Research Article

New Characterizations of k -Uniformly Extremely Convex Banach Spaces

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Two characterizations of k -uniformly extremely convex spaces are given in this paper. As special cases for $k = 1$, two necessary and sufficient conditions of uniform extreme convexity are presented and a characteristic inequality of uniformly extremely convex spaces is given.

1. Introduction

In the geometric theory of Banach spaces, the concept of uniform convexity plays a very significant role and is frequently used in functional analysis. In 2012, this concept has been generalized by Wulede and Ha [1]. They defined the so-called uniform extreme convexity and showed many interesting connections between this concept and several others (see [1–8]). Recently, Wulede et al. [9], generalizing the uniform extreme convexity, introduced the so-called k -uniform extreme convexity. Every k -uniformly extremely convex space is $(k + 1)$ -uniformly extremely convex, but the converse implication is not true (see [9]). It is shown that 1-uniform extreme convexity coincides with uniform extreme convexity (see [9]).

In 1960, Singer [10] introduced k -strictly convex spaces. It is well known that every k -strictly convex space is $(k + 1)$ -strictly convex space. Also, it was shown that 1-strict convexity coincides with strict convexity.

Throughout the sequel, the symbol X denotes a real Banach space and X^* denotes its dual space. $B(X) = \{x : x \in X, \|x\| \leq 1\}$, $S(X) = \{x : x \in X, \|x\| = 1\}$ denote the closed unit ball and the unit sphere in X , respectively. For $x_1, x_2, \dots, x_{k+1} \in S(X)$, the k -dimensional volume enclosed by $x_1, x_2, \dots, x_{k+1}$ is given by

$$V(x_1, x_2, \dots, x_{k+1})$$

$$= \sup \left\{ \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_1) & f_1(x_2) & \cdots & f_1(x_{k+1}) \\ \cdots & \cdots & \cdots & \cdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_{k+1}) \end{vmatrix} : f_i \in S(X^*), i = 1, \dots, k \right\}. \quad (1)$$

In 1972, Daneš [11] proved the so-called “Drop Theorem”. In 1987, modifying the assumption of Daneš drop theorem, Rolewicz [6] began the study of drop property for the closed unit ball. He defined the norm $\|\cdot\|$ to have the drop property, if for every closed set $C \subset X$ disjoint from closed unit ball $B(X)$ there exists a $x \in C$ such that $D(x, B(X)) \cap C = \{x\}$, where the set $D(x, B(X))$, the convex hull of x and $B(X)$, is called the drop generated by $x \notin B(X)$.

Definition 1 (see [1]). A Banach space X is said to be uniformly extremely convex space provided that for any sequences $\{x_n\}, \{y_n\}$ of norm-one elements of X , if there is

a norm-one functional $f \in X^*$ such that $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(y_n) = 1$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Definition 2 (see [9]). A Banach space X is said to be k -uniformly extremely convex space provided that for any sequences $\{x_n^{(i)}\}_{i=1}^{k+1}$ of norm-one elements of X , if there is a norm-one functional $f \in X^*$ such that $\lim_{n \rightarrow \infty} f(x_n^{(i)}) = 1$ for $i = 1, \dots, k+1$, then $\lim_{n \rightarrow \infty} V(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k+1)}) = 0$.

Lemma 3 (see [9]). A Banach space X is k -uniformly extremely convex if and only if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for each norm-one functional $f \in X^*$ and for all norm-one elements $x_1, \dots, x_{k+1}$ of X with $f(x_1 + x_2 + \dots + x_{k+1}) > (k+1) - \delta$, one has $V(x_1, x_2, \dots, x_{k+1}) < \epsilon$.

Lemma 4 (Singer [10], Wulede [9]). Every strictly convex (uniformly extremely convex, respectively) space is k -strictly convex (k -uniformly extremely convex, respectively).

Lemma 5 (Wulede [1, 9]). Let X be a Banach space; let k be an integer with $k \geq 1$. Then, X is k -uniformly extremely convex if and only if X is k -strictly convex and has the drop property.

