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Research Article

Inversion of Riesz Potentials for Dunkl Transform

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The inversion of Riesz potentials for Dunkl transform when $G = \mathbb{Z}_2^d$ is given by using the generalized wavelet transforms. It is also proved that the Riesz potentials I_α^κ are automorphisms on the Semyanistyi-Lizorkin spaces.

1. Introduction

Dunkl transform is a generalization of the Fourier transform associated with a family of weight functions, h_κ, invariant under a finite reflection group. Many papers devote to study the Dunkl transform; see [1–6] and the references therein.

In [7], the Riesz potentials I_α^κ for Dunkl transform were defined by the generalized translation operators, τ_y. The explicit expression and boundedness of τ_y are known only in some special cases such as when $G = \mathbb{Z}_2^d$ and the case when the kernel is a suitable radial function. The boundedness of I_α^κ was given only in the two cases mentioned above. Gorbachev et al. [8] studied the weighted (L^p, L^q)-boundedness properties of Riesz potentials for Dunkl transform represented by the Stein-Weiss inequality. In this paper, we will study the inversion of I_α^κ in the case when $G = \mathbb{Z}_2^d$. The paper is organized as follows. In Section 2, some necessary facts in Dunkl's theory are reviewed. Section 3 is devoted to introduce the Semyanistyi-Lizorkin spaces associated with the reflection-invariant measure $h_\kappa^2(x)dx$. In the final section, the inversion of the Riesz potentials I_α^κ will be given by the generalized wavelet transforms defined by the generalized translation operators.

2. Preliminaries

2.1. Dunkl Operator and Dunkl Transform. Let G be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system R_+, normalized so that $\langle v, v\rangle = 2$ for all $v \in R_+$, where $\langle x, y\rangle$ denotes the usual Euclidean inner product. Let κ be a nonnegative multiplicity function defined on R_+ with the property that $\kappa_u = \kappa_v$ whenever σ_u is conjugate to σ_v in G; then $v \longmapsto \kappa_v$ is a G-invariant function. The weight function is positive homogeneous of degree $\gamma_\kappa := \sum_{v\in R_+} \kappa_v$ defined by

$$h_\kappa(x) = \prod_{v\in R_+} |\langle x, v\rangle|^{\kappa_v}, \quad x \in \mathbb{R}^d. \tag{1}$$

Note that h_κ is invariant under the reflection group G.

Let $\mathcal{D}_j$ be Dunkl's differential-difference operators defined in [1] as

$$\mathcal{D}_j f(x) = \partial_j f(x) + \sum_{v\in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} \langle v, \varepsilon_j\rangle, \quad 1 \le j \le d, \tag{2}$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d$ are the standard unit vectors of $\mathbb{R}^d$ and σ_v denotes the reflection with respect to the hyperplane perpendicular to v, $x\sigma_v := x - 2(\langle x, v\rangle/\|v\|^2)v$, $x \in \mathbb{R}^d$. The operators $\mathcal{D}_j$, $1 \le j \le d$, map $\mathcal{P}_n^d$ to $\mathcal{P}_{n-1}^d$, where $\mathcal{P}_n^d$ denotes the space of homogeneous polynomials of degree n in d variables, and they mutually commute; that is, $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$, $1 \le i, j \le d$. For example, when $d = 1$, the Dunkl operator is

$$\mathcal{D} f(x) = f'(x) + \left(\kappa + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}. \tag{3}$$

The intertwining operator V_κ is a linear operator determined uniquely by

$$V_\kappa \mathscr{P}_n^d \subset \mathscr{P}_n^d, \quad V_\kappa 1 = 1, \qquad \mathscr{D}_i V_\kappa = V_\kappa \partial_i, \quad 1 \le i \le d. \tag{4}$$

Let $E(x, iy) = V_\kappa^{(x)}[e^{i\langle x,y\rangle}]$, $x, y \in \mathbb{R}^d$, where the superscript means that V_κ is applied to the x variable. For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, the Dunkl transform is defined by

$$\widehat{f}(y) = c_h \int_{\mathbb{R}^d} f(x) E(x, -iy) h_\kappa^2(x)\, dx \tag{5}$$

where c_h is the constant defined by $c_h^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(x) e^{-\|x\|^2/2} dx$.

