Inversion of Riesz Potentials for Dunkl Transform

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Abstract

The inversion of Riesz potentials for Dunkl transform when $G = \mathbb{Z}^d$ is given by using the generalized wavelet transforms. It is also proved that the Riesz potentials $I^\alpha_n$ are automorphisms on the Semyanistyi-Lizorkin spaces.

1. Introduction

Dunkl transform is a generalization of the Fourier transform associated with a family of weight functions, $h_\alpha$, invariant under a finite reflection group. Many papers devote to study the Dunkl transform; see [1–6] and the references therein.

In [7], the Riesz potentials $I^\alpha_n$ for Dunkl transform were defined by the generalized translation operators, $\tau_n$. The explicit expression and boundedness of $\tau_n$ are known only in some special cases such as when $G = \mathbb{Z}^d$ and the case when the kernel is a suitable radial function. The boundedness of $I^\alpha_n$ was given only in the two cases mentioned above. Gorbachev et al. [8] studied the weighted $(L^p, L^q)$-boundedness properties of Riesz potentials for Dunkl transform represented by the Stein-Weiss inequality. In this paper, we will study the inversion of $I^\alpha_n$ in the case when $G = \mathbb{Z}^d$. The paper is organized as follows. In Section 2, some necessary facts in Dunkl’s theory are reviewed. Section 3 is devoted to introduce the Semyanistyi-Lizorkin spaces associated with the reflection-invariant measure $h_\alpha(x)dx$. In the final section, the inversion of the Riesz potentials $I^\alpha_n$ will be given by the generalized wavelet transforms defined by the generalized translation operators.

2. Preliminaries

2.1. Dunkl Operator and Dunkl Transform. Let $G$ be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system $R_+$, normalized so that $\langle v, v \rangle = 2$ for all $v \in R_+$, where $\langle x, y \rangle$ denotes the usual Euclidean inner product. Let $\kappa$ be a nonnegative multiplicity function defined on $R_+$ with the property that $\kappa_a = \kappa_+$ whenever $a$ is conjugate to $\sigma_{\kappa_+}$ in $G$; then $\nu \mapsto \kappa_\nu$ is a $G$-invariant function. The weight function is positive homogeneous of degree $\gamma_\nu := \sum_{\nu \in \mathbb{R}} \kappa_\nu$, defined by

$$h_\kappa(x) = \prod_{\nu \in \mathbb{R}_+} |\langle x, \nu \rangle|^{\gamma_\nu}, \quad x \in \mathbb{R}^d. \tag{1}$$

Note that $h_\kappa$ is invariant under the reflection group $G$.

Let $D_i$ be Dunkl’s differential-difference operators defined in [1] as

$$D_i f(x) = \partial_i f(x) + \sum_{\nu \in \mathbb{R}} \kappa_\nu \frac{f(x) - f(x\sigma_{\nu})}{\langle x, \nu \rangle} (\nu, \epsilon_i), \quad 1 \leq j \leq d, \tag{2}$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_d$ are the standard unit vectors of $\mathbb{R}^d$ and $\sigma_{\nu}$ denotes the reflection with respect to the hyperplane perpendicular to $\nu$, $x\sigma_{\nu} := x - 2(\langle x, \nu \rangle / \|\nu\|^2)\nu$, $x \in \mathbb{R}^d$. The operators $D_i$, $1 \leq j \leq d$, map $\mathcal{D}$ to $\mathcal{D}^{-1}$, where $\mathcal{D}$ denotes the space of homogeneous polynomials of degree $n$ in $d$ variables, and they mutually commute; that is, $\mathcal{D}_j \mathcal{D}_i = \mathcal{D}_i \mathcal{D}_j$, $1 \leq i, j \leq d$. For example, when $d = 1$, the Dunkl operator is

$$D f(x) = f'(x) + \left(\kappa + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}. \tag{3}$$
The intertwining operator $V_k$ is a linear operator determined uniquely by
\begin{equation}
V_k \mathcal{D} \mathcal{D}^{-1} \subset \mathcal{D} \mathcal{D}^{-1}, \quad V_k 1 = 1,
\end{equation}
where $\mathcal{D} = \partial_x$. Let $E(x,y) = V(\sqrt{\epsilon}) \mathcal{D}^{-1}$, where the super-
script means that $V_k$ is applied to the $x$ variable. For $f \in L^1(\mathbb{R}^d, h_k^2)$, the Dunkl transform is defined by
\begin{equation}
\hat{f}(y) = c_k \int_{\mathbb{R}^d} f(x) E(x,-iy) h_k^2(x) \, dx
\end{equation}
where $c_k$ is the constant defined by $c_k^{-1} = \int_{\mathbb{R}^d} h_k^2(x) e^{-1|x|^2/2} \, dx$.