Lemma 6 (see [12]). Let $1 < p < +\infty$; then we have

$$\left(\frac{1 + t_1 + \dots + t_k}{k+1} \right)^p \leq \frac{1 + t_1^p + \dots + t_k^p}{k+1}, \quad (2)$$

where $t_1, t_2, \dots, t_k \geq 0$, and the sign of equality holds if and only if $t_1 = t_2 = \dots = t_k = 1$.

2. Characterizations of k -Uniformly Extremely Convex Spaces

Theorem 7. A Banach space X is k -uniformly extremely convex if and only if for any $k+1$ sequences $\{x_n^{(i)}\}_{i=1}^{k+1} \subset X$, if $\lim_{n \rightarrow \infty} \|x_n^{(i)}\| = \tau$, $i = 1, \dots, k+1$ and for some norm-one functional $f \in X^*$, $\lim_{n \rightarrow \infty} f(x_n^{(1)} + x_n^{(2)} + \dots + x_n^{(k+1)}) = (k+1)\tau$, then $\lim_{n \rightarrow \infty} V(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k+1)}) = 0$.

Proof.

Necessity. Without loss of generality, we may assume that $\tau \neq 0$ and suppose that for any $k+1$ sequences $\{x_n^{(i)}\}_{i=1}^{k+1} \subset X$ with $\lim_{n \rightarrow \infty} \|x_n^{(i)}\| = \tau$, $i = 1, \dots, k+1$, and for some norm-one functional $f \in X^*$, $\lim_{n \rightarrow \infty} f(x_n^{(1)} + x_n^{(2)} + \dots + x_n^{(k+1)}) = (k+1)\tau$.

Let $y_n^{(i)} = x_n^{(i)} / \|x_n^{(i)}\|$, $i = 1, \dots, k+1$, and then we have $\|y_n^{(i)}\| = 1$, $i = 1, \dots, k+1$, and

$$\begin{aligned} & \lim_{n \rightarrow \infty} f(y_n^{(1)} + y_n^{(2)} + \dots + y_n^{(k+1)}) \\ &= \lim_{n \rightarrow \infty} f\left(\frac{x_n^{(1)}}{\|x_n^{(1)}\|} + \frac{x_n^{(2)}}{\|x_n^{(2)}\|} + \dots + \frac{x_n^{(k+1)}}{\|x_n^{(k+1)}\|}\right) \\ &= k+1. \end{aligned} \quad (3)$$

It follows that $\lim_{n \rightarrow \infty} f(y_n^{(i)}) = 1$, $i = 1, \dots, k+1$. By the assumption that X is a k -uniformly extremely convex space, we have $\lim_{n \rightarrow \infty} V(y_n^{(1)}, y_n^{(2)}, \dots, y_n^{(k+1)}) = 0$. Hence,

$$\begin{aligned} & \lim_{n \rightarrow \infty} V(y_n^{(1)}, y_n^{(2)}, \dots, y_n^{(k+1)}) \\ &= \lim_{n \rightarrow \infty} V\left(\frac{x_n^{(1)}}{\|x_n^{(1)}\|}, \frac{x_n^{(2)}}{\|x_n^{(2)}\|}, \dots, \frac{x_n^{(k+1)}}{\|x_n^{(k+1)}\|}\right) \\ &= \frac{1}{\tau^k} \lim_{n \rightarrow \infty} V(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k+1)}) = 0. \end{aligned} \quad (4)$$

It follows that $\lim_{n \rightarrow \infty} V(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k+1)}) = 0$.