Define $\mathscr{A}_\kappa(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d, h_\kappa^2) : \widehat{f} \in L^1(\mathbb{R}^d, h_\kappa^2)\}$, and, for the sake of simplicity, set $\langle f, g\rangle_\kappa = \int_{\mathbb{R}^d} f(x) g(x) h_\kappa^2(x) dx$ whenever the integral exists and denote $\lambda_\kappa = 2\gamma_\kappa + d$.

The Dunkl transform shares many of the important properties with the usual Fourier transform, part of which are listed as follows ([2, 3]).

Proposition 1. *(i) If $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, then $\widehat{f} \in C_0(\mathbb{R}^d)$ and $\|\widehat{f}\|_\infty \le \|f\|_{\kappa,1}$.*

(ii) If $f \in \mathscr{A}_\kappa(\mathbb{R}^d)$, then $f(x) = \widehat{\widehat{f}}(-x)$.

(iii) The Dunkl transform $f \longrightarrow \widehat{f}$ is a topological automorphism on $\mathscr{S}(\mathbb{R}^d)$.

(iv) For $f \in \mathscr{S}(\mathbb{R}^d)$, then $\widehat{\mathscr{D}_j f}(\xi) = iM_j \widehat{f}(\xi)$ and $\widehat{M_j f}(\xi) = i\mathscr{D}_j \widehat{f}(\xi)$, where $M_j f(x) = x_j f(x)$, $j = 1, 2, \ldots, d$.

(v) For all $f, g \in L^1(\mathbb{R}^d, h_\kappa^2)$, we have $\langle \widehat{f}, g\rangle_\kappa = \langle f, \widehat{g}\rangle_\kappa$.

(vi) There exists a unique extension of the Dunkl transform to $L^2(\mathbb{R}^d, h_\kappa^2)$ with $\|\widehat{f}\|_{\kappa,2} = \|f\|_{\kappa,2}$.

2.2. Generalized Translation Operator and Generalized Convolution. Let $y \in \mathbb{R}^d$ be given. The generalized translation operator $f \longmapsto \tau_y f$ is defined on $L^2(\mathbb{R}^d, h_\kappa^2)$ by $\widehat{\tau_y f}(x) = E(y, -ix)\widehat{f}(x)$, $x \in \mathbb{R}^d$.

For $f, g \in L^2(\mathbb{R}^d, h_\kappa^2)$, the generalized convolution operator is defined by

$$(f *_\kappa g)(x) = \int_{\mathbb{R}^d} f(y) \tau_x \widetilde{g}(y) h_\kappa^2(y)\, dy, \tag{6}$$

where $\widetilde{g}(y) = g(-y)$. The main properties of the generalized translation operator and the generalized convolution are collected below [6, 9, 10].

Proposition 2. *(i) For $f \in \mathscr{A}_\kappa(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d, h_\kappa^2)$ being bounded, then*

$$\int_{\mathbb{R}^d} \tau_y f(\xi) g(\xi) h_\kappa^2(\xi)\, d\xi = \int_{\mathbb{R}^d} f(\xi) \tau_{-y} g(\xi) h_\kappa^2(\xi)\, d\xi. \tag{7}$$

(ii) For $f \in \mathscr{S}(\mathbb{R}^d)$,

$$\widehat{\tau_y f}(\xi) = E(\xi, iy) \widehat{f}(\xi), \qquad E\widehat{(\cdot, iy) f}(\cdot)(x) = \tau_{-y}\widehat{f}(x). \tag{8}$$

When $G = \mathbb{Z}_2^d$,

(iii) for $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \le p \le \infty$, $\|\tau_y f\|_{\kappa,p} \le \|f\|_{\kappa,p}$

(iv) for $f, g \in L^p(\mathbb{R}^d, h_\kappa^2)$, we have $\widehat{f *_\kappa g} = \widehat{f} \cdot \widehat{g}$ and $f *_\kappa g = g *_\kappa f$

(v) let $p, q, r \ge 1$ and $1/r = 1/p + 1/q - 1$. For $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $g \in L^q(\mathbb{R}^d, h_\kappa^2)$, $\|f *_\kappa g\|_{\kappa,r} \le c\|f\|_{\kappa,p}\|g\|_{\kappa,q}$.

