]$,

$$f(t,\sigma u)\geq\sigma^k f(t,u). \tag{4}$$

It is easy to see that if $\sigma\in(1,+\infty)$, then $f(t,\sigma u)\leq\sigma^k f(t,u)$.

(H_2) : $0\leq A=\int_0^1(1-t)^{q-1}d\Lambda(t)<1,\ q\in(2,3]$,

(H_3) : $\int_0^1 d\Lambda(t)\geq 0$, $\int_0^1 d\Lambda(t)\neq 1$.

2. Preliminaries

For the convenience of the reader, we present here some necessary definitions from fractional calculus theory. These definitions and properties can be found in the recent monograph [23].

Definition 1. The Riemann-Liouville fractional integral of order $q>0$ of a function $x:(0,\infty)\longrightarrow R$ is given by

$$I^q x(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}x(s)\,ds, \tag{5}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2. The Riemann-Liouville fractional derivative of order $q>0$ of a continuous function $x:(0,\infty)\longrightarrow R$ is given by

$$D^q x(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{x(s)}{(t-s)^{q-n+1}}ds, \tag{6}$$

where $n1\leq q<n$, $n=[q]+1, q>0$, provided that the right-hand side is pointwise defined on $(0,\infty)$. In particular,

$$D^n x(t)=x^{(n)}(t),\quad n=1,2,3,\ldots. \tag{7}$$

Lemma 3 (see [13]). *Assume that $(H_1)-(H_3)$ hold. Let $y\in L^1((0,1),[0,+\infty))$. Then boundary value problem*

$$\begin{gathered}D^q u(t)+y(t)=0,\quad t\in(0,1),\\ u(1)=u'(1)=0,\\ u(0)=\int_0^1 u(t)\,d\Lambda(t)\end{gathered} \tag{8}$$

has the unique solution given by the following formula:

$$u(t)=\int_0^1 G_q(t,s)\,y(s)\,ds, \tag{9}$$

where

$$G(t,s)=\begin{cases}(1-t)^{q-1}s^{q-1}, & 0\leq s\leq t\leq 1;\\ (1-t)^{q-1}s^{q-1}-(s-t)^{q-1}, & 0\leq t\leq s\leq 1,\end{cases} \tag{10}$$

$$G_q(t,s)=G(t,s)+\frac{(1-t)^{q-1}}{1-A}\int_0^1 G(t,s)\,d\Lambda(t).$$

One can prove that $G(t,s), G_q(t,s)$ have the following properties.

Lemma 4. *Note that $G_q(t,s)$ is the Green function of problem (8).*

Lemma 5 (see [12]). *For $t,s\in[0,1]$, one has*

$$\begin{gathered}\frac{1}{\Gamma(q)}m(t)k(s)\leq G(t,s)\leq\frac{1}{\Gamma(q-1)}k(s),\\ \frac{1}{\Gamma(q)}m(t)k(s)\leq G(t,s)\leq\frac{1}{\Gamma(q-1)}m(t),\end{gathered} \tag{11}$$

where

$$\begin{gathered}k(s)=s^{q-1}(1-s),\\ m(t)=t(1-t)^{q-1}.\end{gathered} \tag{12}$$

Lemma 6.

$$g(s)(1-t)^{q-1}\leq G_q(t,s)\leq M(1-t)^{q-1} \tag{13}$$

where $M>0$ is a constant and $g(s)\in L^1(0,1)$ is nonnegative for any $s\in(0,1)$.

Proof. We have the estimation

$$\begin{aligned}G_q(t,s)&\leq\frac{1}{\Gamma(q-1)}(1-t)^{q-1}+\frac{(1-t)^{q-1}}{1-A}\\ &\quad\cdot\int_0^1\frac{1}{\Gamma(q-1)}(1-t)^{q-1}d\Lambda(t)\\ &=\frac{1}{\Gamma(q-1)}\left[1+\frac{\int_0^1(1-t)^{q-1}d\Lambda(t)}{1-A}\right](1-t)^{q-1}\\ &=\left[\frac{1}{\Gamma(q-1)}\frac{1}{1-A}\right](1-t)^{q-1}=M(1-t)^{q-1},\end{aligned} \tag{14}$$