Define $\mathcal{S}_k(\mathbb{R}^d) = \{ f \in L^1(\mathbb{R}^d, h_k^2) : \hat{f} \in L^1(\mathbb{R}^d, h_k^2) \}$, and, for the sake of simplicity, set $(f,g)_k = \int_{\mathbb{R}^d} f(x)g(x) h_k^2(x) \, dx$ whenever the integral exists and denote $A_k = 2d + d$.

The Dunkl transform shares many of the important properties with the usual Fourier transform, part of which are listed as follows ([2, 3]).

**Proposition 1.** (i) If $f \in L^1(\mathbb{R}^d, h_k^2)$, then $\hat{f} \in C_0(\mathbb{R}^d)$ and $\| f \|_{L^1} \leq \| \hat{f} \|_{L^\infty}$.

(ii) If $f \in \mathcal{S}_k(\mathbb{R}^d)$, then $f(x) = \hat{\hat{f}}(-x)$.

(iii) The Dunkl transform $f \rightarrow \hat{f}$ is a topological automorphism on $\mathcal{S}(\mathbb{R}^d)$.

(iv) For $f \in \mathcal{S}(\mathbb{R}^d)$, then $\mathcal{D}_i \hat{f} = iM_i \hat{f}$ and $M_j \hat{f} = \mathcal{D}_j \hat{f}$, where $M_j f(x) = x_j f(x)$, $j = 1, 2, \ldots, d$.

(v) For all $f, g \in L^1(\mathbb{R}^d, h_k^2)$, we have $(\hat{f}, \hat{g})_k = (\hat{f}, g)_k$.

(vi) There exists a unique extension of the Dunkl transform to $L^2(\mathbb{R}^d, h_k^2)$ with $\| \hat{f} \|_{L^2} = \| f \|_{L^2}$.

2.2. Generalized Translation Operator and Generalized Con-
volution. Let $y \in \mathbb{R}^d$ be given. The generalized translation operator $f \rightarrow \tau_y f$ is defined on $L^2(\mathbb{R}^d, h_k^2)$ by $\tau_y f = E(y,-ix) f(x)$, $x \in \mathbb{R}^d$.

For $f, g \in L^1(\mathbb{R}^d, h_k^2)$, the generalized convolution operator is defined by
\begin{equation}
(f *_y g)(x) = \int_{\mathbb{R}^d} f(y) \tau_y g(y) h_k^2(y) \, dy
\end{equation}
where $\bar{g}(y) = g(-y)$. The main properties of the generalized translation operator and the generalized convolution are collected below [6, 9, 10].

**Proposition 2.** (i) For $f \in \mathcal{S}_k(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d, h_k^2)$ being bounded, then
\begin{equation}
\int_{\mathbb{R}^d} \tau_y f(\xi) \bar{g}(\xi) h_k^2(\xi) \, d\xi
\end{equation}
(ii) For $f \in \mathcal{S}(\mathbb{R}^d)$,
\begin{equation}
\int_{\mathbb{R}^d} \tau_y f(\xi) \bar{g}(\xi) h_k^2(\xi) \, d\xi
\end{equation}
When $G = \mathbb{Z}^d$,
(iii) for $f \in L^1(\mathbb{R}^d, h_k^2)$, $1 \leq p \leq \infty$, $\| \tau_y f \|_{L^p} \leq \| f \|_{L^p}$,
(iv) for $f, g \in L^p(\mathbb{R}^d, h_k^2)$, we have $\overline{\tau_y f * g} = \overline{\hat{f} \cdot \hat{g}}$ and $f *_y g = f * g$,
(v) let $p, q, r \geq 1$ and $1/p + 1/r = 1/p + 1/q - 1$. For $f \in L^p(\mathbb{R}^d, h_k^2)$, $g \in L^q(\mathbb{R}^d, h_k^2)$, $\| f *_y g \|_{L^r} \leq c\| f \|_{L^p} \| g \|_{L^q}$.