2.3. Dunkl Transform of Distributions. References [5, 11, 12] study the actions of the Dunkl operators and Dunkl transform on the space $\mathscr{S}'(\mathbb{R}^d)$. Reference [4] gives the definition of the Dunkl transform for the local integrable functions under the measure $h_\kappa^2(x)dx$.

Let $f \in L^1_{loc}(\mathbb{R}^d, h_\kappa^2)$; the generalized function associated with f is defined by

$$\langle f, \varphi\rangle_\kappa = \int_{\mathbb{R}^d} f(x) \varphi(x) h_\kappa^2(x)\, dx, \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{9}$$

The Dunkl transform of $f \in \mathscr{S}'(\mathbb{R}^d)$ is defined as

$$\langle \widehat{f}, \varphi\rangle_\kappa = \langle f, \widehat{\varphi}\rangle_\kappa, \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{10}$$

Then, for $f \in L^1_{loc}(\mathbb{R}^d, h_\kappa)$, the Dunkl transform of f is

$$\langle \widehat{f}, \varphi\rangle_\kappa = \int_{\mathbb{R}^d} f(x) \widehat{\varphi}(x) h_\kappa^2(x)\, dx, \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{11}$$

For $f \in \mathscr{S}'(\mathbb{R}^d)$, the dilation transform εf is defined as

$$\langle \varepsilon f, \varphi\rangle_\kappa = \langle f, \varepsilon^{-\lambda_\kappa} \varphi(\varepsilon^{-1}\cdot)\rangle_\kappa, \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{12}$$

Let δ be the Dirac distribution associated with the measure $h_\kappa^2(x)dx$; that is, $\langle \delta, \varphi\rangle_\kappa = \varphi(0)$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$.

Lemma 3. *Let $\varphi \in L^1(\mathbb{R}^d, h_\kappa^2)$ satisfy $\int_{\mathbb{R}^d} \varphi(x) h_\kappa^2(x) dx = 1$. For any $\varepsilon > 0$, define $\phi_\varepsilon(x) = \varepsilon^{-\lambda_\kappa}\phi(x/\varepsilon)$; then ϕ_ε is a δ-sequence; that is,*

$$\lim_{\varepsilon \longrightarrow 0^+} \langle \phi_\varepsilon, \varphi\rangle_\kappa = \varphi(0) = \langle \delta, \varphi\rangle_\kappa, \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{13}$$

Then by (9) and (10), we obtain the Dunkl transform of δ as $\widehat{\delta}(\xi) = 1$.

Define the action of the Dunkl operators $\mathscr{D}_j$, $j = 1, 2, \ldots, d$ on the space $\mathscr{S}'(\mathbb{R}^d)$ as

$$\langle \mathscr{D}_j f, \varphi\rangle_\kappa = -\langle f, \mathscr{D}_j \varphi\rangle_\kappa, \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d). \tag{14}$$

Denote $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{Z}_+^d$ and $\mathscr{D}^\beta = \mathscr{D}_1^{\beta_1} \circ \mathscr{D}_2^{\beta_2} \circ \cdots \circ \mathscr{D}_d^{\beta_d}$. Combining (9), (14), and Proposition 1 (iv), we have

$$\widehat{\mathscr{D}^\beta \delta}(\xi) = (i\xi)^\beta = i^{|\beta|} \xi_1^{\beta_1} \xi_2^{\beta_2} \cdots \xi_d^{\beta_d}, \qquad \widehat{x^\beta} = i^{|\beta|} \mathscr{D}^\beta \delta. \tag{15}$$

2.4. Dunkl Riesz Potentials. For simplicity, we call the Riesz potential for Dunkl transform as the Dunkl Riesz potential $I_\alpha^\kappa f$, which is defined on $\mathcal{S}(\mathbb{R}^d)$ in [7] as

$$I_\alpha^\kappa f(x) = \left(d_\alpha^\kappa\right)^{-1} \int_{\mathbb{R}^d} \tau_y f(x) \frac{1}{|y|^{\lambda_\kappa - \alpha}} h_\kappa^2(y)\, dy, \tag{16}$$

where $0 < \alpha < \lambda_\kappa$ and $d_\alpha^\kappa = 2^{-\lambda_\kappa/2+\alpha}\Gamma(\alpha/2)\Gamma((\lambda_\kappa - \alpha)/2)$. The Dunkl transform and the Hardy-Littlewood-Sobolev theorem of I_α^κ are given in [7].