$$\begin{aligned}G_q(t,s)&\geq\frac{1}{\Gamma(q)}\frac{(1-t)^{q-1}}{1-A}\\ &\quad\cdot\int_0^1 t(1-t)^{q-1}s^{q-1}(1-s)\,d\Lambda(t)\\ &=\frac{1}{\Gamma(q)}\left[\frac{(1-t)^{q-1}}{1-A}\int_0^1 t(1-t)^{q-1}d\Lambda(t)\right]\\ &\quad\cdot s^{q-1}(1-s)\\ &=\left[\frac{1}{\Gamma(q)}\frac{\int_0^1 t(1-t)^{q-1}d\Lambda(t)}{1-A}s^{q-1}(1-s)\right]\\ &\quad\cdot(1-t)^{q-1}=g(s)(1-t)^{q-1},\end{aligned} \tag{15}$$

where $M=(1/\Gamma(q-1))(1/(1-A))$ and $g(s)=(1/\Gamma(q))(\int_0^1 t(1-t)^{q-1}d\Lambda(t)/(1-A))s^{q-1}(1-s)$. Thus, (13) holds. □

3. The Main Results

Throughout this paper, we will work in the space $E = C[0,1]$, which is a Banach space if it is endowed with the norm $\| u \| = \max_{t\in[0,1]}|u(t)|$ for any $u \in E$.

Define the set P in E as follows:

$P = \{u \in E \mid$ there exists positive constants $0 < l_u < 1 < L_u$ such that $l_u(1-t)^{q-1} \le u(t) \le L_u(1-t)^{q-1},\ t \in [0,1]\}$.

And define the operator $T : E \longrightarrow E$.

$$Tu(t) = \int_0^1 G_q(t,s) f(s,u(s))\,ds. \tag{16}$$

Evidently $(1-t)^{q-1} \in P$. Therefore, P is not empty.

Theorem 7. *Assume that* (H_1)-(H_3) *hold. And*

$$0 < \int_0^1 f\left(t,(1-t)^{q-1}\right) dt < \infty. \tag{17}$$

Then BVP (3) has at least one positive solution $u(t)$*, and there exist constants* $0 < l_u < 1 < L_u$ *satisfying*

$$l_u(1-t)^{q-1} \le u(t) \le L_u(1-t)^{q-1}, \quad t \in [0,1]. \tag{18}$$

Proof. It is clear that u is a solution of (3) if and only if u is a fixed point of T.

Claim 1. The operator $T : P \longrightarrow P$ is nondecreasing.

In fact, for $u \in E$, it is obvious that $u \in E$, $Tu(1) = Tu'(1) = 0, Tu(0) = \int_0^1 Tu(t)d\Lambda(t)$, and $Tu(t) > 0$ for $t \in (0,1)$. For any $u \in P$, we have that, for $t \in [0,1]$,

$$\begin{aligned} Tu(t) &= \int_0^1 G_q(t,s) f(s,u(s))\,ds. \\ &\le M(1-t)^{q-1} f\left(s, L_u(1-s)^{q-1}\right) ds \\ &\le \int_0^1 M(1-t)^{q-1} L_u^k f\left(s,(1-s)^{q-1}\right) \\ &\le L_{Tu}(1-t)^{q-1} \end{aligned} \tag{19}$$

and

$$\begin{aligned} Tu(t) &= \int_0^1 G_q(t,s) f(s,u(s))\,ds. \\ &\ge g(s)(1-t)^{q-1} f\left(s, l_u(1-s)^{q-1}\right) ds \\ &\ge (1-t)^{q-1} l_u^k \int_0^1 g(s) f\left(s,(1-s)^{q-1}\right) \\ &= l_{Tu}(1-t)^{q-1}, \end{aligned} \tag{20}$$

where L_{Tu} and l_{Tu} are positive constants satisfying

$$\begin{aligned} L_{Tu} &> \max\left\{1, \int_0^1 ML_u^k f\left(s,(1-s)^{q-1}\right)\right\}, \\ 0 < l_{Tu} &< \min\left\{1, l_u^k \int_0^1 g(s) f\left(s,(1-s)^{q-1}\right)\right\}. \end{aligned} \tag{21}$$

Thus, it follows that there are constants $0 < l_{Tu} < 1 < L_{Tu}$ such that, for $t \in [0,1]$,

$$l_{Tu}(1-t)^{q-1} \le Tu(t) \le L_{Tu}(1-t)^{q-1}. \tag{22}$$

Therefore, for any $u(t) \in P$, $Tu(t) \in P$, i.e., T is the operator $P \longrightarrow P$. From (16), it is easy to see that T is nondecreasing for u. Hence, Claim 1 holds.