2.3. Dunkl Transform of Distributions. References [5, 11, 12] study the actions of the Dunkl operators and Dunkl transform on the space $\mathcal{S}'(\mathbb{R}^d)$. Reference [4] gives the definition of the Dunkl transform for the local integrable functions under the measure $h_k^2(x) \, dx$.

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d, h_k^2)$; the generalized function associated with $f$ is defined by
\begin{equation}
\langle f, \varphi \rangle_k = \int_{\mathbb{R}^d} f(x) \varphi(x) h_k^2(x) \, dx, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).
\end{equation}

The Dunkl transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined as
\begin{equation}
\langle \hat{f}, \varphi \rangle_k = \langle f, \hat{\varphi} \rangle_k, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).
\end{equation}

Then, for $f \in L^1_{\text{loc}}(\mathbb{R}^d, h_k^2)$, the Dunkl transform of $f$ is
\begin{equation}
\langle \hat{f}, \varphi \rangle_k = \int_{\mathbb{R}^d} f(x) \varphi(x) h_k^2(x) \, dx, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).
\end{equation}

For $f \in \mathcal{S}'(\mathbb{R}^d)$, the dilation transform $\varphi f$ is defined as
\begin{equation}
\langle \varphi f, \varphi \rangle_k = \langle f, \varphi \rangle_k, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).
\end{equation}

Let $\delta$ be the Dirac distribution associated with the measure $h_k^2(x) \, dx$; that is, $\langle \delta, \varphi \rangle_k = \varphi(0), \varphi \in \mathcal{S}(\mathbb{R}^d)$.

**Lemma 3.** Let $\varphi \in L^1(\mathbb{R}^d, h_k^2)$ satisfy $\int_{\mathbb{R}^d} \varphi(x) h_k^2(x) \, dx = 1$.

For any $\epsilon > 0$, define $\phi_\epsilon(x) = \epsilon^{-\lambda} \varphi(\epsilon x)$; then $\phi_\epsilon$ is a $\delta$-
sequence; that is,
\begin{equation}
\lim_{\epsilon \rightarrow 0^+} \langle \phi_\epsilon \varphi \rangle_k = \varphi(0), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).
\end{equation}

Then by (9) and (10), we obtain the Dunkl transform of $\delta$ as $\delta(\xi) = 1$.

Define the action of the Dunkl operators $\mathcal{D}_j$, $j = 1, 2, \ldots, d$ on the space $\mathcal{S}'(\mathbb{R}^d)$ as
\begin{equation}
\langle \mathcal{D}_j f, \varphi \rangle_k = -\langle f, \mathcal{D}_j \varphi \rangle_k, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).
\end{equation}

Denote $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{Z}^d$ and $\mathcal{D}^\beta = \mathcal{D}^{\beta_1} \circ \mathcal{D}^{\beta_2} \circ \cdots \circ \mathcal{D}^{\beta_d}$. Combining (9), (14), and Proposition 1 (iv), we have
\begin{equation}
\mathcal{D}^{\beta} \delta(\xi) = \int_{\mathbb{R}^d} \mathcal{D}^{\beta_1} \mathcal{D}^{\beta_2} \cdots \mathcal{D}^{\beta_d} \delta(\xi) = \delta(\xi) \delta(\xi).
\end{equation}
2.4. Dunkl Riesz Potentials. For simplicity, we call the Riesz
potential for Dunkl transform as the Dunkl Riesz potential
$I^α_1 f$, which is defined on $\mathcal{S}(\mathbb{R}^d)$ in [7] as
$$I^α_1 f(x) = (d^α_1)^{-1} \int_{\mathbb{R}^d} \tau_x f(x) \frac{1}{|y|^{\alpha - d}} h^2_{\alpha}(y) \, dy,$$
where $0 < \alpha < \lambda_\kappa$ and $d^α_1 = 2^{-\lambda_\kappa/2} \Gamma(\alpha/2)\Gamma((\lambda_\kappa - \alpha)/2)$. The
Dunkl transform and the Hardy-Littlewood-Sobolev theorem of
$I^α_1 f$ are given in [7].