Proposition 4. *Let $0 < \alpha < \lambda_\kappa$. The identity*

$$\widehat{I_\alpha^\kappa f}(x) = |x|^{-\alpha} \widehat{f}(x) \tag{17}$$

holds in the sense that

$$\begin{aligned} &\int_{\mathbb{R}^d} I_\alpha^\kappa f(x)\, g(x)\, h_\kappa^2(x)\, dx \\ &\quad = \int_{\mathbb{R}^d} \widehat{f}(x)\, |x|^{-\alpha}\, \widehat{g}(x)\, h_\kappa^2(x)\, dx \end{aligned} \tag{18}$$

whenever $f, g \in \mathcal{S}(\mathbb{R}^d)$.

Proposition 5. *Let $G = \mathbb{Z}_2^d$ and $0 < \alpha < \lambda_\kappa$. Let p and q satisfy*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\lambda_\kappa}, \quad 1 \le p < q < \infty. \tag{19}$$

(1) *For $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $p > 1$, $\|I_\alpha^\kappa f\|_{\kappa,q} \le c\|f\|_{\kappa,p}$.*

(2) *For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, the mapping $f \longmapsto I_\alpha^\kappa f$ is of weak type $(1, q)$.*

3. Semyanistyi-Lizorkin Spaces

The following spaces were introduced by Semyanistyi and generalized by Lizorkin and Samko; see [13–15].

Let $\Psi = \Psi(\mathbb{R}^d)$ be the class of functions ψ in $\mathcal{S}(\mathbb{R}^d)$ vanishing at the origin 0 with all their derivatives; that is,

$$\begin{aligned} \Psi = \Big\{ \psi \in \mathcal{S}\left(\mathbb{R}^d\right) : \partial^\beta \psi(0) = 0 \text{ for all } |\beta| \\ = 0, 1, 2, \ldots \Big\}. \end{aligned} \tag{20}$$

The space Ψ is a closed linear subspace of $\mathcal{S}(\mathbb{R}^d)$. It can be regarded as a linear topological space with the induced topology generated by the sequence of norms

$$\begin{aligned} \|\psi\|_m = \max_x (1 + |x|)^m \sum_{|j| \le m} \left| \left(\partial^\beta \psi\right)(x) \right|, \\ m = 0, 1, 2, \ldots. \end{aligned} \tag{21}$$

We claim that $\mathcal{D}^\beta \psi(0) = 0$ for $\psi \in \Psi$, since $\partial_j \psi(0) = 0$ implies $\mathcal{D}_j \psi(0) = 0$ for $j = 1, 2, \ldots, d$.

Let $\Phi = \Phi(\mathbb{R}^d)$ be the image of Ψ under the Dunkl transform; that is, $\Phi = \{\psi \in \Psi : \widehat{\psi}\}$. Since the Dunkl transform is an automorphism of $\mathcal{S}(\mathbb{R}^d)$, the space is a closed linear subspace of $\mathcal{S}(\mathbb{R}^d)$. We equip Φ with the induced topology of the ambient space $\mathcal{S}(\mathbb{R}^d)$. Then Φ becomes a linear topology space which is isomorphic to Ψ under the action of the Dunkl transform. According to the definition of Φ, we conclude that the space Φ consists of all functions φ which are orthogonal to all polynomials as for the measure $h_\kappa^2(x)dx$; that is,

$$\begin{aligned} \Phi = \Big\{ \varphi \in \mathcal{S}\left(\mathbb{R}^d\right) : \int_{\mathbb{R}^d} x^k \varphi(x)\, h_\kappa^2(x)\, dx \\ = 0, \text{ for all } k \in \mathbb{Z}_+^d \Big\}. \end{aligned} \tag{22}$$

In fact, if $\varphi \in \Phi$, then $\widehat{\psi} \in \Phi$, and for any multi-index k, by Proposition 1 (iv), we have

$$\begin{aligned} &\int_{\mathbb{R}^d} x^k \varphi(x)\, h_\kappa^2(x)\, dx \\ &\quad = \int_{\mathbb{R}^d} E(x, -i\xi)\, x^k \varphi(x)\, h_\kappa^2(x)\, dx|_{\xi=0} \\ &\quad = \left(\widehat{M^k \varphi}\right)(0) = i^k \mathcal{D}^k \widehat{\varphi}(0) = 0. \end{aligned} \tag{23}$$

Denote that Φ' and Ψ' are the spaces of all semilinear functionals on Φ and Ψ, respectively. Some properties of Φ and Ψ are given in the following proposition.