Sufficiency. Suppose that $\{x_n^{(i)}\}_{i=1}^{k+1}$ are sequences of norm-one elements of X such that for some norm-one functional $f \in X^*$ the equality $\lim_{n \rightarrow \infty} f(x_n^{(i)}) = 1$ holds for $i = 1, \dots, k+1$. It follows that $\lim_{n \rightarrow \infty} f(x_n^{(1)} + x_n^{(2)} + \dots + x_n^{(k+1)}) = k+1$. From the conditions given here, we have

$$\lim_{n \rightarrow \infty} V(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k+1)}) = 0. \quad (5)$$

This shows that X is k -uniformly extremely convex space. $\square$

Theorem 8. Let $1 < p < +\infty$, X be a Banach space, and M be an arbitrary bounded subset of X . Then X is k -uniformly extremely convex if and only if for any $\epsilon > 0$ there exists $0 < \delta(\epsilon, p) < 1$ such that the inequality

$$\begin{aligned} & \left[f\left(\frac{x_1 + x_2 + \dots + x_{k+1}}{k+1}\right) \right]^p \\ & \leq (1 - \delta(\epsilon, p)) \frac{\|x_1\|^p + \|x_2\|^p + \dots + \|x_{k+1}\|^p}{k+1} \end{aligned} \quad (6)$$

holds for any norm-one functional $f \in X^*$ and all elements $x_1, x_2, \dots, x_{k+1} \in M$ with

$$V(x_1, x_2, \dots, x_{k+1}) \geq \epsilon. \quad (7)$$

Proof.

Sufficiency. Suppose that for every $\epsilon > 0$ there is a $0 < \delta_1 = \delta_1(\epsilon, p) < 1$ such that the inequality

$$\left[f\left(\frac{x_1 + x_2 + \dots + x_{k+1}}{k+1}\right) \right]^p \leq 1 - \delta_1 \quad (8)$$

holds for any norm-one functional $f \in X^*$ and all norm-one elements $x_1, x_2, \dots, x_{k+1}$ of X with $V(x_1, x_2, \dots, x_{k+1}) \geq \epsilon$.

Let us denote $\delta_2 = 1 - (1 - \delta_1)^{1/p} > 0$ and take $\delta(\epsilon) = (k+1)\delta_2$. Then we have

$$f(x_1 + x_2 + \dots + x_{k+1}) \leq (k+1) - \delta. \quad (9)$$

By Lemma 3, we know that X is a k -uniformly extremely convex space.

Necessity. Suppose the inequality (6) is not true. Then there exist $r > 0$ and $\epsilon_0 > 0$ such that for each $n \in \mathbb{N}$ there exist $\{x_n^{(i)}\}_{i=1}^{k+1} \subset X$ satisfying $\|x_n^{(1)}\| \leq r$ for $i = 1, \dots, k+1$, $V(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k+1)}) \geq \epsilon_0$ and

$$\left[f \left(\frac{x_n^{(1)} + x_n^{(2)} + \dots + x_n^{(k+1)}}{k+1} \right) \right]^p > \left(1 - \frac{1}{n} \right) \frac{\|x_n^{(1)}\|^p + \|x_n^{(2)}\|^p + \dots + \|x_n^{(k+1)}\|^p}{k+1}. \quad (10)$$

Put $r_n = \max_{1 \leq i \leq k+1} \|x_n^{(i)}\|$ and take $y_n^{(1)} = x_n^{(1)}/r_n$, $y_n^{(2)} = x_n^{(2)}/r_n, \dots, y_n^{(k+1)} = x_n^{(k+1)}/r_n$, $\epsilon_1 = \epsilon_0/r^k$. Rearranging the vectors, we can assume that $\|y_n^{(1)}\| = 1$. We have $r_n \leq r$ for every n and

$$\begin{aligned} V(y_n^{(1)}, y_n^{(2)}, \dots, y_n^{(k+1)}) &= \frac{1}{r_n^k} V(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k+1)}) \\ &\geq \frac{\epsilon_0}{r^k} = \epsilon_1. \end{aligned} \quad (11)$$

Combining inequality (10) with Lemma 6 we derive that

$$\begin{aligned} 1 - \frac{1}{n} &< \frac{\left[f \left((y_n^{(1)} + y_n^{(2)} + \dots + y_n^{(k+1)}) / (k+1) \right) \right]^p}{\left(\|y_n^{(1)}\|^p + \|y_n^{(2)}\|^p + \dots + \|y_n^{(k+1)}\|^p \right) / (k+1)} \\ &\leq \frac{\left((1 + \|y_n^{(2)}\| + \dots + \|y_n^{(k+1)}\|) / (k+1) \right)^p}{\left(1 + \|y_n^{(2)}\|^p + \dots + \|y_n^{(k+1)}\|^p \right) / (k+1)} \leq 1. \end{aligned} \quad (12)$$