Proposition 4. Let $0 < \alpha < \lambda_\kappa$. The identity
$$I^α_1 f(x) = |x|^{-\alpha} \tilde{f}(x)$$
holds in the sense that
$$\int_{\mathbb{R}^d} I^α_1 f(x) g(x) h^2_{\alpha}(x) \, dx = \int_{\mathbb{R}^d} \tilde{f}(x) |x|^{-\alpha} \tilde{g}(x) h^2_{\alpha}(x) \, dx$$
whenever $f, g \in \mathcal{S}(\mathbb{R}^d)$.

Proposition 5. Let $G = \mathbb{Z}_2^d$ and $0 < \alpha < \lambda_\kappa$. Let $p$ and $q$ satisfy
$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\lambda_\kappa}, \quad 1 \leq p < q < \infty.$$  
(1) For $f \in L^p(\mathbb{R}^d, h^2_{\alpha})$, $p > 1$, $\|I^α_1 f\|_{L^q} \leq c \|f\|_{L^p}$.
(2) For $f \in L^1(\mathbb{R}^d, h^2_{\alpha})$, the mapping $f \mapsto I^α_1 f$ is of weak
    type $(1, q)$.

3. Semyanistyi-Lizorkin Spaces

The following spaces were introduced by Semyanistyi and
generalized by Lizorkin and Samko; see [13–15].

Let $\Psi = \Psi(\mathbb{R}^d)$ be the class of functions $\psi$ in $\mathcal{S}(\mathbb{R}^d)$
vanishing at the origin $0$ with all their derivatives; that is,
$$\Psi = \{ \psi \in \mathcal{S}(\mathbb{R}^d) : \partial^{\beta}_{\psi}(0) = 0 \text{ for all } |\beta| = 0, 1, 2, \ldots \}.$$  
(20)

The space $\Psi$ is a closed linear subspace of $\mathcal{S}(\mathbb{R}^d)$. It can be
regarded as a linear topological space with the induced
topology generated by the sequence of norms
$$\|\psi\|_m = \max \{ 1 + |x|^m \sum_{|\beta| \leq m} \left| (\partial^{\beta}_{\psi})(x) \right| \},$$
(21)
$$m = 0, 1, 2, \ldots.$$  
We claim that $\partial^{\beta}_{\psi}(0) = 0$ for $\psi \in \Psi$, since $\partial_j \psi(0) = 0$ implies
$\partial_j \psi(0) = 0$ for $j = 1, 2, \ldots, d$.

Let $\Phi = \Phi(\mathbb{R}^d)$ be the image of $\Psi$ under the Dunkl
transform; that is, $\Phi = \{ \varphi \in \Psi : \hat{\varphi} \}$. Since the Dunkl
transform is an automorphism of $\mathcal{S}(\mathbb{R}^d)$, the space is a closed
linear subspace of $\mathcal{S}(\mathbb{R}^d)$. We equip $\Phi$ with the induced
topology of the ambient space $\mathcal{S}(\mathbb{R}^d)$. Then $\Phi$ becomes a
linear topology space which is isomorphic to $\Psi$ under the
action of the Dunkl transform. According to the definition of
$\Phi$, we conclude that the space $\Phi$ consists of all functions
$\varphi$ which are orthogonal to all polynomials as for the measure
$h^2_{\alpha}(x) \, dx$; that is,
$$\varphi \in \Phi \iff \int_{\mathbb{R}^d} \varphi(x) h^2_{\alpha}(x) \, dx = 0, \text{ for all } k \in \mathbb{Z}_2^d.$$  
(22)

In fact, if $\varphi \in \Phi$, then $\hat{\varphi} \in \Psi$, and for any multi-index $k$, by
Proposition 1 (iv), we have
$$\int_{\mathbb{R}^d} x^k \varphi(x) h^2_{\alpha}(x) \, dx = 0.$$  
(23)

Denote that $\Phi'$ and $\Psi'$ are the spaces of all semilinear
functionals on $\Phi$ and $\Psi$, respectively. Some properties of $\Phi$
and $\Psi$ are given in the following proposition.