Proposition 6. *(i) The spaces Φ and Ψ are not empty.*

(ii) The space Φ does not contain compactly supported infinitely differentiable functions, rather than 0.

(iii) The space Φ is invariant under the generalized translations.

(iv) The space Φ is dense in $L^p(\mathbb{R}^d, h_\kappa^2)$, $1 < p < \infty$.

(v) $\mathcal{S}'(\mathbb{R}^d)$-distributions that coincide in the Φ'-sense differ from each other by a polynomial.

(vi) Let $f \in L^r(\mathbb{R}^d, h_\kappa^2)$ and $g \in L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \le r, p < \infty$. If $f = g$ in the Φ'-sense, then $f \equiv g$ almost everywhere.

The proof of the this proposition is similar to the ones in [13–15] except with the reflection-invariant measure $h_\kappa^2(x)dx$. Now we sketch it below.

Proof. (i) Choose $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfying that $\operatorname{supp}\psi = \{x \in \mathbb{R}^d : |x| > 1\}$. Then $\psi \in \Psi$.

(ii) Suppose $\varphi \in C_c^\infty(\mathbb{R}^d)$; then we have

$$\widehat{\varphi}(\xi) = \sum_{|\beta|=0}^{\infty} \frac{\xi^\beta}{\beta!} \left(\partial^\beta \widehat{\varphi}\right)(0). \tag{24}$$

If $\varphi \in \Phi$, $\widehat{\varphi} \in \Psi$, and $\partial^\beta \widehat{\varphi}(0) = 0$, so $\widehat{\varphi}(\xi) = 0$ for all $\xi \in \mathbb{R}^d$. Then $\varphi(x) \equiv 0$.

(iii) This conclusion can be obtained by Proposition 2 (iii).

(iv) It suffices to approximate a function $f \in \mathcal{S}(\mathbb{R}^d)$ by functions $f_N \in \Phi$ in the $L^p(\mathbb{R}^d, h_\kappa^2)$ norm. We introduce the functions

$$\psi_N(x) = \mu(N|x|) f(x) \tag{25}$$

where $\mu \in C^\infty([0,\infty])$ such that $\mu(t) = 1$ for $t \geq 2$, $\mu(t) = 0$ for $0 \leq t \leq 1$ and $0 \leq \mu(t) \leq 1$. We define $f_N(x) = \widehat{\psi}_N(x)$. Since $\psi_N \in \mathcal{S}(\mathbb{R}^d)$ and $\psi_N(x) = 0$ as $|x| \leq 1/N$, we have $\psi \in \Psi$ and then $f_N \in \Phi$. In order to show that $f_N(x)$ approximate the function $f(x)$, we represent them as

$$f_N(x) = f(x) - \int_{\mathbb{R}^d} k(y) \tau_{Ny} f(x) h_\kappa^2(y)\, dy \tag{26}$$

where $k(y)$ is the inverse Dunkl transform of the function $1 - \mu(|x|)$. Then by Lemma 7, $\|f - f_N\|_{\kappa,p} \longrightarrow 0$ as $N \longrightarrow \infty$.

(v) Suppose $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $f = g$ in the sense of Φ'; that is, for all $\varphi \in \Phi$, $\langle f - g, \varphi\rangle_k = 0$. Then, for all $\psi \in \Psi$,

$$\left\langle \widehat{f-g}, \psi \right\rangle_\kappa = \langle f - g, \widehat{\psi}\rangle_\kappa = 0. \tag{27}$$

This means that $\operatorname{supp}(\widehat{f-g}) = \{0\}$, which implies that $\widehat{f-g}$ is a finite linear combination of the derivatives of the delta function. Hence, by (15), $f - g$ is a polynomial.