It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\left[f \left((y_n^{(1)} + y_n^{(2)} + \dots + y_n^{(k+1)}) / (k+1) \right) \right]^p}{\left(\|y_n^{(1)}\|^p + \|y_n^{(2)}\|^p + \dots + \|y_n^{(k+1)}\|^p \right) / (k+1)} &= 1 \\ &= \lim_{n \rightarrow \infty} \frac{\left((1 + \|y_n^{(2)}\| + \dots + \|y_n^{(k+1)}\|) / (k+1) \right)^p}{\left(1 + \|y_n^{(2)}\|^p + \dots + \|y_n^{(k+1)}\|^p \right) / (k+1)}. \end{aligned} \quad (13)$$

We shall show that

$$\lim_{n \rightarrow \infty} \|y_n^{(i)}\| = 1 \quad (14)$$

for every $i = 1, \dots, k+1$. To this end, it is enough to show that every subsequence of the sequence $\{\|y_i^{(n)}\|\}$ has a further subsequence converging to 1. Since every bounded sequence has a convergent subsequence, without loss of generality, we may assume that the sequence $\{\|y_i^{(n)}\|\}$ converges to some t_i . The right equality in (13) shows that

$$\left(\frac{1}{k+1} \sum_{i=1}^{k+1} t_i \right)^p = \frac{1}{k+1} \sum_{i=1}^{k+1} t_i \quad (15)$$

where $t_1 = 1$. This is the equality in Hölder's inequality, so it holds only if

$$\frac{1}{k+1} \sum_{m=1}^{k+1} t_m^p = t_i^p \quad (16)$$

for every $i = 1, \dots, k+1$. Consequently, $t_i = t_1 = 1$ for every $i = 1, \dots, k+1$ and the left equality in (13) shows that

$$\lim_{n \rightarrow \infty} f(y_n^{(1)} + y_n^{(2)} + \dots + y_n^{(k+1)}) = k+1. \quad (17)$$

Combining equality (17) with the assumption that X is a k -uniformly extremely convex space, by Theorem 7 we have

$$\lim_{n \rightarrow \infty} V(y_n^{(1)}, y_n^{(2)}, \dots, y_n^{(k+1)}) = 0; \quad (18)$$

this contradicts (11). $\square$

In particular, considering the special cases of Theorems 7 and 8 when $k = 1$, we obtain a necessary and sufficient condition for uniform extreme convexity and a characteristic inequality of uniformly extremely convex spaces.

Corollary 9. *A Banach space X is uniformly extremely convex if and only if for any sequences $\{x_n\}, \{y_n\}$ in X , if $\lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|y_n\| = \tau$ and for some norm-one functional $f \in X^*$, $\lim_{n \rightarrow \infty} f(x_n + y_n) = 2\tau$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Corollary 10. *Let $1 < p < +\infty$, X be a Banach space, and M be an arbitrary bounded subset of X . Then X is uniformly extremely convex if and only if for any $\epsilon > 0$, there exists $0 < \delta(\epsilon, p) < 1$ such that the inequality*

$$\frac{[f(x+y)]^p}{2^p} \leq (1 - \delta(\epsilon, p)) \frac{\|x\|^p + \|y\|^p}{2} \quad (19)$$

holds for any norm-one functional $f \in X^$ and all elements $x, y \in M$ with $\|x - y\| \geq \epsilon$.*

In addition to two characterizations of uniformly extremely convex spaces presented in Lemmas 4 and 5, we can also obtain another feature of uniformly extremely convex spaces as follows.

Corollary 11. *A Banach space X is uniformly extremely convex if and only if X is strictly convex and k -uniformly extremely convex.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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