Proposition 6. (i) The spaces $\Phi$ and $\Psi$ are not empty.
(ii) The space $\Phi$ does not contain compactly supported
infinite differentiable functions, rather than $0$.
(iii) The space $\Phi$ is invariant under the generalized
translations.
(iv) The space $\Phi$ is dense in $L^1(\mathbb{R}^d, h^2_{\alpha})$, $1 < p < \infty$.
(v) $\mathcal{S}'(\mathbb{R}^d)$-distributions that coincide in the $\Phi'$-sense
differ from each other by a polynomial.
(vi) Let $f \in L^1(\mathbb{R}^d, h^2_{\alpha})$ and $g \in L^1(\mathbb{R}^d, h^2_{\alpha})$, $1 \leq r, p < \infty$.
If $f = g$ in the $\Phi'$-sense, then $f \equiv g$ almost everywhere.

The proof of this proposition is similar to the ones in
[13–15] except with the reflection-invariant measure $h^2_{\alpha}(x) \, dx$.
Now we sketch it below.

Proof. (i) Choose $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfying that supp$\psi = \{ x \in \mathbb{R}^d : |x| > 1 \}$. Then $\psi \in \Psi$.
(ii) Suppose $\varphi \in C_{c}^{\infty}(\mathbb{R}^d)$; then we have
$$\hat{\varphi}(\xi) = \sum_{|\beta| \leq 0} \frac{\xi^\beta}{\partial^\beta_{\varphi}(0)}.$$  
(24)

If $\varphi \in \Phi$, $\hat{\varphi} \in \Psi$, and $\partial^\beta_{\varphi}(0) = 0$, so $\hat{\varphi}(\xi) = 0$ for all $\xi \in \mathbb{R}^d$.
Then $\psi(x) = 0$.
(iii) This conclusion can be obtained by Proposition 2 (iii).
(iv) It suffices to approximate a function \( f \in \mathcal{S}(\mathbb{R}^d) \) by functions \( f_N \in \Phi \) in the \( L^1(\mathbb{R}^d, h_N^2) \) norm. We introduce the functions

\[
\psi_N(x) = \mu(N |x|) f(x)
\]

(25)

where \( \mu \in C^\infty([0, \infty)) \) such that \( \mu(t) = 1 \) for \( t \geq 2, \mu(t) = 0 \) for \( 0 \leq t \leq 1 \) and \( 0 \leq \mu \leq 1 \). We define \( f_N(x) = \psi_N(x) \).

Since \( \psi_N \in \mathcal{S}(\mathbb{R}^d) \) and \( \psi_N(x) = 0 \) as \( |x| \leq 1/N \), we have \( \psi \in \Psi \) and then \( f_N \in \Phi \). In order to show that \( f_N(x) \) approximate the function \( f(x) \), we represent them as

\[
f_N(x) = f(x) - \int_{\mathbb{R}^d} k(y) \tau_{Ny} f(x) h_N^2(y) \, dy
\]

(26)

where \( k(y) \) is the inverse Dunkl transform of the function \( 1 - \mu(|x|) \). Then by Lemma 7, \( \| f - f_N \|_{\psi} \to 0 \) as \( N \to \infty \).

(v) Suppose \( f, g \in \mathcal{S}(\mathbb{R}^d) \) and \( f = g \) in the sense of \( \Phi^1 \); that is, for all \( \varphi \in \Phi \), \( \langle f - g, \varphi \rangle = 0 \). Then, for all \( \psi \in \Psi \),

\[
\langle f - g, \psi \rangle_{\psi} = \langle f - g, \hat{\psi} \rangle_{\psi} = 0.
\]

(27)

This means that \( \text{supp} (f - g) = \{0\} \), which implies that \( f - g \) is a finite linear combination of the derivatives of the delta function. Hence, by (15), \( f - g \) is a polynomial.