(vi) For $\gamma > 0$, denote $\mu_f(\gamma)$ and $\mu_g(\gamma)$ as the distributions of f and g, respectively. Then

$$\begin{aligned} \|f\|_{\kappa,r}^r &= \int_{\mathbb{R}^d} |f(x)|^r h_\kappa^2(x)\, dx \\ &\geq \int_{\{x\in\mathbb{R}^d : |f(x)|\geq\gamma\}} |f(x)|^r h_\kappa^2(x)\, dx \\ &\geq \int_{\{x\in\mathbb{R}^d : |f(x)|\geq\gamma\}} \gamma^r h_\kappa^2(x)\, dx = \gamma^r \cdot \mu_f(\gamma). \end{aligned} \tag{28}$$

Then $\mu_f(\gamma)$ is finite since $f \in L^r(\mathbb{R}^d, h_\kappa^2)$. The same argument gives that $\mu_g(\gamma)$ is finite as well. We claim that $f = g$ almost everywhere. In fact, $f(x) = g(x) + P(x)$ by (v), where $P(x)$ is a polynomial. Then, for all $\gamma > 0$,

$$\mu_P(2\gamma) = \mu_{f-g}(2\gamma) \leq \mu_f(\gamma) + \mu_g(\gamma) < +\infty. \tag{29}$$

Thus $P(x) \equiv 0$ a.e. So we have $f = g$ a.e. consequently. □

Lemma 7. *Let $\rho \in L^1(\mathbb{R}^d, h_\kappa^2)$ and $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $1 < p < \infty$. Define*

$$g_N(x) = \int_{\mathbb{R}^d} \rho(y) \tau_{Ny} \widetilde{f}(x) h_\kappa^2(y)\, dy. \tag{30}$$

Then, $\|g_N\|_{\kappa,p} \longrightarrow 0$ as $N \longrightarrow \infty$.

Proof. According to the definition,

$$g_N(x) = N^{-2\lambda_\kappa} \int_{\mathbb{R}^d} \rho\left(\frac{y}{N}\right) \tau_y \widetilde{f}(x) h_\kappa^2(y)\, dy. \tag{31}$$

If $p = 2$,

$$\|g_N\|_{\kappa,2}^2 = N^{-4\lambda_\kappa} \int_{\mathbb{R}^d} \left|\widehat{\rho}\left(\frac{x}{N}\right) \widehat{f}(x)\right|^2 h_\kappa^2(x)\, dx. \tag{32}$$

Then $\|g_N\|_{\kappa,2} \longrightarrow 0$ as $N \longrightarrow \infty$ by Lebesgue's dominated theorem.

When $p \neq 2$, by Proposition 2 (v),

$$\|g_N\|_{\kappa,p} \leq c \|\rho\|_{\kappa,1} \|f\|_{\kappa,p}. \tag{33}$$

It suffices to verify $\|g_N\|_{\kappa,p} \longrightarrow 0$ for $f \in C_0^\infty(\mathbb{R}^d)$. Let $r > 1$ be any number such that $2 < p < r$. By Hölder's inequality, we have

$$\|g_N\|_{\kappa,p} \leq \|g_N\|_{\kappa,r}^{1-\nu} \|g_N\|_{\kappa,2}^{\nu}, \tag{34}$$

where $\nu = 2(r-p)/p(r-2)$. Then, by (33)

$$\|g_N\|_{\kappa,p} \leq c \left(\|\rho\|_{\kappa,1} \|f\|_{\kappa,r}\right)^{1-\nu} \|g_N\|_{\kappa,2}^{\nu} \tag{35}$$

tends to 0 as $N \longrightarrow \infty$. □

Theorem 8. *Let $0 < \alpha < \lambda_\kappa$. The operators I_α^κ are the automorphisms on the space Φ.*

Proof. According to the last equation in the proof for Proposition 4.1 in [7], for $\varphi \in \Phi$,

$$I_\alpha^\kappa \varphi(x) = \widehat{|\cdot|^{-\alpha} \widehat{\varphi}(\cdot)}(x). \tag{36}$$