Claim 2. We take $e(t) = (1-t)^{q-1}$. Let δ and γ be fixed numbers satisfying

$$\begin{aligned} 0 < \delta &\le l_{Te}^{1/(1-k)}, \\ \gamma &\ge L_{Te}^{1/(1-k)}, \end{aligned} \tag{23}$$

and assume that

$$\begin{aligned} u_0 &= \delta e(t), \\ v_0 &= \gamma e(t), \end{aligned} \tag{24}$$

$$\begin{aligned} u_n &= Tu_{n-1}, \\ v_n &= Tv_{n-1}, \\ n &= 1,2,3,\ldots. \end{aligned} \tag{25}$$

Then

$$u_0 \le u_1 \le \ldots \le u_n \le \ldots v_n \le \ldots \le v_1 \le v_0, \tag{26}$$

and there exists $u^* \in P$ such that

$$\begin{aligned} u_n(t) &\longrightarrow u^*(t), \\ v_n(t) &\longrightarrow u^*(t), \end{aligned} \tag{27}$$

uniformly on $[0,1]$.

In fact, $0 < l_{Te} < 1 < L_{Te}$ since $Te \in P$. Therefore, $0 < \delta < 1 < \gamma$. From (24), we have $u_0, v_0 \in P$ and $u_0 \le v_0$.

On the other hand,

$$\begin{aligned} u_1 = Tu_0(t) &= \int_0^1 G_q(t,s) f(s,\delta e(s))\,ds \\ &\ge \delta^k \int_0^1 G_q(t,s) f(s,e(s))\,ds = \delta^k Te \ge \delta^k l_{Te} e(t) \\ &\ge \delta^k \delta^{1-k} e(t) = u_0, \\ v_1 = Tv_0(t) &= \int_0^1 G_q(t,s) f(s,\gamma e(s))\,ds \\ &\le \gamma^k \int_0^1 G_q(t,s) f(s,e(s))\,ds = \gamma^k Te \le \gamma^k l_{Te} e(t) \\ &\le \gamma^k \gamma^{1-k} e(t) = v_0, \end{aligned} \tag{28}$$

and since $u_0 \le v_0$ and T is nondecreasing, by induction, (26) holds.

Let $c_0 = \delta/\gamma$, and then $0 < c_0 < 1$. It follows from (4) that

$$T(c_0 u) \ge c_0^k Tu. \tag{29}$$

And for any natural number n,

$$\begin{aligned} u_n &= Tu_{n-1} = T^n u_0 = T^n(\delta e(t)) = T^n(c_0 \gamma e(t)) \\ &\geq c_0^{k^n} T^n(\gamma e(t)) = c_0^{k^n} v_n. \end{aligned} \tag{30}$$

Thus, for any natural number n and p^*, we have

$$\begin{aligned} 0 &\leq u_{n+p^*} - u_n \leq v_n - u_n \leq \left(1 - c_0^{k^n}\right) v_n \\ &\leq \left(1 - c_0^{k^n}\right) \gamma e(t), \end{aligned} \tag{31}$$

which implies that there exists $u^* \in P$ such that (27) holds and Claim 2 holds.

Letting $n \longrightarrow \infty$ in $u_n = Tu_{n-1}$ and noting the fact that T is continuous, we obtain $u^*(t) = Tu^*(t)$, which is a positive solution of BVP (3). The proof of Theorem 7 is now complete. □

Theorem 8. *Assume that* (H_1)-(H_3) *hold. Then*

(i) BVP (3) has unique positive solution $u^*(t)$*, and there exist constants* l, L *with* $0 < l < 1 < L$ *such that*

$$l(1-t)^{q-1} \leq u^*(t) \leq L(1-t)^{q-1}, \quad t \in [0,1]. \tag{32}$$

(ii) For any initial value $x_0 \in P$*, there exists a sequence* $x_n(t)$ *that uniformly converges to the unique positive solution* $u^*(t)$*, and one has the error estimation*

$$\max\{x_n(t) - u^*(t)\} = \circ\left(1 - \lambda^{k^n}\right), \tag{33}$$

where λ *is a constant with* $0 < \lambda < 1$ *and determined by* x_0.

Proof. Let u_0, v_0, u_n, v_n be defined in (24) and (25).