(vi) For \( y > 0 \), denote \( \mu_\psi(y) \) and \( \mu_\varphi(y) \) as the distributions of \( f \) and \( g \), respectively. Then

\[
\| f \|_{\psi,2} = \int_{\mathbb{R}^d} |f(x)|^2 h_N^2(x) \, dx
\]

\[
\geq \int_{\{x \in \mathbb{R}^d : f(x) \neq 0\}} |f(x)|^2 h_N^2(x) \, dx
\]

\[
\geq \int_{\{x \in \mathbb{R}^d : f(x) \neq 0\}} y^2 h_N^2(x) \, dx = y^2 \cdot \mu_f(y).
\]

(28)

Then \( \mu_f(y) \) is finite since \( f \in L^1(\mathbb{R}^d, h_N^2) \). The same argument gives that \( \mu_\varphi(y) \) is finite as well. We claim that \( f = g \) almost everywhere. In fact, \( f(x) = g(x) + P(x) \) by (v), where \( P(x) \) is a polynomial. Then, for all \( y > 0 \),

\[
\mu_P(2y) = \mu_{f-g}(2y) \leq \mu_f(y) + \mu_\varphi(y) < +\infty.
\]

(29)

Thus \( P(x) \equiv 0 \) a.e. So we have \( f = g \) a.e. consequently.

Lemma 7. Let \( \rho \in L^1(\mathbb{R}^d, h_N^2) \) and \( f \in L^p(\mathbb{R}^d, h_N^2) \), \( 1 < p < \infty \). Define

\[
g_N(x) = \int_{\mathbb{R}^d} \rho(y) \tau_{Ny} f(\frac{x}{N}) h_N^2(y) \, dy.
\]

(30)

Then \( \| g_N \|_{\psi} \to 0 \) as \( N \to \infty \).

Proof. According to the definition,

\[
g_N(x) = N^{-2d} \int_{\mathbb{R}^d} \rho\left(\frac{x}{N}\right) \tau_{Ny} f\left(\frac{x}{N}\right) h_N^2(y) \, dy.
\]

(31)

If \( p = 2 \),

\[
\| g_N \|_{\psi,2} = N^{-4d} \int_{\mathbb{R}^d} \left| \rho\left(\frac{x}{N}\right) \tau_{Ny} f\left(\frac{x}{N}\right) \right|^2 h_N^2(x) \, dx.
\]

(32)

Then \( \| g_N \|_{\psi} \to 0 \) as \( N \to \infty \) by Lebesgue's dominated theorem.

When \( p \neq 2 \), by Proposition 2 (v),

\[
\| g_N \|_{\psi,2} \leq c \| \rho \|_{\psi,1} \| f \|_{\psi,2}.
\]

(33)

It suffices to verify \( \| g_N \|_{\psi} \to 0 \) for \( f \in C_0^\infty(\mathbb{R}^d) \). Let \( r > 1 \) be any number such that \( 2 < p < r \). By H"older's inequality, we have

\[
\| g_N \|_{\psi,p} \leq \| g_N \|_{\psi,2}^{\frac{r}{p}} \| g_N \|_{\psi,r}^{\frac{p}{r}}.
\]

(34)

where \( r = 2(r-p)/p(r-2) \). Then, by (33)

\[
\| g_N \|_{\psi,p} \leq c \| \rho \|_{\psi,1} \| f \|_{\psi,2}^{\frac{r}{p}} \| g_N \|_{\psi,r}^{\frac{p}{r}}.
\]

(35)

tends to 0 as \( N \to \infty \).

Theorem 8. Let \( 0 < \alpha < \lambda \). The operators \( I_\alpha \) are the automorphisms on the space \( \Phi \).

Proof. According to the last equation in the proof for Proposition 4.4 in [7], for \( \varphi \in \Phi \),

\[
I_\alpha \varphi(x) = | \cdot |^{-\alpha} \varphi(\cdot)(x).
\]

(36)

Since \( |y|^{-\alpha} \varphi(y) \) belongs to \( \Psi \) and Dunkl transform maps \( \Psi \) isomorphically onto \( \Phi \), it follows that the map \( \varphi \to | \cdot |^{-\alpha} \varphi(\cdot) \) is continuous from \( \Phi \) to itself. Owing to (36), \( I_\alpha \) is a linear continuous operator from \( \Phi \) to \( \Phi \). Conversely, we claim that \( I_\alpha \) is surjective. In fact, for \( \varphi \in \Phi \), let \( \varphi_0(x) = | \cdot |^{-\alpha} \varphi(\cdot)(x) \). Then \( \varphi_0 \in \Phi \) and \( I_\alpha \varphi_0 = \varphi \). Furthermore, the map \( \varphi \to \varphi_0 \) is continuous in the topology of the space \( \Phi \). This completes the proof.