Since $|y|^{-\alpha}\widehat{\varphi}(y)$ belongs to Ψ and Dunkl transform maps Ψ isomorphically onto Φ, it follows that the map $\varphi \longrightarrow \widehat{|\cdot|^{-\alpha}\widehat{\varphi}(\cdot)}$ is continuous from Φ to itself. Owing to (36), I_α^κ is a linear continuous operator from Φ to Φ. Conversely, we claim that I_α^κ is surjective. In fact, for $\varphi \in \Phi$, let $\varphi_0(x) = \widehat{|\cdot|^{-\alpha}\widehat{\varphi}(\cdot)}(x)$. Then $\varphi_0 \in \Phi$ and $I_\alpha^\kappa \varphi_0 = \varphi$. Furthermore, the map $\varphi \longrightarrow \varphi_0$ is continuous in the topology of the space Φ. This completes the proof. □

4. The Inversion of I_α^κ

Now we can give the main result of this paper, the inversion of the Dunkl Riesz potentials I_α^κ when the group $G = \mathbb{Z}_2^d$. The method follows the idea in [16]. Rubin [17] gave simpler proofs to some elementary approximate and explicit inversion formulae for the classical Riesz potentials.

Theorem 9. *Let $f \in L^2(\mathbb{R}^d, h_\kappa^2) \cap L^p(\mathbb{R}^d, h_\kappa^2)$, $p \geq 1$, $0 < \alpha < \max\{\lambda_\kappa/2, \lambda_\kappa/p\}$. Suppose that w is a bounded radial function in $L^1(\mathbb{R}^d, h_\kappa^2)$ and the integral*

$$\begin{aligned} d_w(\alpha) &= \frac{1}{\sigma_{d-1}} \int_{\mathbb{R}^d} \frac{\widehat{w}(\xi)}{|\xi|^{\lambda_\kappa+\alpha}} h_\kappa^2(\xi)\, d\xi \\ &= \lim_{\varepsilon\longrightarrow 0} \frac{1}{\sigma_{d-1}} \int_{|\xi|>\varepsilon} \frac{\widehat{w}(\xi)}{|\xi|^{\lambda_\kappa+\alpha}} h_\kappa^2(\xi)\, d\xi \end{aligned} \tag{37}$$

is finite. Then

$$\begin{aligned} d_w(\alpha) f(x) &= \int_0^\infty \frac{W_t(I_\alpha^\kappa f)(x)}{t^{1+\alpha}} dt \\ &= \lim_{\varepsilon\longrightarrow 0} \int_\varepsilon^\infty \frac{W_t(I_\alpha^\kappa f)(x)}{t^{1+\alpha}} dt \end{aligned} \tag{38}$$

where

$$(W_t f)(x) = (f *_\kappa w_t)(x) = t^{-\lambda_\kappa} \int_{\mathbb{R}^d} f(y) \tau_y w\left(\frac{\cdot}{t}\right)(x) h_\kappa^2(y)\, dy. \tag{39}$$

Proof. Denote

$$(T_\varepsilon \varphi)(x) = \int_\varepsilon^\infty \frac{(W_t\varphi)(x)}{t^{1+\alpha}} dt, \qquad \psi_\varepsilon(\xi) = \frac{1}{\sigma_{d-1}} \int_{|y|>\varepsilon|\xi|} \frac{\widehat{w}(y)}{|y|^{\lambda_\kappa+\alpha}} h_\kappa^2(y)\, dy. \tag{40}$$

We claim that the operator T_ε is bounded on $L^p(\mathbb{R}^d, h_\kappa^2)$ for any $1 \le p < \infty$. Indeed, according to Proposition 2 (iv), we have

$$\|W_t\varphi\|_{\kappa,p} = \|\varphi *_\kappa w_t\|_{\kappa,p} \le \|\varphi\|_{\kappa,p} \|w\|_{\kappa,1}. \tag{41}$$

Then the generalized Minkowski's inequality gives that

$$\|T_\varepsilon\varphi\|_{\kappa,p} = \left(\int_{\mathbb{R}^d} \left|\int_\varepsilon^\infty \frac{(W_t\varphi)(x)}{t^{1+\alpha}} dt\right|^p h_\kappa^2(x)\, dx\right)^{1/p} = \int_\varepsilon^\infty \frac{\|W_t\varphi\|_{\kappa,p}}{t^{1+\alpha}} dt \le \|\varphi\|_{\kappa,p} \|w\|_{\kappa,1} \int_\varepsilon^\infty \frac{dt}{t^{1+\alpha}}. \tag{42}$$