(i) It follows from Theorem 7 that BVP (3) has a positive solution $u^*(t) \in P$, which implies that there exist constants l and L with $0 < l < L < 1$ such that $u^*(t)$ satisfies (18). Let $v^*(t)$ be another positive solution of BVP (3); then from Theorem 7 we have that there exist constants c_1 and c_2 with $0 < c_1 < 1 < c_2$ such that

$$c_1(1-t)^{q-1} \leq v^*(t) \leq c_2(1-t)^{q-1}, \quad t \in [0,1]. \tag{34}$$

Let δ defined in (23) be small enough such that $\delta < c_1$ and γ defined in (23) be large enough such that $\gamma > c_2$. Then

$$u_0(t) \leq v^*(t) \leq v_0(t), \quad t \in [0,1]. \tag{35}$$

Note that $Tv^* = v^*$ and T is nondecreasing; we have

$$u_n(t) \leq v^*(t) \leq v_n(t), \quad t \in [0,1]. \tag{36}$$

Letting $n \longrightarrow \infty$ in (36), we obtain that $v^* = u^*$. Hence, the positive solution of BVP (3) is unique.

(ii) From (i), we know that the positive solution u^* to BVP (3) is unique. For any $x_0 \in P$, there exist constants l_0 and L_0 with $0 < l_0 < 1 < L_0$ such that

$$l_0(1-t)^{q-1} \leq x_0(t) \leq L_0(1-t)^{q-1}, \quad t \in [0,1]. \tag{37}$$

Similar to (i), we can let δ and γ defined by (23) satisfy $\delta < l_0$ and $\gamma > L_0$. Then

$$u_0(t) \leq x_0(t) \leq v_0(t), \quad t \in [0,1]. \tag{38}$$

Let $x_n = Tx_{n-1}, n = 1, 2, \ldots$. Note that T is nondecreasing; we have

$$u_n(t) \leq x_n(t) \leq v_n(t), \quad t \in [0,1]. \tag{39}$$

Letting $n \longrightarrow \infty$ in (39), it follows that x_n uniformly converges to the unique positive solution u^* for BVP (3), where

$$x_n = \int_0^1 G_q(t,s) f(s, x_{n-1}(s))\, ds, \quad n = 1, 2, \ldots. \tag{40}$$

At the same time, (33) follows from (31). Thus, the proof of the theorem is complete. □

4. An Example

$$\begin{aligned} &\mathrm{D}^{5/2} x(t) + \sin(t)\left(x^{1/2}(t) + x^{1/3}(t)\right) = 0, \\ &\qquad\qquad t \in (0,1), \\ &x(1) = x'(1) = 0, \\ &x(0) = \int_0^1 x(t)\, d\Lambda(t), \end{aligned} \tag{41}$$

where

$$\Lambda(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{2}\right); \\ 2, & t \in \left[\frac{1}{2}, \frac{3}{4}\right); \\ 1, & t \in \left[\frac{3}{4}, 1\right]. \end{cases} \tag{42}$$

Analysis 1. Let

$$\begin{aligned} q &= \frac{5}{2}, \\ f(t,x) &= \sin(t)\left(x^{1/2}(t) + x^{1/3}(t)\right), \end{aligned} \tag{43}$$

and then for any $\sigma \in (0,1)$, we take $k = 1/2$ and have

$$f(t, \sigma u) \geq \sigma^k f(t,u). \tag{44}$$

Then (H_1) holds.

In addition, we have

$$\begin{aligned} 0 &< A = \int_0^1 (1-t)^{q-1}\, d\Lambda(t) = \frac{\sqrt{2}}{2} - \frac{1}{8} < 1. \\ 0 &< \int_0^1 d\Lambda(t) = \frac{3}{4} < 1. \end{aligned} \tag{45}$$

Then (H_2) and (H_3) hold.

And

$$0 < \int_0^1 f\left(t, (1-t)^{5/2-1}\right) dt < \infty. \tag{46}$$

Hence all conditions of Theorem 7 are satisfied, and consequently we have the following corollary.

Corollary 9. *Problem (41) has unique positive solution $x^*(t)$. For any initial value $x_0 \in P$, the successive iterative sequence $x_n(t)$ generated by*

$$x_n(t) = \int_0^1 G_q(t,s) \sin(s) \left(x_{n-1}^{1/2}(s) + x_{n-1}^{1/3}(s)\right) ds, \quad n = 1, 2, \ldots, \tag{47}$$

uniformly converges to the unique positive solution $x^*(t)$ on $[0,1]$. One has the error estimation

$$\max\{x_n(t) - x^*(t)\} = \circ\left(1 - \lambda^{(1/2)^n}\right), \tag{48}$$

where λ is a constant with $0 < \lambda < 1$ and determined by the initial value x_0. And there are constants l, L with $0 < l < 1 < L$ such that

$$l(1-t)^{3/2} \le x^*(t) \le L(1-t)^{3/2}, \quad t \in [0,1]. \tag{49}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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