4. The Inversion of \( I_\alpha \)

Now we can give the main result of this paper, the inversion of the Dunkl Riesz potentials \( I_\alpha \) when the group \( G = \mathbb{Z}^d \). The method follows the idea in [16]. Rubin [17] gave simpler proofs to some elementary approximate and explicit inversion formulæ for the classical Riesz potentials.

Theorem 9. Let \( f \in L^p(\mathbb{R}^d, h_N^2) \cap L^p(\mathbb{R}^d, h_N^2) \), \( p \geq 1, 0 < \alpha < \max\{\lambda_{2s}/2, \lambda_{2s}/p\} \). Suppose that \( w \) is a bounded radial function in \( L^1(\mathbb{R}^d, h_N^2) \) and the integral

\[
d_w(\alpha)(f)(x) = \int_{\mathbb{R}^d} \frac{\tilde{w}(\xi)}{|\xi|^{\lambda_s + \alpha}} h_N^2(\xi) \, d\xi
\]

(37)

is finite. Then

\[
d_w(\alpha)(f)(x) = \int_{\mathbb{R}^d} \frac{W_n(f_N)}{\rho(x)} \, dt
\]

(38)

\[
= \lim_{\lambda \to 0} \int_0^\infty \frac{W_n(f_N)}{\rho(x)} \, dt
\]
where

\[
(W_{\lambda} f)(x) = (f *_{\lambda} u_{\jmath})(x) = t^{-\lambda} \int_{\mathbb{R}^d} f(y) \, r_{\lambda} u \left( x | y \right) h_{\lambda}^2(y) \, dy.
\]  

(39)

**Proof.** Denote

\[
(T_{\varphi} f)(x) = \int_{\mathbb{R}} \left( \frac{W_{\lambda}(\varphi)(x)}{t^{1+\alpha}} \right) \, dt,
\]

\[
\psi_{\varepsilon}(\xi) = \frac{1}{\sigma_{d-1}} \int_{|y|<\varepsilon \xi} |y|^{1+\alpha} h_{\lambda}^2(y) \, dy.
\]

(40)

We claim that the operator \( T_{\varphi} \) is bounded on \( L^p(\mathbb{R}^d, h_{\lambda}^2) \) for any \( 1 \leq p < \infty \). Indeed, according to Proposition 2 (iv), we have

\[
\|W_{\lambda}(\varphi)\|_{L^p(\mathbb{R}^d, h_{\lambda}^2)} = \left( \int_{\mathbb{R}^d} \left| \frac{W_{\lambda}(\varphi)(x)}{t^{1+\alpha}} \right|^p \, |h_{\lambda}^2(x)| \, dx \right)^{1/p} \leq \|\varphi\|_{L^p(\mathbb{R}^d, h_{\lambda}^2)} \cdot \|\psi_{\varepsilon}\|_{L^p(\mathbb{R}^d, h_{\lambda}^2)} \cdot \|\psi_{\varepsilon}\|_{L^p(\mathbb{R}^d, h_{\lambda}^2)}.
\]

(41)

Then the generalized Minkowski’s inequality gives that

\[
\|T_{\varphi} f\|_{L^p(\mathbb{R}^d, h_{\lambda}^2)} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} \left( \frac{W_{\lambda}(\varphi)(x)}{t^{1+\alpha}} \right)^p \, |h_{\lambda}^2(x)| \, dx \right)^{1/p} \, dt \right)^{1/p} \leq \|T_{\varphi} f\|_{L^p(\mathbb{R}^d, h_{\lambda}^2)} \cdot \|\psi_{\varepsilon}\|_{L^p(\mathbb{R}^d, h_{\lambda}^2)} \cdot \|\psi_{\varepsilon}\|_{L^p(\mathbb{R}^d, h_{\lambda}^2)}.
\]

(42)

If we can show that

\[
T_{\varepsilon} I_{\lambda}^\alpha f = (\psi_{\varepsilon} f)^\vee,
\]

(43)

where \( f^\vee \) denotes the inverse Dunkl transform of \( f \), then we have

\[
\|T_{\varepsilon} I_{\lambda}^\alpha f - d_{\alpha} (\alpha) f\|_{L^2(\mathbb{R}^d)} = \|T_{\varepsilon} I_{\lambda}^\alpha f - d_{\alpha} (\alpha) f\|_{L^2(\mathbb{R}^d)} \leq \|\psi_{\varepsilon} f - d_{\alpha} (\alpha) f\|_{L^2(\mathbb{R}^d)}.
\]

(44)

tending to 0 as \( \varepsilon \to 0 \), by Lebesgue’s theorem on dominated convergence.