If we can show that

$$T_\varepsilon I_\alpha^\kappa f = \left(\psi_\varepsilon \widehat{f}\right)^\vee, \tag{43}$$

where $f^\vee$ denotes the inverse Dunkl transform of f, then we have

$$\|T_\varepsilon I_\alpha^\kappa f - d_w(\alpha) f\|_{\kappa,2} = \left\|\widehat{T_\varepsilon I_\alpha^\kappa f} - d_w(\alpha)\widehat{f}\right\|_{\kappa,2} = \left\|\psi_\varepsilon \widehat{f} - d_w(\alpha)\widehat{f}\right\|_{\kappa,2} \tag{44}$$

tending to 0 as $\varepsilon \longrightarrow 0$, by Lebesgue's theorem on dominated convergence.

Recall that $f \in L^2(\mathbb{R}^d, h_\kappa^2) \cap L^p(\mathbb{R}^d, h_\kappa^2)$. Let first $0 < \alpha < \lambda_\kappa/p$, $p > 1$. Then Proposition 4 gives that $I_\alpha^\kappa f \in L^q(\mathbb{R}^d, h_\kappa^2)$ when $1/p - 1/q = \alpha/\lambda_\kappa$. And therefore, $T_\varepsilon I_\alpha^\kappa f \in L^q(\mathbb{R}^d, h_\kappa^2)$ by replacing φ with $I_\alpha^\kappa f$ in (42).

On the other hand, $(\psi_\varepsilon \widehat{f})^\vee \in L^2(\mathbb{R}^d, h_\kappa^2)$. According to Proposition 6 (vii), it suffices to prove (43) in the Φ'-sense; that is,

$$\langle T_\varepsilon I_\alpha^\kappa f, u\rangle_\kappa = \left\langle \left(\psi_\varepsilon \widehat{f}\right)^\vee, u\right\rangle_\kappa \quad \text{for all } u \in \Phi. \tag{45}$$

After changing the order of the integration, the left-hand side of (45) equals

$$\langle I_\alpha^\kappa T_\varepsilon f, u\rangle_\kappa = \langle T_\varepsilon f, I_\alpha^\kappa u\rangle_\kappa = \langle f, T_\varepsilon I_\alpha^\kappa u\rangle_\kappa. \tag{46}$$

Since the Dunkl transform of $I_\alpha^\kappa u$ is $|\xi|^{-\alpha}\widehat{u}(\xi)$, then

$$\langle f, T_\varepsilon I_\alpha^\kappa u\rangle_\kappa = \left\langle \widehat{f}(\xi), |\xi|^{-\alpha}\widehat{u}(\xi) \int_\varepsilon^\infty \frac{\widehat{w}(t\xi)}{t^{1+\alpha}} dt\right\rangle_\kappa. \tag{47}$$

Since w is radial, then

$$\int_\varepsilon^\infty \frac{\widehat{w}(t\xi)}{t^{1+\alpha}} dt = \frac{|\xi|^\alpha}{\sigma_{d-1}} \int_{|y|>\varepsilon|\xi|} \frac{\widehat{w}(y)}{|y|^{\lambda_\kappa+\alpha}} h_\kappa^2(y)\, dy = |\xi|^\alpha \psi_\varepsilon(\xi). \tag{48}$$

Combining the above, we obtain (45) as desired.

When $\lambda_\kappa/p < \alpha < \lambda_\kappa/2$, $p > 1$, the argument is the same as the first case, except with $q = 2\lambda_\kappa/(\lambda_\kappa - 2\alpha)$.

For the case $p = 1$, $f \in L^1(\mathbb{R}^d, h_\kappa^2) \cap L^2(\mathbb{R}^d, h_\kappa^2)$ and $0 < \alpha < \lambda_\kappa$. By interpolation, $f \in L^s(\mathbb{R}^d, h_\kappa^2)$ for all $1 \le s \le 2$. Choosing s in the interval $(1, \min\{2, \lambda_\kappa/\alpha\})$ and $q = \lambda_\kappa s/(\lambda_\kappa - \alpha s)$ in the Hardy-Littlewood-Sobolev theorem, we can get the result by repeating the argument when $p > 1$. Thus we finish the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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