Recall that \( f \in L^2(\mathbb{R}^d, h_{\lambda}^2) \cap L^p(\mathbb{R}^d, h_{\lambda}^2) \). Let first \( 0 < \alpha < \lambda_{\lambda_{\lambda}/p} \), \( p > 1 \). Then Proposition 4 gives that \( I_{\lambda}^\alpha f \in L^p(\mathbb{R}^d, h_{\lambda}^2) \) when \( 1/p - 1/\alpha = \alpha/\lambda_{\lambda_{\lambda}} \). And therefore, \( T_{\varepsilon} I_{\lambda}^\alpha f \in L^p(\mathbb{R}^d, h_{\lambda}^2) \) by replacing \( \varphi \) with \( I_{\lambda}^\alpha f \) in (42).

On the other hand, \( (\psi_{\varepsilon} f)^\vee \in L^2(\mathbb{R}^d, h_{\lambda}^2) \). According to Proposition 6 (vii), it suffices to prove (43) in the \( \Phi' \)-sense; that is,

\[
\langle T_{\varepsilon} I_{\lambda}^\alpha f, u \rangle_{\Phi} = \langle (\psi_{\varepsilon} f)^\vee, u \rangle_{\Phi} \quad \text{for all } u \in \Phi.
\]

(45)

After changing the order of the integration, the left-hand side of (45) equals

\[
\langle I_{\lambda}^\alpha T_{\varepsilon} f, u \rangle_{\Phi} = \langle T_{\varepsilon} f, I_{\lambda}^\alpha u \rangle_{\Phi} = \langle f, T_{\varepsilon} I_{\lambda}^\alpha u \rangle_{\Phi}.
\]

(46)

Since the Dunkl transform of \( I_{\lambda}^\alpha u \) is \( |\xi|^{-\alpha} \tilde{u}(\xi) \), then

\[
\langle f, T_{\varepsilon} I_{\lambda}^\alpha u \rangle_{\Phi} = \left( \langle \tilde{f}(\xi), |\xi|^{-\alpha} \tilde{u}(\xi) \right)_{\mathbb{R}^d} \int_{\varepsilon}^{\infty} \frac{\tilde{w}(\xi)}{t^{1+\alpha}} \, dt \right). \]

(47)

Since \( w \) is radial, then

\[
\int_{\varepsilon}^{\infty} \frac{\tilde{w}(\xi)}{t^{1+\alpha}} \, dt = \frac{\|\hat{\xi}\|_{\mathbb{R}^d}}{\sigma_{d-1}} \int_{|y|<\varepsilon \xi} \frac{|\xi|^{-\alpha} \tilde{u}(y)}{|y|^{1+\alpha}} \, dy = \|\hat{\xi}\|_{\mathbb{R}^d} \psi_{\varepsilon}(\xi).
\]

(48)

Combining the above, we obtain (45) as desired.

When \( \lambda_{\lambda}/p < \alpha < \lambda_{\lambda}/2 \), \( p > 1 \), the argument is the same as the first case, except with \( q = 2\lambda_{\lambda}/(\lambda_{\lambda} - 2\alpha) \).

For the case \( p = 1 \), \( f \in L^1(\mathbb{R}^d, h_{\lambda}^2) \cap L^2(\mathbb{R}^d, h_{\lambda}^2) \) and \( 0 < \alpha < \lambda_{\lambda} \). By interpolation, \( f \in L^s(\mathbb{R}^d, h_{\lambda}^2) \) for all \( 1 < s < 2 \). Choosing \( s \) in the interval \( (1, \min\{2, \lambda_{\lambda}/\alpha\}) \) and \( q = 2s/(\lambda_{\lambda} - \alpha) \) in the Hardy-Littlewood-Sobolev theorem, we can get the result by repeating the argument when \( p > 1 \). Thus we finish the proof.

\[ \square \]

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**


